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# A NUMERICAL APPROACH FOR ESTIMATING THE EFFECTIVE ELASTIC PROPERTIES OF HONEYCOMB STRUCTURES

**Nguyen Trung Kien**[1,*], **Nguyen Anh Minh**[2]
[1] *Ho Chi Minh City University of Technology and Education, Vietnam*
[2] *Ton Duc Thang University, Ho Chi Minh City, Vietnam*
*E-mail: kiennt@hcmute.edu.vn
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**Abstract.** This paper presents a method for calculating effective elastic moduli of honeycomb core of sandwich plates. The theory is based on the homogenization method for periodic composite materials. Abaqus software is used to model the localization elastic problem on a representative volume element and then the balance of micro-macroscopic deformation energy enables to calculate homogenized elastic moduli of the honeycomb core. Numerical examples are carried out and compared to other models.

*Keywords*: Honeycomb sandwich plate, homogenization method, periodic composite materials.

## 1. INTRODUCTION

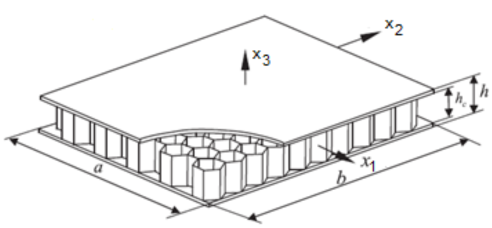
Sandwich structures are used in a wide variety of applications including aircraft, aerospace, naval/marine, construction, and transportation industries where strong stiff and light structures are required. These structures are composed of a core bonded to two face sheets, among which the type-honeycomb core is much used. In practice, the difficulties on the behavior analysis of these honeycomb sandwich structures reside on the calculation of effective elastic properties of honeycomb core. This subject has been attracted several researches. Gibson et al. [1], Masters and Evans [2] proposed an analytical solution for determining the in-plane elastic properties of honeycomb. Abd-el Sayed et al [3] investigated a theoretical approach to the deformation of honeycomb-based composite materials from which in-plane properties of honeycomb are derived. The out-of-plane properties of honeycomb have been studied by Liu and Zhao [4], Shi [5] and Grediac [6] using analytical and numerical methods. Other approaches in this subject can be found in the works of Zhang and Ashby [7] and Nast [8]. A literature review shows that most of studies used analytical approaches which requires hypothesis on the formulation. Alternatively, the homogenization method has been developed in recent years for the behavior analysis of heterogeneous composites [9–12], this approach is simple and efficient in estimating the effective material properties, especially when a commercial package of finite element method is applied.

This paper aims to present a numerical approach for estimating the homogenized elastic properties of periodic honeycomb structures. The theory is based on homogenization method for periodic composite materials. The numerical results are compared to earlier works to verify the accuracy of the present study and to investigated the effects of side-to-thickness ratio of the unit honeycomb cell on the in-plane elastic moduli of honeycomb core.

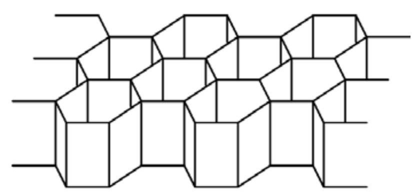
## 2. THEORETICAL FORMULATION

Consider a honeycomb sandwich plate as Fig. 1a in which the honeycomb core is constituted by honeycombs periodic in two directions $x_1$ and $x_2$ of the plate (Fig. 1b). Therefore, it can extract a representative volume cell $Y$ with sides $2l_1 \times 2l_2$ and height $h_c$ as in Figs. 1c and 1d. The calculation of homogenized elastic properties of the honeycomb structure requires the solution field of an elastic localization problem on the unit cell $Y$ that is expressed by

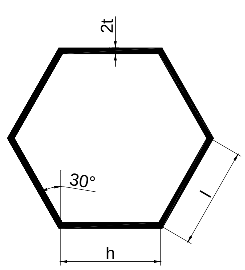
$$\begin{aligned}
&\boldsymbol{\sigma}(\mathbf{x}).\nabla = 0, \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\,\boldsymbol{\varepsilon}(\mathbf{x}),\\
&\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{E} + \mathbf{e}\left(\mathbf{u}^{per}(\mathbf{x})\right),\\
&\mathbf{e}\left(\mathbf{u}^{per}(\mathbf{x})\right) = \mathrm{grad}^s\mathbf{u}^{per}(\mathbf{x}),\\
&\mathbf{u}^{per}(\mathbf{x}) \text{ periodicon } \partial Y_l, \boldsymbol{\sigma}(\mathbf{x}).\mathbf{n} \text{ antiperiodicon } \partial Y_l.
\end{aligned} \tag{1}$$



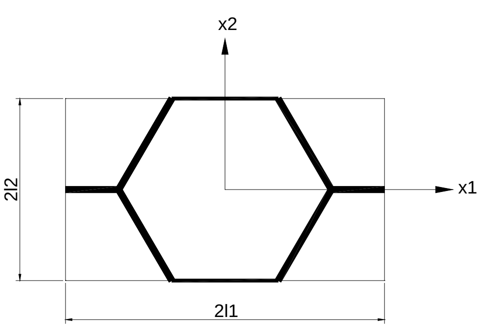

(a) Geometry of a honeycomb sandwich plate



(b) Honeycomb core




(c) Geometry of honeycomb cell




(d) Size of representative volume element

*Fig. 1.* Geometry of a representative volume element of honeycomb core of sandwich plate

where $\nabla$ is nabla operator, $\mathbf{u}^{per}(\mathbf{x})$ is the displacement field periodic on the in-plane boundaries $\partial Y_l$ of $Y$, $\mathbf{e}(\mathbf{u}^{per}(\mathbf{x}))$ the associated strain field, $\boldsymbol{\sigma}(\mathbf{x})$ the stress field, $\mathbf{E}$ the homogeneous in-plane strains. Eq. (1) is the general form of the elastic localization problem of composite materials with periodic boundary conditions.

Moreover, due to the the depth $h_c$ of the honeycomb core is normally much larger than its cell sizes so to simplify the calculation, the problem Eq. (1) can be considered as a plane strain problem. The homogenized elastic constants thus constitute the plane stiffness matrix

$$\mathbf{C}^{\text{hom}} = \begin{bmatrix} C_{11}^{\text{hom}} & C_{12}^{\text{hom}} & 0 \\ C_{12}^{\text{hom}} & C_{22}^{\text{hom}} & 0 \\ 0 & 0 & C_{66}^{\text{hom}} \end{bmatrix} . \tag{2}$$

Once the solution field of Eq. (1) is obtained, the homogenized elastic constants can be derived from the balance of the micro-macroscopic strain energy (Hill's principle) [13] as follows

$$W(\mathbf{E}) = \frac{1}{2}\mathbf{E}\mathbf{C}^{\text{hom}}\mathbf{E} = \min_{\boldsymbol{\varepsilon}\in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{x})\,\mathbf{C}(\mathbf{x})\,\boldsymbol{\varepsilon}(\mathbf{x}) \right\rangle_Y , \tag{3}$$

where $KA$ is used to indicate a kinematic admissible field, the operator $<>$ is the surface average which is followed

$$\langle f(\mathbf{x})\rangle_y = \frac{1}{S_Y}\int_Y f(\mathbf{x})d\mathbf{x} , \tag{4}$$

where $S_Y$ is area of the mid-plane of the unit cell $Y$. Substituting Eq. (2) into Eq. (3) leads to

$$\frac{1}{2}\left(C_{11}^{\text{hom}}E_{11}^2 + C_{22}^{\text{hom}}E_{22}^2 + 2C_{12}^{\text{hom}}E_{11}E_{22} + 4C_{66}^{\text{hom}}E_{12}^2\right) = \min_{\boldsymbol{\varepsilon}\in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{x})\,\mathbf{C}(\mathbf{x})\,\boldsymbol{\varepsilon}(\mathbf{x}) \right\rangle . \tag{5}$$

It can be seen from Eq. (5) that the constants $C_{11}^{\text{hom}}, C_{22}^{\text{hom}}, C_{12}^{\text{hom}}, C_{66}^{\text{hom}}$ can be derived by appropriate choices of the in-plane homogeneous strain field and by the calculation of the average of strain energy on the cell $Y$ subjected to periodic boundary conditions (see Eq. (1)), and then homogenized elastic moduli $E_1^{\text{hom}}, E_2^{\text{hom}}, \nu_{12}^{\text{hom}}, G_{12}^{\text{hom}}$ will be obtained. Concretely, four following cases are considered.

Case 1: $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C_{11}^{\text{hom}}$ is determined by

$$\frac{C_{11}^{\text{hom}}}{2} = \min_{\boldsymbol{\varepsilon}\in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{x})\,\mathbf{C}(\mathbf{x})\,\boldsymbol{\varepsilon}(\mathbf{x}) \right\rangle_Y . \tag{6}$$

Case 2: $\mathbf{E} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C_{22}^{\text{hom}}$ is calculated by

$$\frac{C_{22}^{\text{hom}}}{2} = \min_{\boldsymbol{\varepsilon}\in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{x})\,\mathbf{C}(\mathbf{x})\,\boldsymbol{\varepsilon}(\mathbf{x}) \right\rangle_Y . \tag{7}$$

Case 3: $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_{12}^{\text{hom}}$ is derived from

$$\frac{1}{2}\left(C_{11}^{\text{hom}} + C_{22}^{\text{hom}} + 2C_{12}^{\text{hom}}\right) = \min_{\varepsilon \in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}\left(\mathbf{x}\right)\mathbf{C}\left(\mathbf{x}\right)\boldsymbol{\varepsilon}\left(\mathbf{x}\right)\right\rangle_Y . \tag{8}$$

Case 4: $\mathbf{E} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C_{66}^{\text{hom}}$ is determined by

$$2C_{66}^{\text{hom}} = \min_{\varepsilon \in KA} \left\langle \frac{1}{2}\boldsymbol{\varepsilon}\left(\mathbf{x}\right)\mathbf{C}\left(\mathbf{x}\right)\boldsymbol{\varepsilon}\left(\mathbf{x}\right)\right\rangle_Y . \tag{9}$$

Moreover, it is observed from Fig. 1d that the unit cell $Y$ is symmetric in the in-plane directions, that enables to obtain simplifications, more specifically concerning the boundary conditions. These symmetries lead to the use of "Dirichlet" boundary conditions instead of "periodicity" conditions, the former ones being easier to account for within finite element computations. In general way, if $\mathbf{x} \rightarrow \mathbf{u}\left(\mathbf{x}^*\right)$ is the image of the displacement field $\mathbf{x} \rightarrow \mathbf{u}\left(\mathbf{x}\right)$ by a symmetry $\mathbf{S}$, the field $\mathbf{u}\left(\mathbf{x}^*\right)$ is given by

$$\mathbf{u}\left(\mathbf{x}^*\right) = \mathbf{S}\mathbf{u}\left({}^t\mathbf{S}\mathbf{x}\right) , \tag{10}$$

while the image of a second order tensor by $\mathbf{S}$ is expressed as

$$\mathbf{T}\left(\mathbf{x}^*\right) = \mathbf{S}\mathbf{T}\left({}^t\mathbf{S}\mathbf{x}\right)^t\mathbf{S} . \tag{11}$$

The formulations (10) and (11) imply that the image by $\mathbf{S}$ of the solution field $\left(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}\right)$ of Eq. (1) complies with the following system

$$\begin{aligned} &\boldsymbol{\sigma}^*\left(\mathbf{x}\right).\nabla = 0, \boldsymbol{\sigma}^*\left(\mathbf{x}\right) = \mathbf{C}\left(\mathbf{x}\right)\boldsymbol{\varepsilon}^*\left(\mathbf{x}\right), \\ &\boldsymbol{\varepsilon}^*\left(\mathbf{x}\right) = \mathbf{S}\mathbf{E}\left({}^t\mathbf{S}\mathbf{x}\right)^t\mathbf{S} + \mathbf{S}\mathbf{e}\left({}^t\mathbf{S}\mathbf{x}\right)^t\mathbf{S}, \\ &\mathbf{e}\left(\mathbf{u}^{*per}\left(\mathbf{x}\right)\right) = \text{grad}^s\mathbf{u}^{*per}\left(\mathbf{x}\right), \\ &\mathbf{u}^{*per}\left(\mathbf{x}\right) \text{ periodic on } \partial Y_l, \boldsymbol{\sigma}^*\left(\mathbf{x}\right).\mathbf{n} \text{ antiperiodic on } \partial Y_l. \end{aligned} \tag{12}$$

It can be seen from Eq. (12) that if a symmetry $\mathbf{S}$ and macroscopic strain $\mathbf{E}$ are chosen as: $\mathbf{E} = \mathbf{S}\mathbf{E}\left({}^t\mathbf{S}\mathbf{x}\right)^t\mathbf{S}$, the start and non-start variables will comply with the same system of equations so that $\mathbf{u}^{*per}\left(\mathbf{x}\right) = \mathbf{u}^{per}\left(\mathbf{x}\right), \boldsymbol{\sigma}^*\left(\mathbf{x}\right) = \boldsymbol{\sigma}\left(\mathbf{x}\right)$. Otherwise, if the following condition is satisfied: $\mathbf{E} = -\mathbf{S}\mathbf{E}\left({}^t\mathbf{S}\mathbf{x}\right)^t\mathbf{S}$, $\mathbf{u}^{*per}\left(\mathbf{x}\right) = -\mathbf{u}^{per}\left(\mathbf{x}\right), \boldsymbol{\sigma}^*\left(\mathbf{x}\right) = -\boldsymbol{\sigma}\left(\mathbf{x}\right)$. Practically, the symmetry of the unit cell (Fig. 1d) along the $x_1$- and $x_2$-axis allows to consider a quarter of $Y$ as Fig. 2.

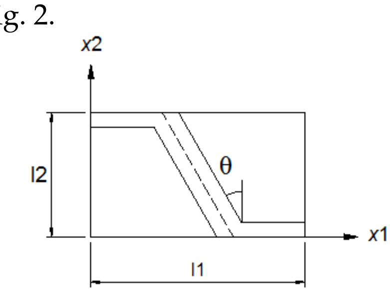



*Fig. 2.* Geometry of a quarter of the unit cell $Y$

For Case 1 (Eq. (6)), the symmetry $\mathbf{S}$ with respect to the $x_1$-axis is given by $\mathbf{S}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the stresses $\boldsymbol{\sigma}^*(\mathbf{x})$ and displacements $\mathbf{u}^{*per}(\mathbf{x})$ are therefore of the form

$$\boldsymbol{\sigma}^*(\mathbf{x}) = \begin{pmatrix} \sigma_{11} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{22} \end{pmatrix} (-x_1, x_2)\ , \mathbf{u}^{*per}(\mathbf{x}) = \begin{Bmatrix} -u_1 \\ u_2 \end{Bmatrix} (-x_1, x_2)\,. \tag{13}$$

Additionally, by using the periodic boundary conditions, the following results are obtained

$$u_1^{per}(0, x_2) = 0, u_1^{per}(l_1, x_2) = u_1^{per}(-l_1, x_2,) = 0, \sigma_{12}(\mathbf{l}_1, x_2) = 0, \mathbf{l}_1 = \{-l_1, 0, l_1\}\,. \tag{14}$$

Similarly, by considering the symmetry with respect to the $x_2$-axis with $\mathbf{S}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the following boundary conditions are obtained

$$u_2^{per}(x_1, \mathbf{l}_2) = 0, \mathbf{l}_2 = \{-l_2, 0, l_2\}\,, \sigma_{12}(x_1, \mathbf{l}_2) = 0, \mathbf{l}_2 = \{-l_2, 0, l_2\}\,. \tag{15}$$

Finally, for the first case, the symmetry of the unit cell $Y$ according to the $x_1$ and $x_2$-axis leads to the following boundary conditions

$$x_1 = 0 : u_1 = 0, \sigma_{12} = 0, x_1 = l_1 : u_1 = l_1, \sigma_{12} = 0, x_2 = 0, l_2 : u_2 = 0, \sigma_{21} = 0. \tag{16}$$

Similarly, the boundary conditions of Case 2 (Eq. (7)) are given by

$$x_2 = 0 : u_2 = 0, \sigma_{21} = 0, x_2 = l_2 : u_2 = l_2, \sigma_{21} = 0, x_1 = 0, l_1 : u_1 = 0, \sigma_{12} = 0. \tag{17}$$

For Case 3 (Eq. (8))

$$\begin{aligned} &x_1 = 0 : u_1 = 0, \sigma_{12} = 0, x_1 = l_1 : u_1 = l_1, \sigma_{12} = 0, \\ &x_2 = 0 : u_2 = 0, \sigma_{21} = 0, x_2 = l_2 : u_2 = l_2, \sigma_{21} = 0. \end{aligned} \tag{18}$$

For Case 4 (Eq. (9))

$$\begin{aligned} &x_1 = 0 : u_2 = 0, \sigma_{11} = 0, x_1 = l_1 : u_2 = l_1, \sigma_{11} = 0, \\ &x_2 = 0 : u_1 = 0, \sigma_{22} = 0, x_2 = l_2 : u_1 = l_2, \sigma_{22} = 0. \end{aligned} \tag{19}$$

It can be seen that the boundary conditions defined in Eqs. (16)-(19) enable to easily model the unit cell $Y$ by using a commercial software.

## 3. NUMERICAL RESULTS AND DISCUSSIONS

A number of numerical examples will be considered in this section to verify the accuracy of the present theory and to investigate the effects of side-to-thickness ratio of the unit honeycomb cell on the in-plane effective elastic moduli. Unless specific mention, the following parameters of the honeycomb cell is used (in [2]): $E = 1$ GPa, $\nu = 0.3$, $l = h$, $t = 0.1$ mm, $\theta = 30^\circ$. The elastic localization problem will be modelized by Abaqus software in which a linear plane strain element CPE4R (Fig. 3) is used with a refined meshing to assure the convergence of the strain energy. Due to the symmetry of the unit cell in the $x_1$- and $x_2$-directions, a quarter of the cell with the prescribed displacement boundary conditions is carried out.

Tab. 1 presents the comparison of homogenized elastic moduli obtained from the present study and those obtained from Gibson et al. [1], Masters and Evans [2]. It can be

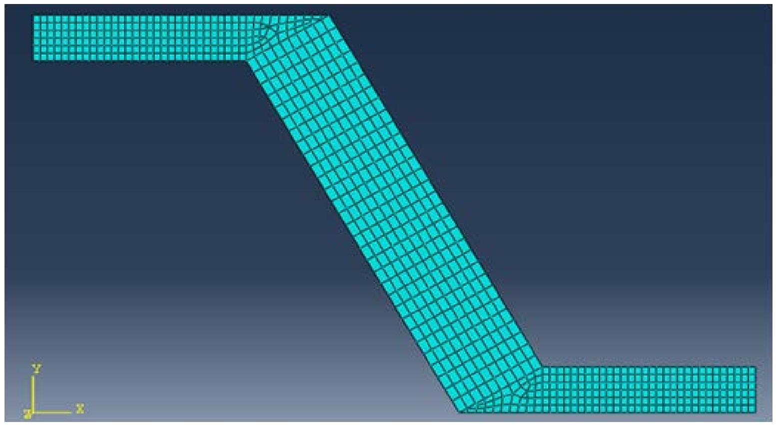



*Fig. 3.* A meshing on a quarter of the unit cell $Y$ using a linear plane strain element CPE4R

*Table 1.* Homogenized elastic moduli with $l/t = 10$

| Method | $E_1^{hom}$ (GPa) | $E_2^{hom}$ (GPa) | $\nu_{12}^{hom}$ | $\nu_{21}^{hom}$ | $G_{12}^{hom}$ (GPa) |
|---|---|---|---|---|---|
| Present | 0.0189 | 0.0189 | 0.8632 | 0.8632 | 0.0051 |
| Gibson et al. [1] | 0.0185 | 0.0185 | 1 | 1 | 0.0046 |
| Master and Evans [2] | 0.0165 | 0.0165 | 0.8571 | 0.8571 | 0.0044 |

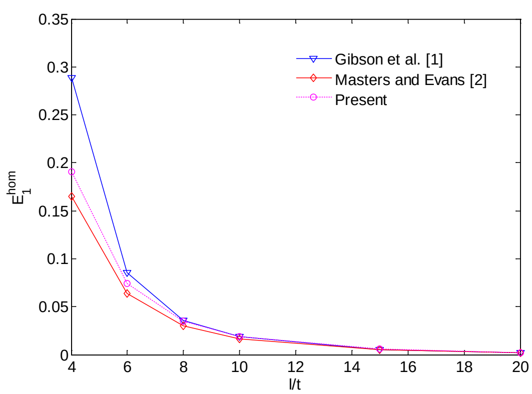



*Fig. 4.* Variation of homogenized Young's modulus $E_1^{hom}$(GPa) with respect to $l/t$

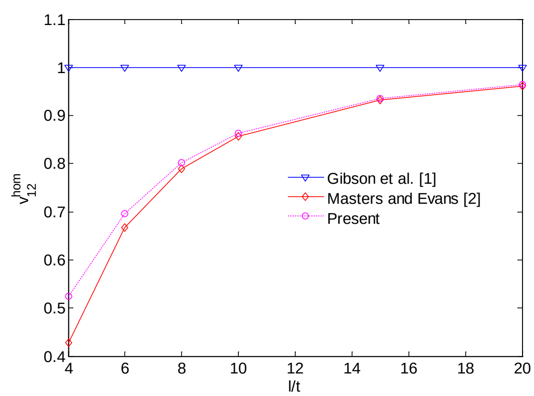



*Fig. 5.* Variation of homogenized Poisson's ratio $\nu_{12}^{hom}$ with respect to $l/t$

seen from this table that the present solution is similar with that of [1] for Young's moduli while the Poisson's ratio derived from the present study is in well agreement with that of [2]. It should be noted that the expression of Poisson's ratio of [1] requires $v_{12}^{hom} = v_{21}^{hom}$, that will be lead to the numerical problems in the calculation of the plane stiffness constants $C_{ij}^{hom}$. Tab. 1 shows that there are minor differences on the shear modulus

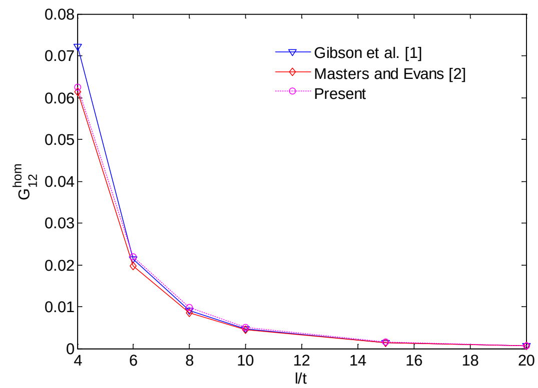



*Fig. 6.* Variation of homogenized shear modulus $G_{12}^{\text{hom}}$(GPa) with respect to $l/t$

between the models. Moreover, in order to investigate the effect of the side-to-thickness ratio of the unit honeycomb cell on the homogenized elastic moduli, Figs. 4-6 display the variation of the homogenized elastic moduli with respect to the $l/t$. It can be seen from Figs. 4-6 that the Young's and shear modulus decrease with an increase of $l/t$, conversely the Poisson's ratio increases with $l/t$ (Fig. 5).

## 4. CONCLUSIONS

This paper proposed a numerical approach for estimating the effective elastic moduli of periodic honeycomb structures. The theory is based on the homogenization method for periodic composite materials. Abaqus software is used to model the localization elastic problem on a representative volume element and then the balance of micro-macroscopic deformation energy enables to calculate homogenized elastic moduli of the honeycomb core. Numerical examples are carried out and compared to other models. The present method is found to be simple and efficient in predicting equivalent elastic moduli of honeycomb structures.

## ACKNOWLEDGEMENTS

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