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VAN DER POL - DUFFING OSCILLATOR UNDER COMBINED HARMONIC AND RANDOM EXCITATIONS

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Abstract. A new technique is proposed to investigate the response of Van der Pol-Duffing (V-D for short) oscillator to a combination of harmonic and random excitations in the primary resonant frequency region. The analytical approach is based on the stochastic averaging method and equivalent linearization method. The stochastic averaging is applied to the original equation transformed into Cartesian coordinates. Then the resulting nonlinear averaged equations are linearized by the equivalent linearization method so that the equations obtained can be solved exactly by the technique of auxiliary function. Numerical results show that the proposed approximate technique is an effective approach to solving the V-D equation. Although the technique has been used for the V-D equation in the paper, however, it can also be used to solve many other nonlinear oscillators.

Keywords: Van der Pol, Duffing, averaging method, equivalent linearization, harmonic excitation, random excitation.

1. INTRODUCTION

Systems under combined harmonic and random excitations have received a flurry of research effort in the past few decades. Since all natural or man-made systems are, more or less, nonlinear and for those systems the exact solutions are known only for a number of special cases, approximate techniques have been necessarily developed to determine the response statistics of nonlinear systems [1–6]. In this paper, we propose an approximate technique combining two well-known methods, namely, the stochastic averaging method and equivalent linearization method to study V-D oscillator. Over years, the stochastic averaging method has proved to be a powerful approximate technique for the prediction of response of weakly nonlinear vibrations subjected to dampings and random excitations. This method was originally given by Krylov and Bogoliubov [7] and then developed by Mitropolskii [4, 8, 9], Stratonovich [10], and Khasminskii [11]. A comprehensive review attesting the success of the stochastic averaging method in random vibration was done by Roberts and Spanos [12]. The success of the stochastic averaging method is mainly due to its two advantages: the equations of motion of a system are much simplified and the dimensions of the response coordinates are often reduced; the averaged response is a diffusive Markov process and the method of Fokker-Planck (FP) equation, which is still hard to solve analytically so far, can be applied. To solve Fokker-Planck equation, some methods were proposed as in references [5, 6, 13–17].

Meanwhile, the equivalent linearization method, extended from the well known harmonic linearization technique to stochastic problems, is one of the most popular approaches in nonlinear random vibration problems. The method consists of optimally approximating the non- linearities in the given system by linear models so that the solution of the resulting equivalent system is available. An original version of this method was proposed by Caughey [18, 19] and has been developed by many authors [20–25].

V-D oscillator is one of the typical mathematical models for dynamical systems having a single unstable fixed point, along with a single stable limit cycle. Various aspects of this problem have been studied up to recent years [26–33].

The objective of the present paper is to propose a technique combining the stochastic averaging method and equivalent linearization method to study V-D oscillator under combined harmonic and random excitations. By using the conventional equivalent linearization method, the nonlinear averaged V-D equations can be replaced by linear ones whose solution can be found exactly. The basic concept of this technique is that the stochastic averaging of V-D equation is carried out in Cartesian coordinates. Finally, the mean square responses of the V-D system obtained by the proposed technique are validated by numerical simulation results, obtained by Monte-Carlo simulation.

The remainder of the present paper proceeds as follows: Section 2 gives the concepts of proposed approximate technique; Section 3 compares responses of the system obtained by proposed technique to ones of numerical simulation. Summary and conclusions are given in Section 4.

2. APPROXIMATE TECHNIQUE

In this section we are concerned with an equation of motion of the V-D oscillator under combined harmonic and random excitations

$$\ddot{x} - \varepsilon\left(\alpha - \beta x^2\right)\dot{x} + \varepsilon\gamma x^3 + \omega^2 x = \varepsilon P\cos\nu t + \sqrt{\varepsilon}\,\sigma\xi\left(t\right), \tag{1}$$

where α, β, γ, ω, P, ν, σ are positive parameters, ε is a small positive parameter, and function $\xi\left(t\right)$ is a Gaussian white noise process of unit intensity with the correlation function $R_\xi\left(\tau\right) = \left\langle \xi\left(t\right)\xi\left(t+\tau\right)\right\rangle = \delta\left(\tau\right)$, where $\delta\left(\tau\right)$ is the Dirac delta function, and notation $\langle . \rangle$ denotes the mathematical expectation operator. We consider Eq. (1) in the primary resonant frequency region, i.e. parameters ω and ν have the relation

$$\omega^2 - \nu^2 = \varepsilon\Delta, \tag{2}$$

where Δ is a detuning parameter. Substituting (2) into Eq. (1) yields

$$\ddot{x} + \nu^2 x = \varepsilon f\left(x, \dot{x}, \nu t\right) + \sqrt{\varepsilon}\sigma\xi\left(t\right), \tag{3}$$

where

$$f(x, \dot{x}, \nu t) = -\Delta x + \left(\alpha - \beta x^2\right)\dot{x} - \gamma x^3 + P\cos\nu t. \tag{4}$$

We seek the solution of Eq. (3) in the form of [3,4].

$$\begin{aligned} x &= b\cos\nu t + d\sin\nu t, \\ \dot{x} &= -b\nu\sin\nu t + d\nu\cos\nu t, \end{aligned} \tag{5}$$

where b and d are slowly varying random processes satisfying an additional condition

$$\dot{b}\cos\nu t + \dot{d}\sin\nu t = 0. \tag{6}$$

Substituting (5) into Eq. (3) and then solving the resulting equation and Eq. (6) with respect to the derivatives $\dot{b}$ and $\dot{d}$ yield

$$\begin{aligned} \dot{b} &= -\frac{1}{\nu}\left(\varepsilon f + \sqrt{\varepsilon}\sigma\xi(t)\right)\sin\nu t, \\ \dot{d} &= \frac{1}{\nu}\left(\varepsilon f + \sqrt{\varepsilon}\sigma\xi(t)\right)\cos\nu t, \end{aligned} \tag{7}$$

where

$$f = f\left(b\cos\nu t + d\sin\nu t,\ -b\nu\sin\nu t + d\nu\cos\nu t\right). \tag{8}$$

Solving the system (7) is a difficult problem. Thus, an approximate method is required for this system.

2.1. Stochastic averaging method

The system (7) can be simplified by using the stochastic averaging method [4]

$$\begin{aligned} \dot{b} &= \varepsilon H_1(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_1(t), \\ \dot{d} &= \varepsilon H_2(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_2(t), \end{aligned} \tag{9}$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent Gaussian white noises with unit intensity, and the drift coefficients $H_1(b,d)$ and $H_2(b,d)$ are determined as follows

$$H_1(b,d) = -\frac{1}{\nu}\left\langle f\ \sin\nu t\right\rangle_t,\ H_2(b,d) = \frac{1}{\nu}\left\langle f\ \cos\nu t\right\rangle_t, \tag{10}$$

where $\langle . \rangle_t$ represents the time-averaging over one period defined by

$$\langle . \rangle_t = \frac{1}{T}\int_0^T (.)\,dt. \tag{11}$$

Substituting (4) into (10), noting (5), yields the drift coefficients of the system (9)

$$\begin{aligned} H_1(b,d) &= \frac{\alpha}{2}b + \frac{\Delta}{2\nu}d + g_1(b,d), \\ H_2(b,d) &= -\frac{\Delta}{2\nu}b + \frac{\alpha}{2}d + \frac{P}{2\nu} + g_2(b,d), \end{aligned} \tag{12}$$

where

$$g_1(b,d) = -\frac{1}{8\nu}\left(\beta\nu b^3 - 3\gamma b^2 d + \beta\nu b d^2 - 3\gamma d^3\right),$$
$$g_2(b,d) = -\frac{1}{8\nu}\left(3\gamma b^3 + \beta\nu b^2 d + 3\gamma b d^2 + \beta\nu d^3\right). \tag{13}$$

The solution of the FP equation written for the stationary probability density function (PDF) associated with the system (9) is still a difficult problem since $H_1(b,d)$ and $H_2(b,d)$ are nonlinear functions. Thus, in order to overcome this difficulty, the equivalent linearization method is employed.

2.2. Equivalent linearization method and technique of auxiliary function

Following the equivalent linearization method, the nonlinear terms (13) are replaced by

$$\bar{g}_1(b,d) = \eta_{11}b + \eta_{12}d + \eta_{13},$$
$$\bar{g}_2(b,d) = \eta_{21}b + \eta_{22}d + \eta_{23}, \tag{14}$$

where coefficients $\eta_{ij}, i = 1,2; j = 1,2,3$ are to be determined by an optimization criterion. Thus, the functions $H_i, i = 1,2$ in (12) are replaced by linear functions $\bar{H}_i, i = 1,2$ given by

$$\bar{H}_1(b,d) = \alpha_1 b + \beta_1 d + \lambda_1,$$
$$\bar{H}_2(b,d) = \alpha_2 b + \alpha_2 d + \lambda_2, \tag{15}$$

where

$$\alpha_1 = \frac{\alpha}{2} + \eta_{11}, \quad \beta_1 = \frac{\Delta}{2\nu} + \eta_{12},\ \lambda_1 = \eta_{13},$$
$$\alpha_2 = -\frac{\Delta}{2\nu} + \eta_{21},\ \beta_2 = \frac{\alpha}{2} + \eta_{22},\ \lambda_2 = \frac{P}{2\nu} + \eta_{23}, \tag{16}$$

or, equivalently, the system (9) is replaced by

$$\dot{b} = \varepsilon\bar{H}_1(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_1(t),$$
$$\dot{d} = \varepsilon\bar{H}_2(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_2(t). \tag{17}$$

There are some criteria for determining the coefficients $\eta_{ij}, i = 1,2; j = 1,2,3$. The most extensively used criterion is the mean square error criterion which requires that the mean square of errors be minimum [18]. From (12)-(16), errors between H_i and $\bar{H}_i, i = 1,2$ are

$$e_i = g_i(b,d) - (\eta_{i1}b + \eta_{i2}d + \eta_{i3}),\ i = 1,2. \tag{18}$$

So, the mean square error criterion leads to

$$\left\langle e_i^2\right\rangle = \left\langle \left[g_i(b,d) - (\eta_{i1}b + \eta_{i2}d + \eta_{i3})\right]^2\right\rangle \to \min_{\eta_{ij}}. \tag{19}$$

From

$$\frac{\partial}{\partial\eta_{ij}}\left\langle e_i^2\right\rangle = 0,\ \ i = 1,2;\ j = 1,2,3, \tag{20}$$

it follows that

$$\begin{aligned}
&\langle b\, g_1(b,d)\rangle - \langle b^2\rangle \eta_{11} - \langle bd\rangle \eta_{12} - \langle b\rangle \eta_{13} = 0,\\
&\langle d\, g_1(b,d)\rangle - \langle bd\rangle \eta_{11} - \langle d^2\rangle \eta_{12} - \langle d\rangle \eta_{13} = 0,\\
&\langle g_1(b,d)\rangle - \langle b\rangle \eta_{11} - \langle d\rangle \eta_{12} - \eta_{13} = 0,\\
&\langle b\, g_2(b,d)\rangle - \langle b^2\rangle \eta_{21} - \langle bd\rangle \eta_{22} - \langle b\rangle \eta_{23} = 0,\\
&\langle d\, g_2(b,d)\rangle - \langle bd\rangle \eta_{21} - \langle d^2\rangle \eta_{22} - \langle d\rangle \eta_{23} = 0,\\
&\langle g_2(b,d)\rangle - \langle b\rangle \eta_{21} - \langle d\rangle \eta_{22} - \eta_{23} = 0,
\end{aligned} \tag{21}$$

where $g_1(b,d)$, $g_2(b,d)$ are given by (13). Since the linear system (17) is under Gaussian process excitation, one gets that b and d are jointly Gaussian. Hence, all higher moments $\langle g_i\rangle$, $\langle b\, g_i\rangle$, $\langle d\, g_i\rangle$ can be expressed in terms of first and second moments of b and d by below properties of a Gaussian random vector $\vec{X} = (X_1, X_2) = (b, d)$ [32]

$$\begin{aligned}
&\langle X_i^{n+1}\rangle = \langle X_i\rangle \langle X_i^n\rangle + n\sigma_{X_i}^2 \langle X_i^{n-1}\rangle,\\
&\langle X_i X_1^{n_1} X_2^{n_2}\rangle = \langle X_i\rangle \langle X_1^{n_1} X_2^{n_2}\rangle + n_1 k_{X_i X_1} \langle X_1^{n_1-1} X_2^{n_2}\rangle + n_2 k_{X_i X_2} \langle X_1^{n_1} X_2^{n_2-1}\rangle,\\
&\qquad i = 1,2.
\end{aligned} \tag{22}$$

Here $\sigma_{X_i}^2$ is variance of X_i, $k_{X_i X_j}$ denotes covariance of X_i and X_j, and n, n_1 and $n_2 = 0, 1, 2, 3, \ldots$ Thus, from (13) and (22) one gets

$$\begin{aligned}
\eta_{11} &= -\frac{1}{8\nu}\left[6\gamma \langle bd\rangle + 3\beta\nu \langle b^2\rangle + \beta\nu \langle d^2\rangle\right],\\
\eta_{12} &= -\frac{1}{8\nu}\left(2\beta\nu \langle bd\rangle + 3\gamma \langle b^2\rangle + 9\gamma \langle d^2\rangle\right),\\
\eta_{13} &= \frac{1}{4\nu}\left(\langle b\rangle^2 + \langle d\rangle^2\right)\left(\beta\nu \langle b\rangle + 3\gamma \langle d\rangle\right),\\
\eta_{21} &= \frac{1}{8\nu}\left(-2\beta\nu \langle bd\rangle + 9\gamma \langle b^2\rangle + 3\gamma \langle d^2\rangle\right),\\
\eta_{22} &= \frac{1}{8\nu}\left[6\gamma \langle bd\rangle - \beta\nu \langle b^2\rangle - 3\beta\nu \langle d^2\rangle\right],\\
\eta_{23} &= -\frac{1}{4\nu}\left(3\gamma \langle b\rangle - \beta\nu \langle d\rangle\right)\left(\langle b\rangle^2 + \langle d\rangle^2\right).
\end{aligned} \tag{23}$$

From (22), the relation (23) results in six algebraic equations for eleven unknowns: $\eta_{ij}, i = 1,2;\ j = 1,2,3,\ \langle b\rangle, \langle d\rangle, \sigma_b^2, \sigma_d^2, k_{bd}$. In order to close the system (23), it is noted that the FP equation written for the stationary PDF $W(b,d)$ associated with the system (17) takes the form

$$\frac{\partial}{\partial b}\left(\bar{H}_1 W\right) + \frac{\partial}{\partial d}\left(\bar{H}_2 W\right) = \frac{\sigma^2}{4\nu^2}\left[\frac{\partial^2}{\partial b^2} W + \frac{\partial^2}{\partial d^2} W\right], \tag{24}$$

which can be solved exactly by the technique of auxiliary function. In order to integrate Eq. (24), we introduce an auxiliary function $u(b,d)$ with derivatives up to the second order as follows (see [13] for details)

$$\frac{\partial}{\partial b}\left\{\left(\bar{H}_1 W\right) - \frac{\sigma^2}{4\nu^2}\frac{\partial}{\partial b} W + \frac{\partial}{\partial d}(uW)\right\} + \frac{\partial}{\partial d}\left\{\left(\bar{H}_2 W\right) - \frac{\sigma^2}{4\nu^2}\frac{\partial}{\partial d} W - \frac{\partial}{\partial b}(uW)\right\} = 0. \tag{25}$$

We will choose the function $u(b,d)$ so that the equalities below are fulfilled

$$\begin{aligned} &\bar{H}_1 W - \frac{\sigma^2}{4\nu^2}\frac{\partial}{\partial b}W + \frac{\partial}{\partial d}(uW) = 0,\\ &\bar{H}_2 W - \frac{\sigma^2}{4\nu^2}\frac{\partial}{\partial d}W - \frac{\partial}{\partial b}(uW) = 0. \end{aligned} \tag{26}$$

With the notation

$$\Phi(b,d) = \ln W(b,d), \tag{27}$$

the system (26) now reads

$$\begin{aligned} &\bar{H}_1 - \frac{\sigma^2}{4\nu^2}\frac{\partial \Phi}{\partial b} + \frac{\partial u}{\partial d} + u\frac{\partial \Phi}{\partial d} = 0,\\ &\bar{H}_2 - \frac{\sigma^2}{4\nu^2}\frac{\partial \Phi}{\partial d} - \frac{\partial u}{\partial b} - u\frac{\partial \Phi}{\partial b} = 0, \end{aligned} \tag{28}$$

In [13], the author showed that under some particular types of $u(b,d)$ the exact non-trivial solution of the FP Eq. (25) can be found in quadratures. Thus, for Eq. (25), we consider the simple case where the auxiliary function $u(b,d)$ is a constant

$$u(b,d) = u_0 = const. \tag{29}$$

Substituting (15) and (29) into (28) and solving the system in (28) in $\dfrac{\partial \Phi}{\partial b}$ and $\dfrac{\partial \Phi}{\partial d}$ yields

$$\begin{aligned} &\frac{\partial \Phi}{\partial b} = M(b,d,u_0),\\ &\frac{\partial \Phi}{\partial d} = N(b,d,u_0), \end{aligned} \tag{30}$$

where

$$\begin{aligned} M(b,d,u_0) &= \frac{\dfrac{\sigma^2}{4\nu^2}(\alpha_1 b + \beta_1 d + \lambda_1) + u_0(\alpha_2 b + \beta_2 d + \lambda_2)}{u_0^2 + \dfrac{\sigma^4}{16\nu^4}},\\ N(b,d,u_0) &= \frac{\dfrac{\sigma^2}{4\nu^2}(\alpha_2 b + \beta_2 d + \lambda_2) - u_0(\alpha_1 b + \beta_1 d + \lambda_1)}{u_0^2 + \dfrac{\sigma^4}{16\nu^4}}. \end{aligned} \tag{31}$$

Eliminating $\Phi(b,d)$ from the system (30) gives the equation for the constant u_0

$$\frac{\partial M(b,d,u_0)}{\partial d} = \frac{\partial N(b,d,u_0)}{\partial b}. \tag{32}$$

Substituting (31) into Eq. (32) one gets

$$\frac{\sigma^2}{4\nu^2}\beta_1 + \beta_2 u_0 = \frac{\sigma^2}{4\nu^2}\alpha_2 - \alpha_1 u_0. \tag{33}$$

Hence, under the condition (33), the solution $W(b,d)$ of (24) can be found from (27), (30) and (31) by the quadrature

$$\bar{W}(b,d) = \mathrm{C}\exp\left\{\int M(b,d,u_0)\,db + \int N(b,d,u_0)\,dd\right\}, \tag{34}$$

where C is a normalization constant. In case of $\alpha_1 + \beta_2 \neq 0$, the constant u_0 can be determined from Eq. (33) as follows

$$u_0 = \frac{\sigma^2}{4\nu^2}\frac{(\alpha_2 - \beta_1)}{\alpha_1 + \beta_2}. \tag{35}$$

Substituting (31) and (35) into (34), one obtains the solution of FP equation (24)

$$W(b,d) = C\exp\left\{-\zeta_1 b^2 - \zeta_2 d^2 + \zeta_3 bd + \zeta_4 b + \zeta_5 d\right\}, \tag{36}$$

where coefficients $\zeta_i, i = \overline{1,5}$ are determined as follows

$$\begin{aligned}
\zeta_1 &= -\Psi\left(\left(\frac{\alpha}{2} + \eta_{11}\right)(\alpha + \eta_{11} + \eta_{22}) + \left(-\frac{\Delta}{2\nu} + \eta_{21}\right)\left(-\frac{\Delta}{\nu} + \eta_{21} - \eta_{12}\right)\right),\\
\zeta_2 &= -\Psi\left((\alpha + \eta_{11} + \eta_{22})\left(\frac{\alpha}{2} + \eta_{22}\right) - (\eta_{21} + \eta_{12})\left(\frac{\Delta}{2\nu} + \eta_{12}\right)\right),\\
\zeta_3 &= 2\Psi\left(\left(-\frac{\Delta}{2\nu} + \eta_{21}\right)\left(\frac{\alpha}{2} + \eta_{22}\right) + \left(\frac{\Delta}{2\nu} + \eta_{12}\right)\left(\frac{\alpha}{2} + \eta_{11}\right)\right),\\
\zeta_4 &= 2\Psi\left(\eta_{13}(\alpha + \eta_{11} + \eta_{22}) + \left(\frac{P}{2\nu} + \eta_{23}\right)\left(-\frac{\Delta}{\nu} + \eta_{21} - \eta_{12}\right)\right),\\
\zeta_5 &= 2\Psi\left(\left(\frac{\Delta}{\nu} + \eta_{21} - \eta_{21}\right)\eta_{13} + (\alpha + \eta_{11} + \eta_{22})\left(\frac{P}{2\nu} + \eta_{23}\right)\right),
\end{aligned} \tag{37}$$

where

$$\Psi = \frac{2\nu^2(\alpha + \eta_{11} + \eta_{22})}{\sigma^2\left[\left(-\frac{\Delta}{\nu} + \eta_{21} - \eta_{12}\right)^2 + (\alpha + \eta_{11} + \eta_{22})^2\right]}. \tag{38}$$

It is noted that the coefficients ζ_1 and ζ_2 must be positive so that the PDF $W(b,d)$ (37) has a finite integral. Thus, from the stationary PDF (36), moments $\langle b\rangle$, $\langle d\rangle$, σ_b^2, σ_d^2, k_{bd} can be derived in terms of ζ_i, $i = \overline{1,5}$

$$\begin{aligned}
&\langle b\rangle = \frac{2\zeta_2\zeta_4 + \zeta_3\zeta_5}{4\zeta_1\zeta_2 - \zeta_3^2}, \quad \langle d\rangle = \frac{2\zeta_1\zeta_5 + \zeta_3\zeta_4}{4\zeta_1\zeta_2 - \zeta_3^2},\\
&\sigma_b^2 = \frac{2\zeta_2}{4\zeta_1\zeta_2 - \zeta_3^2}, \sigma_d^2 = \frac{2\zeta_1}{4\zeta_1\zeta_2 - \zeta_3^2}, k_{bd} = \frac{\zeta_3}{4\zeta_1\zeta_2 - \zeta_3^2}.
\end{aligned} \tag{39}$$

Hence, relations (23), (37) and (39) give us a closed system. After being found by solving system (23), with noting (37) and (39), the numerical results of coefficients η_{ij}, $i = 1,2$; $j = 1,2,3$ are substituted into (36) to obtain the approximate stationary PDF in b and d of V-D equation (1).

3. NUMERICAL RESULTS

By squaring both sides of the first equation in (5) and then taking mathematical expectation, one obtains

$$\left\langle x^2(t)\right\rangle = \left\langle b^2\right\rangle \cos^2 \nu t + \left\langle d^2\right\rangle \sin^2 \nu t + \langle bd\rangle \sin 2\nu t. \tag{40}$$

Thus, in this approach the mean square response is time varying. Taking averaging Eq. (40) with respect to time yields the following expression

$$\left\langle\left\langle x^2(t)\right\rangle\right\rangle_t = \frac{1}{2}\left(\left\langle b^2\right\rangle + \left\langle d^2\right\rangle\right) = \frac{1}{2}\left(\langle b\rangle^2 + \sigma_b^2 + \langle d\rangle^2 + \sigma_d^2\right). \tag{41}$$

Substituting (39) into (41) and reducing the obtained result yield the time-averaging of mean square response to be

$$\left\langle\left\langle x^2(t)\right\rangle\right\rangle_t = \frac{\left(2\zeta_2\zeta_4 + \zeta_3\zeta_5\right)^2 + \left(2\zeta_1\zeta_5 + \zeta_3\zeta_4\right)^2}{2\left(4\zeta_1\zeta_2 - \zeta_3^2\right)^2} + \frac{\zeta_1 + \zeta_2}{4\zeta_1\zeta_2 - \zeta_3^2}, \tag{42}$$

where $\zeta_i, i = \overline{1,5}$ are given by (37). It is noted from (42) that the approximate time-averaging value of mean square response of V-D oscillator is calculated algebraically. In order to check the accuracy of the present technique, the various values of response of V-D equation (1) are compared to the numerical simulation results versus the particular parameter. The numerical simulation of the mean square response, denoted by $\left\langle x^2\right\rangle_{sim}$, is obtained by 10,000-realization Monte Carlo simulation. In Tab. 1, time-averaging values of mean-square response of the system is performed for computation with various values of the parameter ε. The system parameters are chosen to be $\alpha = 1, \beta = 4, \omega = 1, P = 2, \gamma = 1, \sigma^2 = 0.1, \varepsilon = 0.05$. Tab. 2 presents time-averaging values of mean-square response of the system evaluated versus the parameter ν in the primary resonant region with the system parameters chosen to be $\alpha = 1, \beta = 4, \omega = 1, P = 2, \gamma = 1, \sigma^2 = 0.1, \nu = 1.01$. The two tables show that the proposed technique gives a good prediction when the parameter ε is small and the frequency ν of the periodic excitation is near to the natural frequency ω of the system. In Tab. 3, time-averaging values of mean square response for various values of are performed with the system parameter chosen to be $\alpha = 1, \beta = 4, \omega = 1, P = 2, \gamma = 0.05, \sigma^2 = 0.1, \nu = 1.01$. The responses are evaluated versus the amplitude P of the periodic excitation and the parameter σ^2 of the random excitation in Tab. 4 and in Tab. 5, respectively. Tab. 3 and Tab. 4 show that the proposed technique gives a good prediction. Meanwhile, Tab. 5 shows that the error of the present technique, in general, increases when random intensity σ^2 increases. For small values of σ^2, however, the proposed technique gives a good prediction. The error in the tables is defined as

$$\text{Error} = \frac{\left|\left\langle x^2\right\rangle_{sim} - \left\langle x^2\right\rangle_{present}\right|}{\left\langle x^2\right\rangle_{sim}} \times 100\%, \tag{43}$$

where $\left\langle x^2\right\rangle_{present}$ denotes the time-averaging values of mean square response by the present technique.

Table 1. The error between the simulation result and approximate values of the time-averaging of mean square response $\langle x^2(t)\rangle$ versus the parameter ε $\left(\alpha = 1, \beta = 4, \omega = 1, P = 2, \gamma = 1, \sigma^2 = 0.1, \nu = 1.01\right)$

ε	$\langle x^2\rangle_{sim}$	$\langle x^2\rangle_{present}$	Error (%)
0.05	0.9374	0.9736	3.86
0.1	0.9100	0.9248	1.63
0.2	0.8996	0.8989	0.08
0.3	0.9025	0.8900	1.38

Table 2. The error between the simulation result and approximate values of the time-averaging of mean square response $\langle x^2(t)\rangle$ versus the parameter ν $\left(\alpha = 1, \beta = 4, \omega = 1, P = 2, \gamma = 1, \sigma^2 = 0.1, \varepsilon = 0.05\right)$

ν	$\langle x^2\rangle_{sim}$	$\langle x^2\rangle_{present}$	Error (%)
1.01	0.9371	0.9736	3.89
1.02	1.0098	1.0546	4.44
1.03	1.0602	1.1107	4.76
1.04	1.0834	1.1343	4.70
1.05	1.0669	1.1088	3.93

Table 3. The error between the simulation result and approximate values of the time-averaging of mean square response $\langle x^2(t)\rangle$ versus the parameter γ $\left(\alpha = 1, \beta = 4, \omega = 1, P = 2, \varepsilon = 0.05, \sigma^2 = 0.1, \nu = 1.01\right)$

γ	$\langle x^2\rangle_{sim}$	$\langle x^2\rangle_{present}$	Error (%)
0.1	1.0997	1.1428	3.92
0.5	1.0687	1.1217	4.96
1	0.9373	0.9736	3.87
2	0.6769	0.6851	1.21
5	0.3696	0.3562	3.63

Table 4. The error between the simulation result and approximate values of the time-averaging of mean square response $\langle x^2(t)\rangle$ versus the parameter P $\left(\alpha = 1, \beta = 4, \omega = 1, \sigma^2 = 0.1, \varepsilon = 0.05, \gamma = 0.5, \nu = 1.01\right)$

P	$\langle x^2\rangle_{sim}$	$\langle x^2\rangle_{present}$	Error (%)
0.1	0.5140	0.5259	2.32
0.5	0.6457	0.6926	7.26
1	0.8037	0.8546	6.33
2	1.0692	1.1217	4.91
5	1.6905	1.7455	3.25
10	2.5013	2.5564	2.20

Table 5. The error between the simulation result and approximate values of the time-averaging of mean square response $\langle x^2(t) \rangle$ versus the parameter σ^2 $(\alpha = 1, \beta = 4, \omega = 1, P = 2, \sigma^2 = 0.1, \gamma = 1, \nu = 1.01)$

σ^2	$\langle x^2 \rangle_{sim}$	$\langle x^2 \rangle_{present}$	Error (%)
0.1	0.9376	0.9736	3.84
0.5	0.9349	0.9616	2.86
1.0	0.9348	0.9429	0.87
1.5	0.9461	0.9238	2.36
2.0	0.9659	0.9086	5.93
2.5	0.9885	0.9022	8.73
3.0	1.0177	0.9073	10.85

4. CONCLUSIONS

Using the stochastic averaging method to investigate a nonlinear vibration leads to solve a FP equation whose solution is still a difficult problem. To overcome this, the present paper proposes a new approximate approach to find the response of the V-D oscillator by combining two typical methods, namely, stochastic averaging method and equivalent linearization method. As shown, the stochastic averaging of V-D equation is carried out in Cartesian coordinates. It is obtained that the drift coefficients of the averaged equations in the system (9) are polynomial forms in b and d which give an advantageous context to apply stochastic equivalent linearization method. The FP equation associated with the equivalent linearized system (17) can be solved exactly by the technique of auxiliary function. Numerical result shows that, for the V-D oscillator, the present technique gives a good prediction when the frequency ν of the periodic excitation is near to the natural frequency ω of the system; nonlinear coefficient γ and value of random intensity σ^2 are small.

It is noted that although the proposed technique is used to solve the V-D equation only, it is applicable to other nonlinear systems with weak nonlinearity and weak excitations. Further, since the accuracy of the proposed technique depends on criterion of equivalence adopted, the technique can be improved by using advanced optimization criteria (e.g., [22, 25, 33]) during the equivalent linearization procedure. These notes will be studied in future.

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