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CAST3M IMPLEMENTATION OF THE EXTENDED FINITE ELEMENT METHOD FOR COHESIVE CRACK

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Abstract. In this paper, the finite element for cohesive crack for quasi-brittle materials is constructed by XFEM, which allows incorporation of the displacement discontinuities in the element. The algorithm of construction and procedures for involving this finite element into code Cast3M are presented. The numerical calculations in fracture mechanics are presented to demonstrate the benefits of the proposed implementation.

Key words: Fracture, cohesive crack, quasi-brittle material, XFEM, Cast3M.

1. INTRODUCTION

The extended finite element method (XFEM) was first developed for two-dimensional linear elastic fracture mechanics [1, 2, 3], then it was used for crack growth, intersecting cracks and fluid mechanics. In the recent years, XFEM has been also applied to the cohesive crack model [4], [6]-[9]. The model which Hillerborg introduced in [5] has been used extensively in non-linear fracture mechanics of quasi-brittle materials and is applied in this paper. There are other developments of XFEM to existing codes. An implementation of the extended finite element method for linear elastic fracture mechanics problem within the finite element software Abaqus is introduced in [11]. In the same interest, an object enriched finite element library, named OpenXFEM++ which has been built and used to solve various 2D linear elastic fracture problems with great success, is presented in [12]. The finite element for the brittle crack by XFEM is constructed and involved into code Cast3M [3]. In order to produce extendable code, the objective of this paper is to present the algorithm of construction of new extended cohesive crack element 4 nodes and the way to add it into an existing finite element code Cast3M.

2. COHESIVE CRACK FOR QUASI-BRITTLE MATERIALS

The fracture process in quasi-brittle materials like concrete, rocks, ceramics, fiber reinforced composites etc., is characterized by the contribution of the resistance from aggregate interlocking, and another inelastic effect. At the crack tip region there exists a narrow strip, where occur non-linear phenomena, such as plasticity yielding, micro- cracks, etc., however the rest of the body presents an elastic behaviour (Fig. 1a). These inelastic located processes can be represented by cohesive models, which simulate the damage region close to the crack tip. These models use to represent those non-linear phenomena such as the "cohesive forces", which are applied along the "cohesive crack". These forces are defined by relations between stresses and displacements, the material resistance decreases proportionally to the increase of the displacements [4].

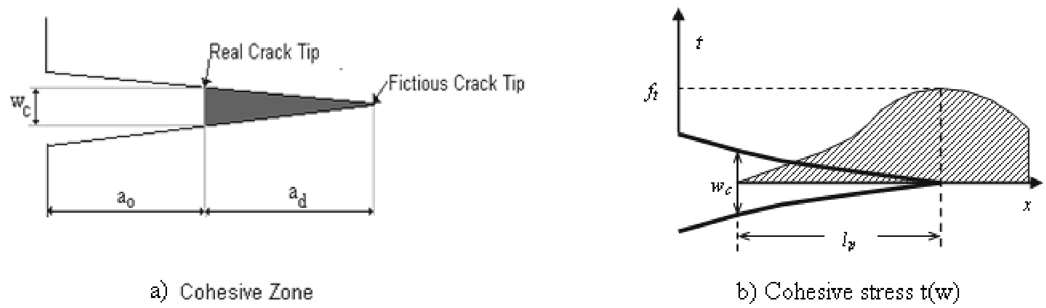


Fig. 1. Sketch showing the process zone of a cohesive law with smooth crack closure Cohesive stress $t(w)$ used by Hillerborg [5]

The stress closing or cohesive stress $t(w)$ is applied to the crack surface. These cohesive stresses depend on the crack opening w, and vary from zero at a characteristic crack opening w_c, to the tensile strength of the material f_t at the tip of the crack (Figure. 1b). It is assumed that the cohesive stresses close the crack smoothly, thus even when the un-cracked material is considered as linear elastic, and often used in the modelling of cementitious materials, stresses are finite in the un-cracked material at the crack tip. In other words, during Mode I crack propagation, the stress intensity factor K_I is zero and the condition for crack propagation is that the stress at the crack tip has reached the tensile strength f_t. The zone in which the cohesive stresses are located was originally called the micro-cracked zone [5], and later the *process zone* or *fracture zone* [6].

Mathematical formulation for cohesive crack problem

Consider a domain Ω as shown in Fig. 2, containing a crack Γ_d. The part of the crack where a cohesive law acted is denoted by Γ_{coh}. Prescribed tractions F are imposed on the boundary Γ_F whereas prescribed displacements $\bar{\boldsymbol{u}}$ (assumed to be zero for simplicity) are imposed on Γ_u. The stress field inside the domain, $\boldsymbol{\sigma}$ is related to the external loading $\boldsymbol{F}$ and the tractions $\boldsymbol{t+}, \boldsymbol{t-}$ in the cohesive zone through the equilibrium equations.

The equilibrium conditions for the quasi-brittle materials having cohesive crack without body force are [7]:

$$\begin{aligned}
&\nabla \cdot \boldsymbol{\sigma} = 0, \\
&\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{F} \text{ on } \Gamma_F, \\
&\boldsymbol{\sigma} \cdot \boldsymbol{m} = -\boldsymbol{t}^+ \text{ on } \Gamma^+_{\mathrm{coh}}, \boldsymbol{\sigma} \cdot \boldsymbol{m} = \boldsymbol{t}^- \text{ on } \Gamma^-_{\mathrm{coh}},
\end{aligned} \tag{1}$$

$\boldsymbol{m}, \boldsymbol{n}$ are the normal vectors for Γ_F and Γ_{coh}, respectively.

The equilibrium condition across surface Γ_{coh} is :

$$\boldsymbol{t} \equiv \boldsymbol{t}^- = -\boldsymbol{t}^+$$

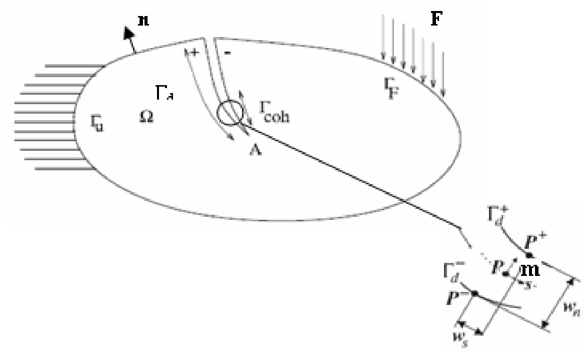


Fig. 2. Geometry of modelled domain and notation

The displacement discontinuity $\boldsymbol{w}$ across Γ_d can be expressed in terms of the displacement vector $\boldsymbol{u}$ computed on two sides of the discontinuity:

$$\boldsymbol{w} = \boldsymbol{u}\Big|_{\Gamma_d^+} - \boldsymbol{u}\Big|_{\Gamma_d^-}$$

Let U be the space of admissible displacements $\boldsymbol{u}$ in Ω; i.e. such that $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_u, $\boldsymbol{u}$ possibly discontinuous on Γ_d and $\boldsymbol{u} \in C^0$ (C^0 is the space of Sobolev) in $\Omega \setminus \Gamma_d$. By introducing the test functions $\boldsymbol{v} \in U_0$ (the displacements are equal zero on Γ_u), the weak form of the equilibrium equations reads [7]:

$$\begin{aligned}
&\int_\Omega \sigma(u) : \varepsilon(v)\, d\Omega = \int_{\Gamma_F} F \cdot v d\Gamma + \int_{\Gamma_d^+} t^+ \cdot v d\Gamma + \int_{\Gamma_d^-} t^- \cdot v d\Gamma \quad \forall v \in U_0 \\
&\int_\Omega \sigma(u) : \varepsilon(v)\, d\Omega = \int_{\Gamma_F} F \cdot v d\Gamma - \int_{\Gamma_d} t \cdot w d\Gamma \\
&\int_\Omega \sigma(u) : \varepsilon(v)\, d\Omega + \int_{\Gamma_d} t \cdot w d\Gamma = \int_{\Gamma_F} F \cdot v d\Gamma
\end{aligned} \tag{2}$$

The equation (2) is weak form of (1) and it includes cohesive stresses $\boldsymbol{t}$ on the two crack faces.

3. THE EXTENDED FINITE ELEMENT METHOD

Partition of unity

A collection of functions $\varphi_i(x)$, each belonging to a node, defined over a body $\Omega(x \in \Omega)$ forms a partition of unity [1] if:

$$\sum_{i=1}^{n} \varphi_i(x) = 1 \tag{3}$$

n is the number of nodal points. The displacement approximation using partition of unity can be formed by:

$$u^h(x) = \sum_{i=1}^{n} \varphi_i(x) \left(\sum_{\alpha=1}^{m} \psi_\alpha(x)\, a_i^\alpha \right) \tag{4}$$

where ψ_α are enrichment functions, m is the number of enrichment functions.

Extended finite element method

Finite element shape functions N_i are also a partition of unity because

$$\sum_{i=1}^{n} N_i(x) = 1$$

Therefore, finite element shape functions are chosen as the partition of unity functions. From equation (4), we note that the finite element space ($\psi_1 = 1, \psi_\alpha = 0(\alpha \neq 1)$) is a subspace of the enriched space u^h. The enriched displacement approximation is written as [2]

$$u^h(x) = \sum_{I\in N} N_I(x) \left[u_I + \underbrace{H(x)\, a_I}_{I\in N_\Gamma} + \underbrace{\sum_{\alpha=1}^{4} F_\alpha(x)\, b_I^\alpha}_{I\in N_\Lambda} \right] \tag{5}$$

where u_I is the nodal displacement vector associated with the continuous part of the finite element, a_I is the nodal enriched degree of freedom vector associated with the Heaviside function, and b_I^α is the nodal enriched degree vector associated with the elastic asymptotic crack tip function. N is the set of all nodes in the mesh; N_Γ is the set of nodes whose shape function support is cut by the crack interior; and N_Λ is the set of nodes whose shape function support is cut by the crack tip ($N_\Gamma \cap N_\Lambda = \emptyset$) (see [3]).

The interior of crack is modeled by the generalized Heaviside enrichment function H, where H takes the value +1 above the crack and -1 below the crack [2]

$$H(x,y) = \begin{cases} 1 \text{ if } y > 0 \\ -1 \text{ if } y < 0 \end{cases} \tag{6}$$

The functions F_α are defined as:

$$\{F_\alpha(r,\theta)\} = \left\{ \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\cos\theta, \sqrt{r}\cos\left(\frac{\theta}{2}\right)\cos\theta \right\} \tag{7}$$

where (r,θ) are the local polar coordinates at the crack tip. Note that the first function in (7), $\sqrt{r}\sin(\theta/2)$, is the discontinuous across the crack faces whereas the last three functions are continuous.

4. DISCRETISED EQUATIONS AND FINITE ELEMENT FOR THE COHESIVE CRACK

The numerical simulation of the cohesive crack is hereby performed within the extended finite element method which is presented in the previous section. The displacement field $\boldsymbol{u}$ of the body can be additively decomposed into a continuous part $\boldsymbol{u}_{\text{cont}}$ and a discontinuous part $\boldsymbol{u}_{\text{disc}}$:

$$\boldsymbol{u}(x) = \boldsymbol{u}_{\text{cont}}(x) + \boldsymbol{u}_{\text{disc}}(x). \tag{8}$$

Inserting the Heaviside function into the discontinuous part, the displacement field in element where the extra degrees of freedom are acted can be written as

$$\boldsymbol{u}(x) = N(x)\boldsymbol{a} + HN(x)\boldsymbol{b},$$

vectors $\boldsymbol{a}$ has the standard degrees and $\boldsymbol{b}$ has the extra degrees.

The displacement discontinuity across Γ_d can be given

$$\boldsymbol{w} = 2N(x)\boldsymbol{b} \tag{9}$$

The strain field and stress field in element can be expressed

$$\begin{aligned} \boldsymbol{\varepsilon} &= B\boldsymbol{a} + HB\boldsymbol{b} \\ \boldsymbol{\sigma} &= \boldsymbol{D}(B\boldsymbol{a} + HB\boldsymbol{b}) \end{aligned} \tag{10}$$

where $\boldsymbol{D}$ relates the stress and strain.

If T is called the relation between the traction and crack displacement, the traction can be written as

$$\boldsymbol{t} = \boldsymbol{T}.\boldsymbol{w} \text{ or } \left\{ \begin{array}{c} t_n \\ t_s \end{array} \right\} = [\boldsymbol{T}] \left\{ \begin{array}{c} w_n \\ w_s \end{array} \right\} \tag{11}$$

Here, we used the cohesive law $\boldsymbol{T}$ which proposed by Hillerborg [5], the normal traction force t_n transmitted across a discontinuity is made by an exponentially decaying function of the history parameter:

$$t_n = f_t \exp\left(-\frac{f_t}{G_f}k\right),$$

where f_t is the tensile strength of the material and G_f is the fracture energy and k (internal variable) is the largest value of the normal separation w_n. The crack shear stiffness is also made by a function of the history parameter. The shear traction t_s acting on the discontinuity surface is calculated from

$$t_s = d_{int} \exp\left(h_s k\right) w_s,$$

where d_{int} is the initial crack shear stiffness (k= 0), w_s is the crack sliding displacement and h_s is equal to:

$$h_s = \ln\left(d_{k=1.0}/d_{int}\right)$$

where $d_{k=1.0}$ is crack shear stiffness when $k = 1.0$.

Replace the displacement field (9), (10), (11) into the equation (2), the discrete equations can be obtained as follow:

$$\left[\begin{array}{cc} \int_\Omega B^T DB d\Omega & \int_\Omega HB^T DB d\Omega \\ \int_\Omega HB^T B d\Omega & \int_\Omega HB^T DB d\Omega \end{array} \right] \left\{ \begin{array}{c} a \\ b \end{array} \right\} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 4\int_{\Gamma_d} N^T T N d\Gamma \end{array} \right] \left\{ \begin{array}{c} a \\ b \end{array} \right\} = \int_{\Gamma_F} N^T F d\Gamma$$

The discretised form of virtual work for incremental displacements (da, db) is written as:

$$\begin{bmatrix} \int_\Omega B^T DBd\Omega & \int_\Omega HB^T DBd\Omega \\ \int_\Omega HB^T Bd\Omega & \int_\Omega HB^T DBd\Omega + 4\int_{\Gamma_d} N^T TNd\Gamma \end{bmatrix} \begin{Bmatrix} da \\ db \end{Bmatrix} = \begin{Bmatrix} f^{ext} \\ 0 \end{Bmatrix} - \begin{Bmatrix} f_a^{int} \\ f_b^{int} \end{Bmatrix} \tag{12}$$

So, the element matrix can be written:

$$K_e = \begin{bmatrix} \int_\Omega B^T DBd\Omega & \int_\Omega HB^T DBd\Omega \\ \int_\Omega HB^T Bd\Omega & \int_\Omega HB^T DBd\Omega + 4\int_{\Gamma_d} N^T TNd\Gamma \end{bmatrix}, \tag{13}$$

where the matrices of shape functions derivatives B for node i are:

$$B_i = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \tag{14}$$

$$f^{\text{ext}} = \int_{\Gamma_F} N^T Fd\Gamma \tag{15}$$

$$\begin{array}{l} f_a^{\text{int}} = \int_\Omega B^T \sigma d\Omega, \\ f_b^{\text{int}} = \int_\Omega HB^T \sigma d\Omega + 2\int_{\Gamma_d} N^T td\Gamma \end{array} \tag{16}$$

Numerical integration for the element with the cohesive crack

As in the standard FEM, it is necessary to perform numerical integration over the element domain for computing the element stiffness matrix (13). The elements which contain the crack include a displacement discontinuity due to the X-FEM formulation. These elements must be subdivided into sub-domains in which the crack is one of the sub-domain boundaries to carry out the numerical integrations. Therefore, the elements non-crossed by a discontinuity are integrated by standard four-point Gauss quadrature. When an element is crossed by a discontinuity, the domains Ω^+ and Ω^- on either side of the discontinuity are triangulated into sub-domains [3]. Within each triangular sub-domain, three-point Gauss quadrature is applied. For the cohesive crack sub-domain, two integration points are positioned on the discontinuity in order to integrate the traction forces.

5. CAST3M IMPLEMENTATION FOR THE COHESIVE ELEMENT

In this part, a procedure for the implementation of the XFEM within the FE code Cast3M [10] for two-dimensional fracture problems is presented. The quadrangular element 4 nodes CRCOH is constructed and involved into Cast3M. The implementation was based on the use of this element and performed in source code of Cast3M by language FORTRAN and enables in modeling of different crack locations and orientations. In order to develop XFEM in numerical calculation, a number of subroutines contain XFEM formulation and some operators represented below such as RIGI for matrix stiffness K of cohesive element, EPSI for the displacement $\boldsymbol{\varepsilon}$, COMP for stress $\boldsymbol{\sigma}$, BSIG for nodal force,... are inserted into software Cast3M .

5.1. Algorithm

- Step 0: Initially, a mesh for quadrilateral element, the location of the crack and the parameters of concrete are inputted.

- Step 1: Using level set to choose crack tip element and crack element. For the element without crack, the standard FEM is applied.

- Step 2: Calculating of the matrix stiffness K for cohesive elements by using (13) and then assembling in the matrix stiffness standard.

- Step 3: Applying a given loading step to solve for the displacement field $\left\{ \begin{array}{c} da \\ db \end{array} \right\}$ using Eq. (12).

- Step 4: Calculating the incremental deformation $d\boldsymbol{\varepsilon}$, and then determinating the stress $\boldsymbol{\sigma}$. Note that the cohesive stresses are local stresses, leading to the co-ordinate system changing.

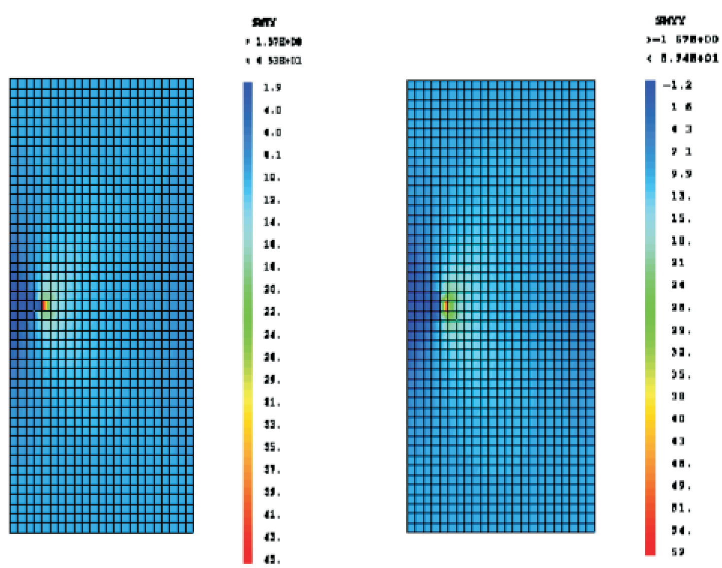
- Step 5: Computing the nodal forces through the program BSIG using Eq. (15) and (16).

- Step 6: Verifying the condition of convergence. If it isn't satisfied, go to step 3.

On the contrast, if the condition is satisfied, the numerical results as the stress, deformation, and displacement for each the loading step can be obtained.

5.2. Numerical example and discussion

Example 1: Consider a plate of width $w = 100$mm and height $L = 200$mm with an edge crack of length $a = 20$mm. The tensile stress $\sigma = 10$MPa. The concretes parameters $E_c = 35000$MPa, $\nu = 0.3$, $f_t = 3$MPa, $G_f = 0.1$N/mm.



σ_{yy} of the cohesive crack $\qquad$ σ_{yy} of the linear elastic crack

Fig. 3. Comparison of the stresses

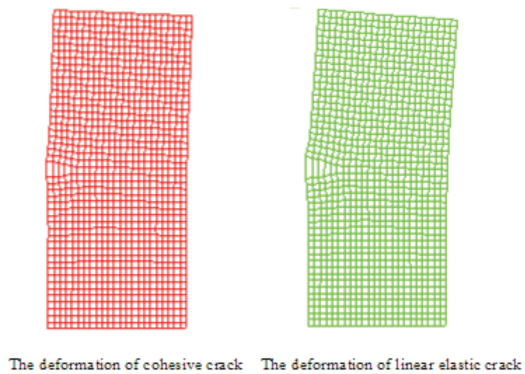


Fig. 4. Comparison of the deformations

Remark: Separation of crack in the linear case (=3.677x10^{-2} mm) is larger than that of cohesive crack (=2.73086x10^{-2} mm). The stress at a crack-tip (= 45MPa) in cohesive crack is also represents a contribution of the cohesive stress in comparison to stress intensity (= 57 MPa) of the linear elastic crack.

Example 2: We use the same dimension of specimen for the incline crack $\alpha = 45^\circ$, length $a = 20$mm.

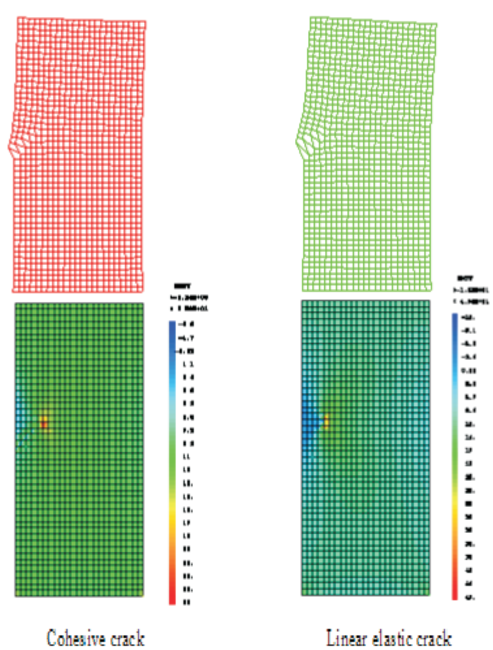


Fig. 5. Comparison of the deformations and the σ_{yy} for incline crack

Remark: In the incline crack case, the influence of the quasi-brittle material on the crack side displacements is presented (see Fig. 5). Clearly, the stress at a crack-tip of cohesive crack ($\boldsymbol{\sigma}_{\max} = 26$ MPa) are smaller than that of linear elastic crack ($\boldsymbol{\sigma}_{\max} = 47$ MPa).

6. CONCLUSIONS

In this paper, a procedure for the implementation of the XFEM within the FE code Cast3M for two-dimensional fracture problems is presented. The quadrangular finite element CRCOH for cohesive crack are constructed and involved in to Cast3M for quasi-brittle materials. This finite element is enriched by the cohesive stress which is related with a separation of the crack sides through the law of Hillerborg. The obtained results are compared with the results of the linear elastic crack showing that the existence of cohesive stress in quasi-brittle materials reduces the separation of crack and stress at crack-tip. So, using the above mentioned model with cohesive crack and given finite element CRCOH is necessary. In the near future, relied upon this framework, the growth of cohesive crack will be continued to develop.

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