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MULTIPLE CRACK IDENTIFICATION IN STEPPED BEAM BY MEASUREMENTS OF NATURAL FREQUENCIES

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Abstract. A new approach is proposed for calculating natural frequencies and crack detection in a stepped cantilever beam with arbitrary number of cracks. This is based an explicit expression of the natural frequencies in term of crack parameter derived in the form similar to the so-called Rayleigh quotient for vibrating beam. The obtained simple relationship between natural frequencies and crack parameters enables not only accurate calculating the natural frequencies but also to develop an efficient procedure for detecting multiple cracks from given natural frequencies. The proposed technique called crack scanning method is illustrated and validated by numerical results.

Keywords: Multi stepped beam, Rayleigh quotient, multi-crack detection, frequency based method, modal analysis.

1. INTRODUCTION

Early detecting damage in engineering structures such as cracks is vitally important to prevent catastrophe that may lead to loss of either material or human lives. The task helps also to extend the structure lifetime by prompt maintenance and repair. A lot of methods have been proposed to detect cracks in structures and most of them are based on the crack-induced change in the dynamic characteristics of structure under consideration. This is because of the fact that crack in a structure member certainly modify the structure dynamic properties that can be usually obtained as results of the dynamic testing. The core of the vibration-based crack detection is the change in the structure modal parameters as a function of crack parameters that is usually subject of the forward problem. Since the natural frequencies are overall parameter of structure that can be most easily and accurately measured or calculated, the change in natural frequencies due to cracks in structures and its application to crack identification still nowadays are of interest.

Adams et al [1] first have shown that ratio of change in two natural frequencies of a bar with single crack represented by a translation spring is dependent only on the crack position so that the ratio can be used for localization of single crack in bar. Later, Liang et al [2] extended the result for beam with single crack represented by a rotational spring. By using the perturbation method Morassi [3] demonstrated that the change in eigenfrequency of a cracked beam is proportional to the curvature at the crack position and this result has been then applied to crack identification in simply supported beam from measured natural frequencies in Morassi and Rollo, [4] and Rubio, [5]. A compact form of the characteristic equation, the most important relationship between natural frequencies and crack parameters, has been obtained in Narkis, [6] for both the longitudinal and bending vibration of simply supported beam. For small crack size the equation allows obtaining analytical solution for crack position and this result has been later validated by an experiment in Sayyad and Kumar, [7]. The characteristic equation for a rotating cracked Timoshenko beam was then derived by Al-Said et al, [8], who have shown influence of crack, rotating speed and shear deformation on the natural frequencies of the beam. Ostachovicz and Krawczuk [9] accomplished an analysis of variation of natural frequencies in cantilever beam with two cracks using derived characteristic equation. Bahera et al [10] obtained fundamental frequency of a shaft with two cracks in viscous liquid. The natural frequencies of beam with arbitrary number of cracks were investigated in dependence on crack parameters by Shifrin and Ruotolo [11]; Li [12]; Zheng and Fan [13]; Khiem and Lien [14]; Aydin [15] and Caddemi and Caliò [16]. However, in the foregoing works the natural frequencies of multiple cracked beam are found from a complicated equation that may contain singularities sometimes troubling numerical solving the equation. Though the simplified characteristic equation obtained in Khiem and Lien, [14] has been used for multi-crack detection in beam by Zhang et al [17], the implicit relationship between the natural frequencies and crack parameters may cause serious problems for crack detection. Therefore, a simple and explicit expression of natural frequencies in term of crack parameters for multiple cracked beam would be surely helpful for the crack detection problem. Liang et al [18]; Patil and Maiti [19] and Li [20] have derived approximate systems of linear algebraic equations relating the change in natural frequencies to damage magnitudes that provide an efficient tool for damage detection in beam. Fernandez-Saez et al [21] obtained an explicit expression of fundamental frequency of beam with single crack through crack position and size based on the classical Rayleigh formulae. Objective of this paper is to develop an explicit expression of natural frequencies in term of crack position and size for a multiple cracked beam and make use of the obtained expression for multi-crack identification problem. First, a novel form of the Rayleigh quotient is derived for multiple stepped beam with a number of cracks and arbitrary boundary conditions. Then, the obtained Rayleigh quotient for multiple cracked uniform beam is applied to determine quantity, position and depth of multiple cracks from given natural frequencies. The theoretical development is illustrated and validated by both numerical and experimental results.

2. THE RAYLEIGH QUOTIENT

Consider a cantilever beam consisting of N spans with the material and geometrical constants: Young's modulus E_j, mass density ρ_j, length L_j, cross section area and moment of inertia F_j, I_j of j-th beam segment $(x_{j-1}, x_j), j = 1, \ldots, n; x_0 = 0;\ x_N = 1 = L_1 + \ldots + L_n$. Suppose furthermore that in the segment (x_{j-1}, x_j) there exist n_j cracks of depth a_{ji} at positions $e_{ji}, i = 1, \ldots, n_j$ $(x_{j-1} \prec e_{j1} \prec \ldots \prec e_{jn_j} \prec x_j)$.

It is well known that k-th mode shape in the beam segment (x_{j-1}, x_j) denoted by $\phi_{kj}(x)$ satisfies the equation

$$-\omega_k^2 m_j \phi_{kj}(x) + S_j d^4\phi_{kj}(x)/dx^4 = 0, \quad x \in (x_{j-1}, x_j), \tag{1}$$

with $m_j = \rho_j F_j; S_j = E_j I_j$ and ω_k is k-th and the following conditions at cracks

$$\begin{gathered}\phi_{kj}(e_{ji}^-) = \phi_{kj}(e_{ji}^+); \quad \phi''_{kj}(e_{ji}^-) = \phi''_{kj}(e_{ji}^+) = \phi''_{kj}(e_{ji}); \quad \phi'''_{kj}(e_{ji}^-) = \phi'''_{kj}(e_{ji}^+); \\ [\phi'_{kj}(e_{ji}^+) - \phi'_{kj}(e_{ji}^-)] = \gamma_{ji}\phi''_{kj}(e_{ji}); \quad i = 1, \ldots, n_j\end{gathered} \tag{2}$$

where $\gamma_{ji} = I_c(a_{ji}/h_j)$, the function of depth of the crack at position e_{ji}, denotes the crack magnitude and is determined accordingly to the Fracture Mechanics. Multiplying both sides of Eqs. (1) by $\phi_{kj}(x)$ and taking integration along the interval (x_{j-1}, x_j), one obtains

$$\omega_k^2 m_j \int_{x_{j-1}}^{x_{j-1}} \phi_{kj}^2(x)dx = S_j \int_{x_{j-1}}^{x_{j-1}} \frac{d^4\phi_{kj}(x)}{dx^4}\phi_{kj}(x)dx;\ j = 1, \ldots, N, k = 1, 2, \ldots \tag{3}$$

Note that for the functions $\phi_{kj}, \phi'_{kj}, \phi''_{kj}, \phi'''_{kj}$ continuous in the segment (a, b), it would be easily to verify that

$$\int_a^b \frac{d^4\phi_{kj}(x)}{dx^4}\phi_{kj}(x)dx = \int_a^b {\phi''_{kj}}^2(x)dx + [B_{kj}(b) - B_{kj}(a)], \tag{4}$$

where

$$B_{kj}(x) = \phi'''_{kj}(x)\phi_{kj}(x) - \phi''_{kj}(x)\phi'_{kj}(x). \tag{5}$$

Therefore, one has

$$\int_{x_{j-1}}^{x_{j-1}} \frac{d^4\phi_{kj}(x)}{dx^4}\phi_{kj}(x)dx = \int_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx + [B_{kj}(x_j) - B_{kj}(x_{j-1})] + \sum_{i=1}^{n_j}[B_{kj}(e_{ji}^-) - B_{kj}(e_{ji}^+)]. \tag{6}$$

Taking account of the conditions (2) the latter equation can be rewritten as

$$\int_{x_{j-1}}^{x_{j-1}} \frac{d^4\phi_{kj}(x)}{dx^4}\phi_{kj}(x)dx = \int_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx + [B_{kj}(x_j) - B_{kj}(x_{j-1})] + \sum_{i=1}^{n_j}\gamma_{ji}{\phi''_{kj}}^2(e_{ji}) \tag{7}$$

and, as consequence, one obtains

$$\begin{aligned}\sum_{j=1}^{N} S_j \int\limits_{x_{j-1}}^{x_{j-1}} \frac{d^4\phi_{kj}(x)}{dx^4}\phi_{kj}(x)dx = \sum_{j=1}^{N} S_j \left\{ \int\limits_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx + \sum_{i=1}^{n_j} \gamma_{ji}{\phi''_{kj}}^2(e_{ji}) \right\} + \\ + \sum_{j=1}^{N-1} [S_j B_{kj}(x_j) - S_{j+1}B_{k,j+1}(x_j)] + [B_{kN}(L)S_N - B_{k1}(0)S_1].\end{aligned} \tag{8}$$

It can be easily verified that under the continuity conditions at step joints x_j

$$\begin{aligned}&\phi_{kj}(x_j) = \phi_{k,j+1}(x_j); && \phi'_{kj}(x_j) = \phi'_{k,j+1}(x_j);\\ &S_j\phi''_{kj}(x_j) = S_{j+1}\phi''_{k,j+1}(x_j); && S_j\phi'''_{kj}(x_j) = S_{j+1}\phi'''_{k,j+1}(x_j),\end{aligned} \tag{9}$$

the second sum in the right hand side of Eq. (8) would be vanished. So that one obtains

$$\omega_k^2 = \frac{\sum\limits_{j=1}^{N} S_j \left\{ \int\limits_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx + \sum\limits_{i=1}^{n_j} \gamma_{ji}{\phi''_{kj}}^2(e_{ji}) \right\} + [B_{kN}(L)S_N - B_{k1}(0)S_1]}{\sum\limits_{j=1}^{N} m_j \int\limits_{x_{j-1}}^{x_j} {\phi_{kj}}^2(x)dx}. \tag{10}$$

This is Rayleigh's quotient for multiple cracked and stepped beam with arbitrary boundary conditions. Furthermore, for all the homogeneous classical boundary conditions (including the simply supports, cantilever, fixed and free ends) the last term in numerator of quotient (10) is vanish. In such the case one has

$$\omega_k^2 = \left\{ \sum_{j=1}^{N} S_j \left[\int\limits_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx + \sum_{i=1}^{n_j} \gamma_{ji}{\phi''_{kj}}^2(e_{ji}) \right] \right\} / \left\{ \sum_{j=1}^{N} m_j \left[\int\limits_{x_{j-1}}^{x_j} {\phi_{kj}}^2(x)dx \right] \right\}. \tag{11}$$

In particularity, if the beam is uniform the Eq. (14) becomes

$$\omega_k^2 = \frac{EI}{\rho F} \left\{ \int\limits_0^L \phi_k''^2(x)dx + \sum_{j=1}^{n} \gamma_j \phi_k''^2(e_j) \right\} / \left\{ \int\limits_0^L {\phi_k}^2(x)dx \right\}, \tag{12}$$

with $n = n_1 + \ldots + n_N$. For multi-stepped undamaged beam the Rayleigh quotient is

$$\omega_k^2 = \left\{ \sum_{j=1}^{N} S_j \left[\int\limits_{x_{j-1}}^{x_j} {\phi''_{kj}}^2(x)dx \right] \right\} / \left\{ \sum_{j=1}^{N} m_j \left[\int\limits_{x_{j-1}}^{x_j} {\phi_{kj}}^2(x)dx \right] \right\}. \tag{13}$$

As the classical Rayleigh quotient, the Eqs. (11)-(13) is simplest tool for calculating natural frequencies by a properly chosen shape functions $\phi_{kj}(x)$. Moreover, the obtained Rayleigh quotient can be engaged to develop a new procedure for multi-crack detection in stepped beam by measurements of natural frequencies.

3. CALCULATION OF NATURAL FREQUENCIES

The mode shape functions are chosen in the form

$$\phi_{kj}(x) = \varphi_{kj0}(x) + A_{kj}x^3 + B_{kj}x^2 + C_{kj}x + D_{kj} + \sum_{i=1}^{n_j} \gamma_{ji}\phi''_{kj}(e_{ji})K(x - e_{ji}), \quad (14)$$

where $\varphi_{kj}(x)$ is k-th mode shapes of undamaged beam in the segment (x_{j-1}, x_j), the constants $A_{kj}, B_{kj}, C_{kj}, D_{kj}$ would be determined latter from boundary and step joint conditions and

$$K(x) = \begin{cases} x, & x \geq 0; \\ 0, & x \leq 0. \end{cases}$$

Obviously, substituting shape function (14) into first two conditions in (9) leads to

$$\varphi_{k,j+1}(x_j) = \varphi_{kj}(x_j), \quad \varphi'_{k,j+1}(x_j) = \varphi'_{kj}(x_j) \quad (15)$$

and

$$A_{k,j+1} = A_{kj}, B_{k,j+1} = B_{kj}, C_{k,j+1} = C_{kj} + \sum_{i=1}^{n_j} \gamma_{ji}\phi''_{kj}(e_{ji}), D_{k,j+1} = D_{kj} - \sum_{i=1}^{n_j} \gamma_{ji}\phi''_{kj}(e_{ji})e_{ji}$$

or

$$A_{kj} = A_k, B_{kj} = B_k, C_{kj} = C_k + \sum_{r=1}^{j-1}\sum_{i=1}^{n_r} \gamma_{ri}\phi''_{kr}(e_{ri}), D_{kj} = D_k - \sum_{r=1}^{j-1}\sum_{i=1}^{n_r} \gamma_{ri}\phi''_{kr}(e_{ri})e_{ri}. \quad (16)$$

The latter equations show that all the functions (14) would be completely determined for $j = 1, ..., N$ by choosing four constants A_k, B_k, C_k, D_k and functions $\phi_{kj}(x)$ which are defined for different classical boundary conditions as follow.

Namely, the boundary conditions of both the cantilever, $\phi_k(0) = \phi'_k(0) = \phi''_k(1) = \phi'''_k(1) = 0$ and beam with free ends, $\phi''_k(0) = \phi'''_k(0) = \phi''_k(1) = \phi'''_k(1) = 0$, would be satisfied by choosing

$$A_k = B_k = C_k = D_k = 0, \quad (17)$$

in combination respectively with

$$\varphi_{k1}(0) = \varphi'_{k1}(0) = \varphi''_{kN}(1) = \varphi'''_{kN}(1) = 0, \quad (18)$$

and

$$\varphi''_{k1}(0) = \varphi'''_{k1}(0) = \varphi''_{kN}(1) = \varphi'''_{kN}(1) = 0. \quad (19)$$

Also, for simply supported beam, to satisfy the conditions $\phi_k(0) = \phi''_k(0) = \phi_k(1) = \phi''_k(1) = 0$, it is simply to choose

$$A_k = B_k = D_k = 0; \quad C_k = \sum_{j=1}^{N}\sum_{i=1}^{n_j} \gamma_{ji}\phi''_{kj}(e_{ji})(e_{ji} - 1); \quad (20)$$

$$\varphi_{k1}(0) = \varphi''_{k1}(0) = \varphi_{kN}(1) = \varphi''_{kN}(1) = 0. \quad (21)$$

Finally, the clamped ends conditions would be fulfilled if

$$C_k = D_k = 0;\, B_k = \sum_{j=1}^{N}\sum_{i=1}^{n_j}\gamma_{ji}\phi''_{kj}(e_{ji})(3e_{ji}-2);\, A_k = -\sum_{j=1}^{N}\sum_{i=1}^{n_j}\gamma_{ji}\phi''_{kj}(e_{ji})(2e_{ji}-1); \quad (22)$$

$$\varphi_{k1}(0) = \varphi'_{k1}(0) = \varphi_{kN}(1) = \varphi'_{kN}(1) = 0. \quad (23)$$

Such chosen constants A_k, B_k, C_k, D_k lead the functions $\{\varphi_{kj}(x), j = 1, \ldots, N\}$ to be chosen as mode shape of undamaged beam satisfying the given boundary conditions for cracked beam under consideration. It can be seen that trivial values of the constants $A_k = B_k = 0$ including three cases of boundary conditions for simply supports, cantilevered and free-free ends lead to much simplified calculation. By the reason, only beam with the listed above boundary conditions is investigated in this paper except the case of clamped ends that requires a special study.

For the cases of boundary conditions, because of $\phi''_{kj}(x) = \varphi''_{kj}(x);\ \phi'''_{kj}(x) = \varphi''_{kj}(x)$, the two last conditions in (9) are equivalent to

$$S_j\varphi''_{kj}(x_j) = S_{j+1}\varphi''_{k,j+1}(x_j);\ S_j\varphi'''_{kj}(x_j) = S_{j+1}\varphi'''_{k,j+1}(x_j). \quad (24)$$

Thus, the problem now remained to construct the mode shape of intact stepped beam $\{\varphi_{kj}(x), j = 1, \ldots, N\}$ that is accomplished by using the transfer matrix method as follows. It was well known that the mode shape of a multiple stepped beam in a uniform segment $(x_{j-1}, x_j), j = 1, \ldots, N$ and boundary conditions at the beam ends can be expressed as

$$\varphi_j(x) = d_{j1}\cosh\lambda_j x + d_{j2}\sinh\lambda_j x + d_{j3}\cos\lambda_j x + d_{j4}\sin\lambda_j x; \quad (25)$$

$$x \in (x_{j-1}, x_j);\, \lambda_j^4 = m_j\omega^2/S_j; \quad (26)$$

$$[\mathbf{B}_0]\cdot\{d_{11}, d_{12}, d_{13}, d_{14}\}^T = 0;\ [\mathbf{B}_1]\cdot\{d_{N1}, d_{N2}, d_{N3}, d_{N4}\}^T = 0, \quad (27)$$

where $\mathbf{B}_0, \mathbf{B}_1$ are given (2×4)-dimensional matrices. The continuity conditions at step joints x_j can be represented as

$$[\mathbf{T}_j]\cdot\{d_{j1}, d_{j2}, d_{j3}, d_{j4}\}^T = [\mathbf{T}_{j+1}]\cdot\{d_{j+1,1}, d_{j+1,2}, d_{j+1,3}, d_{j+1,4}\}^T,$$

or

$$\{d_{j+1,1}, d_{j+1,2}, d_{j+1,3}, d_{j+1,4}\}^T = [\mathbf{T}_{j+1}]^{-1}\cdot[\mathbf{T}_j]\cdot\{d_{j1}, d_{j2}, d_{j3}, d_{j4}\}^T, \quad (28)$$

where $\mathbf{T}_j = \mathbf{T}(x_j, S_j, \lambda_j) = \mathbf{T}_j(x_j), \mathbf{T}_{j+1} = \mathbf{T}(x_j, S_{j+1}, \lambda_{j+1}) = \mathbf{T}_{j+1}(x_j)$ and

$$\mathbf{T}(x, S, \lambda) = \begin{bmatrix} \cosh\lambda x & \sinh\lambda x & \cos\lambda x & \sin\lambda x \\ \lambda\sinh\lambda x & \lambda\cosh\lambda x & -\lambda\sin\lambda x & \lambda\cos\lambda x \\ \lambda^2 S\cosh\lambda x & \lambda^2 S\sinh\lambda x & -\lambda^2 S\cos\lambda x & -\lambda^2 S\sin\lambda x \\ \lambda^3 S\sinh\lambda x & \lambda^3 S\cosh\lambda x & \lambda^3 S\sin\lambda x & -\lambda^3 S\cos\lambda x \end{bmatrix}.$$

Based the recurrent relationship (28) one obtains

$$\{d_{j1}, d_{j2}, d_{j3}, d_{j4}\}^T = [\mathbf{H}_j]\cdot\{d_{11}, d_{12}, d_{13}, d_{14}\}^T, \quad (29)$$

where $[\mathbf{H}_j] = [\mathbf{T}_j(x_{j-1})]^{-1}[\mathbf{T}_{j-1}(x_{j-1})]\ldots[\mathbf{T}_2(x_1)]^{-1}[\mathbf{T}_1(x_1)]$ and

$$\{d_{N1}, d_{N2}, d_{N3}, d_{N4}\}^T = [\mathbf{H}]\cdot\{d_{11}, d_{12}, d_{13}, d_{14}\}^T;\mathbf{H} = \mathbf{H}_N(x_1, x_2, \ldots, x_{N-1}). \quad (30)$$

Combining (27) with (30) yields

$$[\mathbf{B}] \cdot \{d_{11}, d_{12}, d_{13}, d_{14}\}^T = 0, \tag{31}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_1\mathbf{H} \end{bmatrix}. \tag{32}$$

The condition for existence of nontrivial solution of Eq. (31) is

$$\det \mathbf{B}(\omega) = 0, \tag{33}$$

that is the characteristic equation allowing finding the natural frequencies of multi-stepped beam ($\omega_{k0}, k = 1, 2, 3, \ldots$). Every natural frequency ω_{k0} is associated with a nontrivial solution of Eq. (31) $\{d_{11}, d_{12}, d_{13}, d_{14}\}^T = d_k\{a_{11}, a_{12}, a_{13}, a_{14}\}^T$ that contains an arbitrary constant d_k. Hence, mode shape in segment $(x_{j-1}, x_j), j = 2, 3, \ldots$ can be found through the vector

$$\{d_{j1}, d_{j2}, d_{j3}, d_{j4}\}^T = d_k[\mathbf{H}_j] \cdot \{a_{11}, a_{12}, a_{13}, a_{14}\}$$

by using expression (25). Thus, the undamaged mode shape functions $\{\phi_{kj}(x), j = 1, \ldots, N\}$ are constructed.

Now we can calculate the natural frequencies of multiple cracked and stepped beam by using Eq. (11) and shape function (14) reduced to

$$\phi_{kj}(x) = \varphi_{kj}(x) + C_{kj}x + D_{kj} + \sum_{i=1}^{n_j} \gamma_{ji}\varphi''_{kj}(e_{ji})K(x - e_{ji}).$$

Note first that the numerator of quotient (11) in the cases is

$$\begin{aligned} R_N &= \sum_{j=1}^{N} S_j \left[\int_{x_{j-1}}^{x_j} \varphi''^{\,2}_{kj}(x)dx + \sum_{i=1}^{n_j} \gamma_{ji}\varphi''^{\,2}_{kj}(e_{ji}) \right] \\ &= \sum_{j=1}^{N} S_j \int_{x_{j-1}}^{x_j} \varphi''^{\,2}_{kj}(x)dx + \sum_{j=1}^{N} S_j \sum_{i=1}^{n_j} \gamma_{ji}\varphi''^{\,2}_{kj}(e_{ji}) \end{aligned} \tag{34}$$

and the denominator can be calculated as follows.

$$\begin{aligned} R_D &= \sum_{j=1}^{N} m_j \int_{x_{j-1}}^{x_j} \phi^2_{kj}(x)dx \\ &= \sum_{j=1}^{N} m_j \left[\int_{x_{j-1}}^{x_j} \varphi^2_{kj}(x)dx + \int_{x_{j-1}}^{x_j} 2\varphi_{kj}(x)(C_{kj}x + D_{kj})dx + \int_{x_{j-1}}^{x_j} (C_{kj}x + D_{kj})^2dx \right] + \\ &+ \sum_{j=1}^{N} m_j \left\{ \sum_{i=1}^{n_j} \left[2\gamma_{ji}\varphi''_{kj}(e_{ji}) \int_{e_{ji}}^{x_j} [\varphi_{kj}(x) + C_{kj}x + D_{kj}](x - e_{ji})dx \right] \right\} + \end{aligned}$$

$$+\sum_{j=1}^{N} m_j \left\{ \sum_{i,\ell=1}^{n_j} \left[\gamma_{ji}\gamma_{j\ell}\varphi''_{kj}(e_{ji})\varphi''_{kj}(e_{j\ell}) \int_{e_{ji\ell}}^{x_j} (x-e_{ji})(x-e_{j\ell})dx \right] \right\}$$
$$= \sum_{j=1}^{N} m_j \left[\int_{x_{j-1}}^{x_j} \varphi_{kj}^2(x)dx \right] + 2\Psi_1 + \omega_{k0}^{-2}\Psi_2, \tag{35}$$

where $e_{ji\ell} = e_{ji}$ for $i \geq \ell$ or $e_{j\ell}$ if $i \leq \ell$,

$$\Psi_1 = \sum_{j=1}^{N} m_j \left\{ \int_{x_{j-1}}^{x_j} \varphi_{kj}(x)(C_{kj}x + D_{kj})dx + \sum_{i=1}^{n_j} \left[\gamma_{ji}\varphi''_{kj}(e_{ji}) \int_{e_{ji}}^{x_j} \varphi_{kj}(x)(x-e_{ji})dx \right] \right\}$$
$$= \omega_{k0}^{-2} \sum_{j=1}^{N} \sum_{i=1}^{n_j} S_j \gamma_{ji} {\varphi''_{kj}}^2(e_{ji}); \tag{36}$$

$$\Psi_2 = \sum_{j=1}^{N} m_j \left\{ P_j^3 C_{kj}^2 + 2P_j^2 C_{kj}D_{kj} + P_j^1 D_{kj}^2 + \sum_{i=1}^{n_j} [C_{kj}\alpha_j(e_{ji}) + D_{kj}\beta_j(e_{ji})]\gamma_{ji}\varphi''_{kj}(e_{ji}) \right.$$
$$\left. + \sum_{i=1}^{n_j}\sum_{\ell=1}^{n_j} \gamma_{ji}\varphi''_{kj}(e_{ji})\gamma_{j\ell}\varphi''_{kj}(e_{j\ell}) q_j(e_{ji}, e_{j\ell}) \right\} \tag{37}$$
$$= \sum_{j=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{i=1}^{n_p}\sum_{\ell=1}^{n_q} m_j Q_j(e_{pi}, e_{q\ell})\gamma_{pi}\gamma_{q\ell}\varphi''_{kp}(e_{pi})\varphi''_{kq}(e_{q\ell})$$

with $P_j^n = (x_j^n - x_{j-1}^n)/n, n = 1, 2, 3; \alpha_j(e_{ji}) = (x_j - e_{ji})^2(2x_j + e_{ji})/6; \beta_j(e_{ji}) = (x_j - e_{ji})^2/2$ and coefficients $Q_j(e_{pi}, e_{q\ell})$ given below for different boundary conditions under consideration. Namely, for cantilever and beam with free ends one has

$$Q_j(e_{pi}, e_{q\ell}) = \begin{cases} \dfrac{1}{3}\begin{cases}(x_j - e_{ji})^3 : \ i \geq \ell; \\ (x_j - e_{j\ell})^3 : \ i \leq \ell; \end{cases} & p = q = j; \\ [\alpha_j(e_{ji}) - e_{q\ell}\beta_j(e_{ji})]/2 : & q \prec p = j; \\ 0 : & p, q \succ j; \\ [\alpha_j(e_{j\ell}) - e_{pi}\beta_j(e_{j\ell})]/2 : & p \prec q = j; \\ P_j^3 - P_j^2(e_{pi} + e_{q\ell})/2 + P_j^1 e_{pi}e_{q\ell} : & p, q \prec j. \end{cases} \tag{38}$$

For the case of simply supported beam

$$Q_j(e_{pi}, e_{q\ell}) = \begin{cases} P_j^3(1-e_{pi})(1-e_{q\ell}) + (e_{pi} + e_{q\ell})/2 + \dfrac{1}{3}\begin{cases}(x_j - e_{ji})^3 : i \geq \ell; \\ (x_j - e_{j\ell})^3 : i \leq \ell; \end{cases} & p = q = j; \\ (P_j^2 - P_j^3)(1-e_{pi})e_{q\ell} + [\alpha_j(e_{ji}) - e_{q\ell}\beta_j(e_{ji}) + e_{q\ell} - 1]/2 : & q \prec p = j; \\ P_j^3(1-e_{pi})(1-e_{q\ell}) : & p, q \succ j; \\ (P_j^2 - P_j^3)(1-e_{q\ell})e_{pi} + [\alpha_j(e_{j\ell}) - e_{pi}\beta_j(e_{j\ell}) + e_{pi} - 1]/2 : & p \prec q = j; \\ (P_j^3 - 2P_j^2 + P_j^1)e_{pi}e_{q\ell} + P_j^2(e_{pi} + e_{q\ell})/2 : & p, q \prec j. \end{cases} \tag{39}$$

Thus, Rayleigh quotient for multiple cracked and stepped beam with simply supported, fixed-free and free-free ends is expressed as

$$\omega_k^2 = R_N/R_D = \left[\sum_{j=1}^{N} S_j \int_{x_{j-1}}^{x_j} {\varphi''_{kj}}^2(x)dx + {\omega_{k0}}^2\Psi_1\right] / \left[\sum_{j=1}^{N} m_j \int_{x_{j-1}}^{x_j} {\varphi_{kj}}^2(x)dx + 2\Psi_1 + \Psi_2\right]. \tag{40}$$

Obviously, natural frequencies of undamaged beam can be obtained

$$\omega_{k0}^2 = \left[\sum_{j=1}^{N} S_j \int_{x_{j-1}}^{x_j} {\varphi''_{kj}}^2(x)dx\right] / \left[\sum_{j=1}^{N} m_j \int_{x_{j-1}}^{x_j} {\varphi_{kj}}^2(x)dx\right]. \tag{41}$$

Selecting $\varphi_{kj}(x) = d_k\bar{\varphi}_{kj}(x)$ with $\bar{\varphi}_{kj0}(x)$ being the undamaged mode shape normalized by

$$\sum_{j=1}^{N} m_j \left[\int_{x_{j-1}}^{x_j} \varphi_{kj}^2(x)dx\right] = 1, \tag{42}$$

one obtains

$${\omega_{k0}}^2 = \sum_{j=1}^{N} S_j \left[\int_{x_{j-1}}^{x_j} \bar{\varphi''}_{kj}^2(x)dx\right] \tag{43}$$

and

$$\frac{\omega_k^2}{\omega_{k0}^2} = \frac{1 + {\omega_{k0}}^{-2}\sum\limits_{j=1}^{N}\sum\limits_{i=1}^{n_j} S_j\gamma_{ji}\bar{\varphi''}_{kj}^2(e_{ji})}{1 + 2{\omega_{k0}}^{-2}\sum\limits_{j=1}^{N}\sum\limits_{i=1}^{n_j} S_j\gamma_{ji}\bar{\varphi''}_{kj}^2 + \sum\limits_{j=1}^{N}\sum\limits_{p=1}^{N}\sum\limits_{q=1}^{N}\sum\limits_{i=1}^{n_p}\sum\limits_{\ell=1}^{n_q} m_j Q_j(e_{pi}, e_{q\ell})\gamma_{pi}\gamma_{q\ell}\bar{\varphi}''_{kp}(e_{pi})\bar{\varphi''}_{kq}(e_{q\ell})}. \tag{44}$$

This is an explicit expression of natural frequencies for a stepped beam with arbitrary number of cracks that can be used not only for modal analysis of the multiple cracked stepped beam but also provides an important tool for crack identification in the beam.

4. CRACK IDENTIFICATION PROCEDURE

Assuming that m natural frequencies $(\bar{\omega}_1, \ldots, \bar{\omega}_m)$ of an N-stepped beam have been given, the problem is to predict position and size of cracks probably occurred in the beam.

To solve the problem the so-called crack scanning method (CSM) proposed by Khiem and Tran [22] is used. Accordingly to the method a mesh $(0 \le e_1 \prec e_2 \prec \ldots . \prec e_n \prec L)$ of multiple crack positions is initialed for determining unknown depths $(a_1, \ldots, a_n)$. For the evaluated crack depth vector the positions of probable cracks could be determined by the peaks on the map of estimated crack depth versus scanning crack positions. The desired depth of the cracks detected to appear at the positions may be corrected by repeating the crack depth assessment with the new mesh consisting from the detected crack positions.

Introducing the notations

$$\gamma_r = \gamma_{ji}, \quad e_r = e_{ji};$$

$$r = n_1 + \ldots + n_{j-1} + i = 1, \ldots, n = n_1 + \ldots + n_N; j = 1, \ldots, N, i = 1, \ldots, n_j,$$

the quotient (47) can be rewritten as

$$\omega_k^2/\omega_{k0}^2 = \left[1 + \sum_{r=1}^{n} a_{kr}\gamma_r\right] / \left[1 + 2\sum_{r=1}^{n} a_{kr}\gamma_r + \sum_{r_1,r_2=1}^{n} b_{kr_1r_2}\gamma_{r_1}\gamma_{r_2}\right] \tag{45}$$

where

$$a_{kr} = \omega_{k0}^{-2} S_j \bar{\varphi''}_{kj}^2(e_r);\ b_{kr_1r_2} = \sum_{j=1}^{N} m_j Q_j(e_{r_1}, e_{r_2})\bar{\varphi}''_{kp}(e_{r_1})\bar{\varphi}''_{kq}(e_{r_2}) \tag{46}$$

with

$$\begin{cases} n_1 + n_2 + \ldots + n_{j-1} \prec r \le n_1 + n_2 + \ldots + n_j; \\ n_1 + n_2 + \ldots + n_{p-1} \prec r_1 \le n_1 + n_2 + \ldots + n_p; \\ n_1 + n_2 + \ldots + n_{q-1} \prec r_2 \le n_1 + n_2 + \ldots + n_q. \end{cases} \tag{47}$$

Using the given natural frequencies $(\bar{\omega}_1, \ldots, \bar{\omega}_m)$ the Eq. (45) can be rewritten in the form

$$\sum_{j=1}^{n} [A_{kj} + \sum_{\ell=1}^{n} B_{kj\ell}\gamma_\ell]\gamma_j = d_k, k = 1, \ldots, m$$

or

$$[\mathbf{A} + \mathbf{B}(\gamma)]\{\gamma\} = \{\mathbf{d}\} \tag{48}$$

with

$$\mathbf{A} = [(2\delta_k - 1)a_{kj}, k = 1, \ldots, m; j = 1, \ldots, n]; \mathbf{B} = [\delta_k b_{kj\ell}, k = 1, \ldots, m; j, \ell = 1, \ldots, n];$$

$$d_k = 1 - \delta_k; \quad \delta_k = \bar{\omega}_k^2/\omega_{k0}^2. \tag{49}$$

Eqs. (47) can be solved with respect to unknowns $(\gamma_1, \ldots, \gamma_n)$ by using the iteration method

$$[\mathbf{A} + \mathbf{B}(\gamma^{(i-1)})]\{\gamma^{(i)}\} = \{\mathbf{b}\},\ \{\gamma^{(0)}\} = \{\mathbf{0}\}. \tag{50}$$

5. NUMERICAL RESULTS

First, for comparison, especially, with the experimental results the model studied in Ruoloto and Surace, 1997 is adopted here. That is a cantilever beam of length $L = 0{,}8$ m; cross section $H \times B = 0.02\,\mathrm{m} \times 0.02\,\mathrm{m}$; material constants $E = 1.81 \times 10^{11}\,\mathrm{Pa}; \rho = 7860\,\mathrm{kg/m3}$. Double crack have been made at the positions $\bar{e}_1 = e_1/L = 0.3175; \bar{e}_2 = e_2/L = 0.6812$ with various scenarios of relative crack depth C1 (20% and 20%); C2 (20% and 30%) and C3 (30% and 20%). Tab. 1 shows ratio (damaged to undamaged) of the first five frequencies obtained by (a) solution of the characteristic equation established in Khiem and Tran, [22]; (b) the experiment accomplished by Ruotolo and Surace [23] and (c) calculation using the Rayleigh Quotient (29). The measured (b) and calculated (c) frequency ratios compared to the theoretical ones (a) give rise to discrepancies presented in the adjacent columns. Obviously, the discrepancies are all insignificant (less than one percent) so that usefulness of the Rayleigh Quotient derived above for calculating natural frequencies of multiple cracked beam is thus validated.

Table 1. Comparison with the theoretical and experimental results

Frequency Number	Characteristic(a) Equation	Experiment (b)		Rayleigh Quotient (c)	
		Measured	Deviation (%)	Calculated	Deviation (%)
Scenario C1: $a_1/h = 0.2$, $a_2/h = 0.2$ ($\bar{e}_1 = e_1/L = 0.3175\,;\bar{e}_2 = e_2/L = 0.6812$)					
First	0.9929	0.9924	0.05	0.9945	0.16
Second	0.9908	0.9907	0.01	0.9927	0.19
Third	0.9804	0.9814	0.10	0.9846	0.36
Fourth	0.9965	0.9966	0.01	0.9970	0.42
Fifth	0.9942	N/A	N/A	0.9950	0.08
Scenario C2: $a_1/h = 0.2$, $a_2/h = 0.3$($\bar{e}_1 = e_1/L = 0.3175\,;\bar{e}_2 = e_2/L = 0.6812$)					
First	0.9901	0.9945	0.40	0.9940	0.39
Second	0.9810	0.9813	0.03	0.9864	0.55
Third	0.9668	0.9642	0.25	0.9746	0.80
Fourth	0.9942	0.9926	0.16	0.9950	0.08
Fifth	0.9907	N/A	N/A	0.9921	0,15
Scenario C3: $a_1/h = 0.3$, $a_2/h = 0.2$ ($\bar{e}_1 = e_1/L = 0.3175\,;\bar{e}_2 = e_2/L = 0.6812$)					
First	0.9844	0.9895	0.52	0.9880	0.36
Second	0.9873	0.9893	0.20	0.9897	0.24
Third	0.9698	0.9692	0.06	0.9755	0.58
Fourth	0.9945	0.9952	0.07	0.9947	0.02
Fifth	0.9907	N/A	N/A	0.9906	0.01

To demonstrate also applicability of the Rayleigh Quotient to the multi-crack detection problem the crack detection procedure proposed above is running by using the measured natural frequencies given by Ruotolo and Surace [23] (the second column in Tab. 1). Results of the crack magnitude estimation obtained as solution of the Eq. (34) with number of scanning crack points $N = 25$ from the clamped end to the free one are plotted and shown in Figs. 1-3. The three plots give rise consistently the same peaks at the positions $\hat{e}_1 = 0.04$; $\hat{e}_2 = 0.32$; $\hat{e}_3 = 0.68$ where cracks may occur. Following the crack scanning method, the first estimated crack positions would be taken as a new scanning mesh for correcting crack size. This correction has been accomplished and one obtains in result the following corrected crack magnitude

$$\text{Scenario C1: } \hat{\gamma}_1 = 0.0; \hat{\gamma}_2 = 0.0105; \hat{\gamma}_3 = 0.00935;$$
$$\text{Scenario C2: } \hat{\gamma}_1 = 0.0; \hat{\gamma}_2 = 0.0124; \hat{\gamma}_3 = 0.0289;$$
$$\text{Scenario C3: } \hat{\gamma}_1 = 0.0; \hat{\gamma}_2 = 0.0252; \hat{\gamma}_3 = 0.0103.$$

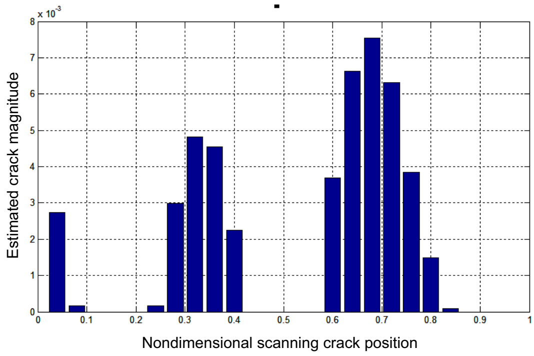

Fig. 1. Estimated crack magnitude from the natural frequencies measured by Ruotolo and Surace in the case of actual cracks are at 0.3175 (depth 20%) and 0.6812 (depth 30%)

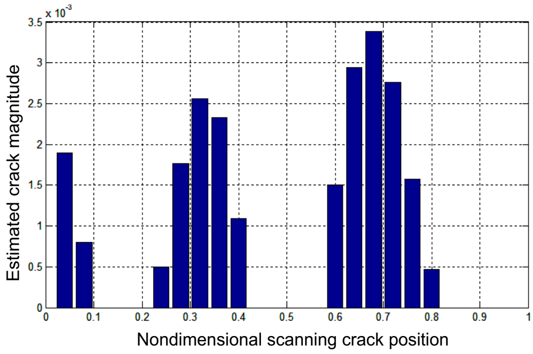

Fig. 2. Estimated crack magnitude from the natural frequencies measured by Ruotolo and Surace in the case of actual cracks are at 0.3175 (depth 20%) and 0.6812 (depth 20%)

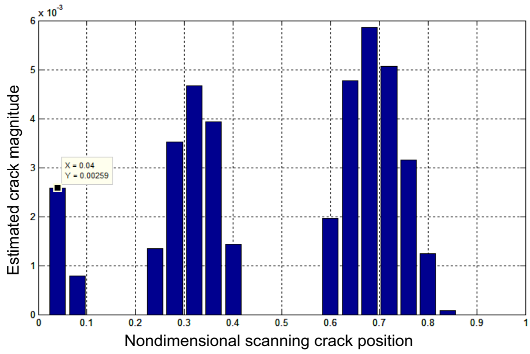

Fig. 3. Estimated crack magnitude from the natural frequencies measured by Ruotolo and Surace in the case of actual cracks are at 0.3175 (depth 30%) and 0.6812 (depth 20%)

The latter allow us to make a conclusion that the number of cracks detected equal to two and the crack positions are $\bar{e}_1 = 0.32$; $\bar{e}_2 = 0.68$. Final results of crack identification and its accuracy are presented in Tab. 2.

Table 2. Results of crack detection

Actual crack scenarios	Identified crack position and depth (error in %)	
C1: $a_1 = a_2 = 0.02$	$a_1 = 0.2218$ (10.9%)	$a_2 = 0.2105$ (5.25%)
C2: $a_1 = 0.2$; $a_2 = 0.3$	$a_1 = 0.2427$ (21.3%)	$a_2 = 0.3588$ (19.6%)
C3: $a_1 = 0.3$; $a_2 = 0.2$	$a_1 = 0.3416$ (13.8%)	$a_2 = 0.2218$ (10.9%)
$e_1 = 0.3175$; $e_2 = 0.6812$	$e_1 = 0.32$ (0.78%)	$e_2 = 0.68$ (0.17%)

Tab. 2 shows that the proposed herein procedure enables exact crack localization (within accuracy of frequency measurement error) and error in crack extent estimation can be within 10% for the cracks depth less than 20% beam thickness.

6. CONCLUSION

In the present study the well known Rayleigh Quotient has been developed for multi-stepped beam with arbitrary number of cracks and boundary conditions. This is an explicit expression of natural frequencies in term of crack parameters that provides a simplified method for calculating natural frequencies of the beam. Moreover, the Rayleigh Quotient obtained in more generalized form can be usefully applied also for identification of multi-stepped beam and boundary condition evaluation from measured natural frequencies. Specifically, based on the explicit expression there has been conducted a simple procedure for multiple crack detection of uniform Euler-Bernoulli beam that allows determining not only the crack parameters but also the amount of cracks in a beam from measured natural frequencies. The procedure is a further development of the so-called crack scanning method for the case of available natural frequencies. The theoretical development has been illustrated and validated by either numerical or experimental results. Namely, the natural frequencies calculated by the Rayleigh Quotient very well agreed with those computed from the characteristic equation and measured from an experiment. The proposed herein crack detection procedure applied with the aforementioned measured natural frequencies allow exact localization of double cracks in beam and estimating also crack depth with error less than 20%. The further study is intended to develop the crack detection procedure in the case of stepped beam with unknown boundary conditions.

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REFERENCES

[1] R. Adams, P. Cawley, C. Pye, and B. Stone. A vibration technique for non-destructively assessing the integrity of structures. *Journal of Mechanical Engineering Science*, **20**(2), (1978), pp. 93–100.

[2] R. Y. Liang, J. Hu, and F. Choy. Theoretical study of crack-induced eigenfrequency changes on beam structures. *Journal of Engineering Mechanics*, **118**(2), (1992), pp. 384–396.

[3] A. Morassi. Crack-induced changes in eigenparameters of beam structures. *Journal of Engineering Mechanics*, **119**(9), (1993), pp. 1798–1803.

[4] A. Morassi and M. Rollo. Identification of two cracks in a simply supported beam from minimal frequency measurements. *Journal of Vibration and Control*, **7**(5), (2001), pp. 729–739.

[5] L. Rubio. An efficient method for crack identification in simply supported Euler-Bernoulli beams. *Journal of Vibration and Acoustics*, **131**(5), (2009), DOI: 10.1115/1.3142876.

[6] Y. Narkis. Identification of crack location in vibrating simply supported beams. *Journal of Sound and Vibration*, **172**(4), (1994), pp. 549–558.

[7] F. B. Sayyad and B. Kumar. Identification of crack location and crack size in a simply supported beam by measurement of natural frequencies. *Journal of Vibration and Control*, **138**(3), (2012), pp. 381–388.

[8] S. M. Al-Said, M. Naji, and A. A. Al-Shukry. Flexural vibration of rotating cracked Timoshenko beam. *Journal of Vibration and Control*, **12**(11), (2006), pp. 1271–1287.

[9] W. Ostachowicz and M. Krawczuk. Analysis of the effect of cracks on the natural frequencies of a cantilever beam. *Journal of Sound and Vibration*, **150**(2), (1991), pp. 191–201.

[10] R. Behera, D. Parhi, and S. Sahu. Vibration analysis of a cracked rotor surrounded by viscous liquid. *Journal of Vibration and Control*, **12**(5), (2006), pp. 465–494.

[11] E. Shifrin and R. Ruotolo. Natural frequencies of a beam with an arbitrary number of cracks. *Journal of Sound and Vibration*, **222**(3), (1999), pp. 409–423.

[12] Q. Li. Vibratory characteristics of multi-step beams with an arbitrary number of cracks and concentrated masses. *Applied Acoustics*, **62**(6), (2001), pp. 691–706.

[13] D. Zheng and S. Fan. Natural frequencies of a non-uniform beam with multiple cracks via modified Fourier series. *Journal of sound and vibration*, **242**(4), (2001), pp. 701–717.

[14] N. Khiem and T. Lien. A simplified method for natural frequency analysis of a multiple cracked beam. *Journal of Sound and Vibration*, **245**(4), (2001), pp. 737–751.

[15] K. Aydin. Vibratory characteristics of Euler-Bernoulli beams with an arbitrary number of cracks subjected to axial load. *Journal of Vibration and Control*, **14**(4), (2008), pp. 485–510.

[16] S. Caddemi and I. Caliò. Exact closed-form solution for the vibration modes of the Euler-Bernoulli beam with multiple open cracks. *Journal of Sound and Vibration*, **327**(3), (2009), pp. 473–489.

[17] Z. Xiaoqing, H. Qiang, and L. Feng. Analytical approach for detection of multiple cracks in a beam. *Journal of engineering mechanics*, **136**(3), (2010), pp. 345–357.

[18] R. Y. Liang, J. Hu, and F. Choy. Quantitative NDE technique for assessing damages in beam structures. *Journal of Engineering Mechanics*, **118**(7), (1992), pp. 1468–1487.

[19] D. Patil and S. Maiti. Detection of multiple cracks using frequency measurements. *Engineering Fracture Mechanics*, **70**(12), (2003), pp. 1553–1572.

[20] Q. Li. Exact solutions for free longitudinal vibrations of non-uniform rods. *Journal of Sound and Vibration*, **234**(1), (2000), pp. 1–19.

[21] J. Fernandez-Saez, L. Rubio, and C. Navarro. Approximate calculation of the fundamental frequency for bending vibrations of cracked beams. *Journal of Sound and Vibration*, **225**(2), (1999), pp. 345–352.

[22] N. T. Khiem and H. T. Tran. A procedure for multiple crack identification in beam-like structure from natural vibration mode. *Journal of Vibration and Control*, (2013). DOI: 10.1177/1077546312470478.

[23] R. Ruotolo and C. Surace. Damage assessment of multiple cracked beams: numerical results and experimental validation. *Journal of Sound and Vibration*, **206**(4), (1997), pp. 567–588.

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