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METHOD OF FIRST INTEGRALS AND INTERFACE, SURFACE WAVES

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Abstract. The method of first integrals (MFI) based on the equation of motion for the displacement vector, or based on the one for the traction vector was introduced recently in order to find explicit secular equations of Rayleigh waves whose characteristic equations (i.e the equations determining the attenuation factor) are fully quartic or are of higher order (then the classical approach is not applicable). In this paper it is shown that, not only to Rayleigh waves, the MFI can be applicable also to other waves by running it on the equations for mixed vectors. In particular: (i) By applying the MFI to the equations for the displacement-traction vector we get the explicit dispersion equations of Stoneley waves in twinned crystals (ii) Running the MFI on the equations for the traction-electric induction vector and the traction-electrical potential vector provides the explicit dispersion equations of SH-waves in piezoelastic materials. The obtained dispersion equations are identical with the ones previously derived using the method of polarization vector, but the procedure of deriving them is more simple.

1. INTRODUCTION

Elastic surface waves and elastic interface waves, discovered by, respectively, L. Rayleigh [1] in 1885 and R. Stoneley [2] in 1924, have been studied extensively and exploited in a wide range of applications in seismology, acoustics, geophysics, telecommunications industry and materials science, for example. It would not be far-fetched to say that Rayleigh's study of surface waves upon an elastic half-space has had fundamental and far-reaching effects upon modern life and many things that we take for granted today, stretching from mobile phones through to the study of earthquakes, as addressed by Adams [3].

For the Rayleigh waves and Stoneley waves, their dispersion equations in the explicit form are very significant in practical applications. They can be used for solving the direct problems: studying effects of material parameters on the wave velocities, and especially for the inverse problems: determining material parameters from the measured wave speeds. Thus, the secular equations in the explicit form are always the main purpose of investigations related to Rayleigh waves and Stoneley waves.

When characteristic equations of waves (i.e. the equation determining the attenuation factor) are biquadratic, such as those of Rayleigh and Stoneley waves in isotropic elastic solids, the analytical expressions of their roots are easily obtained, and explicit secular equations are then derived (from the boundary conditions) using these expressions (see, for instance, [4]). However, when the characteristic equations are fully quartic or are of higher order, this approach is no longer applicable, because either the expressions of the roots are too complicated or, according Galois' theory [5], the roots can not be expressed by the operations: addition, subtraction, multiplication, division and radicals.

In order to overcome this difficulty, some methods have been proposed such as the method of the polarization vector [6]-[10], Ting's method [11], [12], and the method of first integrals (MFI) [13], [14], and with these methods one can obtain explicit secular equations of Rayleigh waves and Stoneley waves without using the characteristic equations. The method of the polarization vector and Ting's method are applicable to any kind of anisotropic materials, however, the deriving of explicit dispersion equations by these two approaches is not simple as that by the MFI. The method of first integrals was first introduced by Mozhaev [13] in 1995, which is based on the equations of motion for displacement components. In order to directly use the traction-free conditions, Destrade [14] introduced this method to the equations for traction components, and he has successfully employed this approach to the case of monoclinic materials with the symmetric plane $x_3 = 0$, where the plane strain still exists, and corresponding Rayleigh waves is a two-component surface wave.

In the present paper, we show that, not only to Rayleigh waves, the MFI can be applicable also to other waves, in particular: (i) Stoneley waves in twinned crystals (ii) SH-waves in piezoelastic materials. The MFI to be used in this paper is based on the equations for mixed vectors, such as the displacement-traction vector, the traction-electrical potential vector. The obtained dispersion equations are identical with the ones previously derived using the method of polarization vector, but the procedure of deriving them is more simple.

2. THE METHOD OF FIRST INTEGRALS

For deriving explicit secular equation of Rayleigh waves, the MFI based on the equations for the traction components is more convenient than the MFI based on those for displacement components, because when using the equation for the traction vector, the traction-free conditions can be taken into account directly. Hence, in this section we present briefly the MFI through the deriving the explicit secular equation of Rayleigh waves in monoclinic elastic materials with the symmetry plane at $x_3 = 0$. For detail the reader is referred to the paper by Destrade [14].

Consider a compressible elastic body of monoclinic material with the symmetry plane at $x_3 = 0$ (see [15]), and suppose that it occupies the half space $x_2 \geq 0$. In this case we can consider the plane strain (see [15]):

$$u_i = u_i(x_1, x_2, t), i = 1, 2;\ u_3 \equiv 0. \tag{1}$$

where u_i are the displacement components. The equations of motion are then of the form:

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} = \rho \ddot{u}_1 \\ \sigma_{12,1} + \sigma_{22,2} = \rho \ddot{u}_2 \end{cases} \tag{2}$$

in which ρ is the mass density of material, a superposed dot signifies the differentiation with respect to the time t, commas indicate the differentiation with respect to the spatial variables x_i. The stresses are related to strains by:

$$\begin{cases} \sigma_{11} = c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + 2c_{16}\epsilon_{12} \\ \sigma_{12} = c_{16}\epsilon_{11} + c_{26}\epsilon_{22} + 2c_{66}\epsilon_{12} \\ \sigma_{22} = c_{12}\epsilon_{11} + c_{22}\epsilon_{22} + 2c_{26}\epsilon_{12} \end{cases} \tag{3}$$

where c_{ij} are the elastic stiffness constants of material and the components of strain ϵ_{ij} are defined as:

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} \tag{4}$$

In addition to equation (2) - (4), the traction-free condition at the plane $x_2 = 0$:

$$\sigma_{12}(0) = \sigma_{22}(0) = 0 \tag{5}$$

and the decay condition at the infinity:

$$u_1(+\infty) = u_2(+\infty) = \sigma_{12}(+\infty) = \sigma_{22}(+\infty) = 0 \tag{6}$$

are required to be satisfied.

Suppose that Rayleigh wave propagates in the x_1-direction with speed c, and decays in the x_2-direction. Then the displacement and stress components are sought in the form:

$$u_i = U_i(y)e^{ik(x_1-ct)},\ \sigma_{i2} = ikt_i(y)e^{ik(x_1-ct)}, i = 1, 2. \tag{7}$$

where k is the wave number, and y $= kx_2$.

Introducing (7) into (2) and (3), and taking into account (4) we have:

$$\xi' = iN\xi \tag{8}$$

where:

$$\begin{aligned} &\xi = \begin{bmatrix} U \\ \tau \end{bmatrix} \text{ with } U = [U_1, U_2]^T, \tau = [t_1, t_2]^T \\ &N = \begin{bmatrix} N_1 & N_2 \\ K & N_1^T \end{bmatrix};\ N_1 = \begin{bmatrix} -r_6 & -1 \\ -r_2 & 0 \end{bmatrix} \\ &N_2 = \begin{bmatrix} n_{66} & n_{26} \\ n_{26} & n_{22} \end{bmatrix};\ K = \begin{bmatrix} -(\eta - X) & 0 \\ 0 & X \end{bmatrix} \end{aligned} \tag{9}$$

and:

$$\begin{aligned} r_2 = \tfrac{1}{\Delta}(c_{12}c_{66} - c_{16}c_{26}),\ r_6 = -\tfrac{1}{\Delta}(c_{12}c_{26} - c_{22}c_{16}),\ n_{22} = c_{66}/\Delta \\ n_{26} = -c_{26}/\Delta,\ n_{66} = c_{22}/\Delta,\ \Delta = c_{22}c_{66} - c_{26}^2 \end{aligned} \tag{10}$$

$\eta = c_{11} - c_{12}r_{12} - c_{16}r_6,\ X = \rho c^2$, the prime stands for the derivative with respect to $y = kx_2$, and the symbol T indicates the transpose of matrices. From (9), (10), it is not difficult to verify that the characteristic equation $|N - pI| = 0$ of (8) is (see also [16]):

$$\omega_4 p^4 - 2\omega_3 p^3 + \omega_2 p^2 - 2\omega_1 p + \omega_0 = 0 \tag{11}$$

where:

$$\begin{aligned}
&\omega_4 = s'_{11},\ \omega_3 = s'_{16},\ \omega_2 = s'_{66} + 2s'_{12} - [s'_{11}(s'_{22} + s'_{66}) - s'^2_{12} - s'^2_{16}]X \\
&\omega_1 = s'_{26} + [s'_{16}(s'_{22} - s'_{12}) + s'_{26}(s'_{11} - s'_{12})]X \\
&\omega_0 = s'_{22} - [s'_{22}(s'_{11} + s'_{66}) - s'^2_{12} - s'^2_{16}]X + X^2/\det C
\end{aligned} \tag{12}$$

here s'_{ij} $(i, j = 1, 2, 6)$ are elements of the matrix S' (called the reduced compliance matrix) that is the inverse of the matrix $C = (c_{ij})$ $(i, j = 1, 2, 6)$ (called the stiffness matrix). Since the characteristic equation (11) is fully quartic, the explicit secular equation of the wave could not be derived by using the traditional approach.

Eliminating U from (8), we have:

$$\widehat{\alpha}\tau'' - i\widehat{\beta}\tau' - \widehat{\gamma}\tau = 0 \tag{13}$$

where:

$$\begin{aligned}
&\widehat{\alpha} = K^{-1} = \begin{bmatrix} \dfrac{-1}{\eta - \rho c^2} & 0 \\ 0 & \dfrac{1}{\rho c^2} \end{bmatrix} \\
&\widehat{\beta} = K^{-1}N_1^T + N_1K^{-1} = \begin{bmatrix} \dfrac{2r_6}{\eta - \rho c^2} & \dfrac{r_2}{\eta - \rho c^2} - \dfrac{1}{\rho c^2} \\ \dfrac{r_2}{\eta - \rho c^2} - \dfrac{1}{\rho c^2} & 0 \end{bmatrix} \\
&\widehat{\gamma} = N_1K^{-1}N_1^T - N_2 = \begin{bmatrix} \dfrac{-r_6^2}{\eta - \rho c^2} - n_{66} & \dfrac{-r_6 r_2}{\eta - \rho c^2} - n_{26} \\ \dfrac{-r_6 r_2}{\eta - \rho c^2} - n_{26} & \dfrac{-r_2^2}{\eta - \rho c^2} - n_{22} \end{bmatrix}
\end{aligned} \tag{14}$$

Note that $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ are symmetric real matrices. In the component form Eq. (13) is written as follows:

$$\widehat{\alpha}_{kl}t''_l - i\widehat{\beta}_{kl}t'_l - \widehat{\gamma}_{kl}t_l = 0,\ \ (k, l = 1, 2) \tag{15}$$

Multiplying two sides of Eq. (15) by $i\overline{t_m}$ and then adding the resulting equation to its conjugation give:

$$\widehat{\alpha}_{kl}(it''_l\overline{t_m} + \overline{it''_l}t_m) + \widehat{\beta}_{kl}(t'_l\overline{t_m} + \overline{t'_l}t_m) + \widehat{\gamma}_{kl}(t_l\overline{it_m} + \overline{t_l}it_m) = 0 \tag{16}$$

where the bar indicates the conjugation. Now we introduce 2×2-matrices D, E, F whose elements are defined as follows:

$$D_{lm} =< it''_l, t_m >;\ \ E_{lm} =< t'_l, t_m >;\ \ F_{lm} =< t_l, it_m >;\ \ (l, m = 1, 2) \tag{17}$$

where: $< \varphi, g >= \int_0^{+\infty} (\varphi\overline{g} + \overline{\varphi}g)dy$. From (5), (6) and (17), we find out that D, E, F being antisymmetric, i. e. :

$$D = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix},\quad E = \begin{bmatrix} 0 & e \\ -e & 0 \end{bmatrix},\quad F = \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} \tag{18}$$

Now, integrating Eq. (16) from 0 to $+\infty$ provides:

$$\widehat{\alpha}D + \widehat{\beta}E + \widehat{\gamma}F = 0 \tag{19}$$

From (18), (19) it follows:

$$\begin{cases} \widehat{\alpha}_{11}d + \widehat{\beta}_{11}e + \widehat{\gamma}_{11}f = 0 \\ \widehat{\alpha}_{12}d + \widehat{\beta}_{12}e + \widehat{\gamma}_{12}f = 0 \\ \widehat{\alpha}_{22}d + \widehat{\beta}_{22}e + \widehat{\gamma}_{22}f = 0 \end{cases} \tag{20}$$

Thus, we have:

$$\begin{vmatrix} \widehat{\alpha}_{11} & \widehat{\beta}_{11} & \widehat{\gamma}_{11} \\ \widehat{\alpha}_{12} & \widehat{\beta}_{12} & \widehat{\gamma}_{12} \\ \widehat{\alpha}_{22} & \widehat{\beta}_{22} & \widehat{\gamma}_{22} \end{vmatrix} = 0 \tag{21}$$

in order that the homogeneous linear system (20) has a non trivial solution. Introducing (14) into (21) yields (see Eq. (28) in [14]):

$$\begin{aligned} [\eta - (1 - r_2)X]\{(\eta - X)[(\eta - X)(n_{66}(X - 1) + r_6^2X] + X^2[(\eta - X)n_{22} + r_2^2]\} \\ + 2r_6X^2(\eta - X)[(\eta - X)n_{26} + r_2r_6] = 0 \end{aligned} \tag{22}$$

Equation (22) is the expected secular equation and it is a quartic equation for $X = \rho c^2$.

Remark 1: Starting from the equation for traction components (13) the boundary conditions (5), (6) have been taken directly, and the secular equation has been derived without using the characteristic equation.

3. STONELEY WAVES IN TWINNED CRYSTALS

Consider Stoneley waves traveling with the velocity c and the wave number k at the interface of a bimaterial made of two perfectly bonded orthotropic media. Both half-spaces are made of the same crystal (mass density ρ, non-zero reduced compliances $s'_{11}, s'_{22}, s'_{12}, s'_{44}, s'_{55}, s'_{66}$, see for instance [15]). However, the axis Ox_2 normal to the interface and the direction of propagation Ox_1 are inclined at an angle θ to the crystallographic axes Oy and Ox of the lower ($x_2 > 0$) half-space and at an angle $-\theta$ for the upper half-space ($x_2 < 0$).

In the lower half-space, the strain-stress relation is $\epsilon_{ij} = s_{ik}\sigma_{ik}$, where the reduced compliances s_{ik} are given by (see, for instance, [16] or [17]):

$$\begin{aligned} s_{11} &= s'_{11}\cos^4\theta + 2(s'_{12} + s'_{66})\cos^2\theta\sin^2\theta + s'_{22}\sin^4\theta \\ s_{22} &= s'_{22}\cos^4\theta + 2(s'_{12} + s'_{66})\cos^2\theta\sin^2\theta + s'_{11}\sin^4\theta \\ s_{12} &= s'_{12} + (s'_{11} + s'_{22} - 2s'_{12} - s'_{66})\cos^2\theta\sin^2\theta \\ s_{66} &= s'_{66} + 4(s'_{11} + s'_{22} - 2s'_{12} - s'_{66})\cos^2\theta\sin^2\theta \\ s_{16} &= [2s'_{22}\sin^2\theta - 2s'_{11}\cos^2\theta + 2(s'_{12} + s'_{66})(\cos^2\theta - \sin^2\theta)]\cos\theta\sin\theta \\ s_{26} &= [2s'_{22}\cos^2\theta - 2s'_{11}\sin^2\theta - 2(s'_{12} + s'_{66})(\cos^2\theta - \sin^2\theta)]\cos\theta\sin\theta \end{aligned} \tag{23}$$

For the upper half-space we have the formulas similar to (23) in which θ is replaced by $-\theta$.

Since the material of the half-spaces is orthotropic, we can consider the plane strain (see [15]), for which the displacement components are of the form (1). In this case, the stresses are related to the strains by Eqs. (2), and the equations of motion are Eqs. (3), where the matrix $C = (c_{ij}),\ i, j = 1, 2, 6$, is the inverse of the matrix $S = (s_{ij}),\ i, j = 1, 2, 6$, whose elements are defined by (23). Further, the displacement components and stress components are required to be continuous at the plane $x_2 = 0$ and must vanish at $\pm\infty$.

Suppose that the Stoneley wave propagates in x_1-direction and attenuates in in x_2-direction. Then, for the lower half-space ($x_2 > 0$) the displacement and stress components are sought in the form (7). Substituting (7) into (2) and (3) yield an equation determining the vector $\xi = [U_1(y)\ U_2(y)\ t_1(y)\ t_2(y)]^T$. It is Eq. (8), in which the matrix N is defined by (9), (10). Note that here the matrix C is the inverse of S (not of S' as in Section 2). Similarly, for the upper half-space ($x_2 < 0$), the solution is sought in the form:

$$u_i^* = U_i^*(y)e^{ik(x_1-ct)},\ \sigma_{i2}^* = ikt_i^*(y)e^{ik(x_1-ct)}, i = 1, 2 \tag{24}$$

in which the vector $\xi_* = [U_1^*(y)\ U_2^*(y)\ t_1^*(y)\ t_2^*(y)]^T$ satisfies:

$$\xi_*' = iN_*\xi_* \tag{25}$$

where N_* is defined also by (9), (10), but θ is replaced by $-\theta$ in the formulas (23) when calculating s_{ij}^*. Note that same quantities related to lower and upper half-spaces have the same symbol but are systematically distinguished by an asterisk if pertaining to upper half-space. In addition to the equations (8) and (25) the following conditions must be satisfied:

$$\xi(0) = \xi_*(0),\ \xi(+\infty) = \xi_*(-\infty) = 0 \tag{26}$$

in order to ensue that the continuous condition at the plane $x_2 = 0$ and the decay condition at $\pm\infty$ are satisfied.

Remark 2:

i) It is not difficult to verify that $s_{11} = s_{11}^*$, $s_{22} = s_{66}^*$, $s_{66} = s_{66}^*$, $s_{12} = s_{12}^*$, $s_{16} = -s_{16}^*$, $s_{26} = -s_{26}^*$ using (23).

ii) The characteristic equations $|N - pI| = 0$ and $|N_* - pI| = 0$ (of (8) and (25), respectively), are of the form (11), and they are both fully quartic. Note that $\omega_0 = \omega_0^*$, $\omega_1 = -\omega_1^*$, $\omega_2 = \omega_2^*$, $\omega_3 = -\omega_3^*$, $\omega_4 = \omega_4^*$.

ii) One can show that if λ is a root of $|N - pI| = 0$, then $-\lambda$ is a root of $|N_* - pI| = 0$.

It is not difficult to demonstrate that the solutions of (8) and (25) are:

$$\xi(y) = \beta_1\Gamma_1 e^{ip_1y} + \beta_2\Gamma_2 e^{ip_2y},\ \xi_*(y) = \beta_1^*\Gamma_1^* e^{-ip_1y} + \beta_2^*\Gamma_2^* e^{-ip_2y} \tag{27}$$

where β_1, β_2 are constants to be determined, p_1, p_2 are the roots of 11 having negative imaginary parts in order to ensure the decay condition, and:

$$\Gamma_k = [a_k\ b_k\ m_k\ n_k]^T,\ \Gamma_k^* = [-a_k\ b_k\ m_k\ -n_k]^T,\ k = 1, 2 \tag{28}$$

in which:

$$\begin{aligned} a_k &= [(1+r_2)X-\eta]p_k + r_2 r_6 X + n_{26}(\eta - X)X \\ b_k &= -Xp_k^2 - 2r_6 X p_k - [1 + r_6^2 + n_{66}(\eta - X)]X + \eta \\ m_k &= p_k^2 + (r_6 - n_{26}X)p_k - r_2(1 - n_{66}X) - r_6 n_{26} X \\ n_k &= -p_k^3 - 2r_6 p_k^2 + [r_2 - r_6^2 - n_{66}(\eta - X)]p_k + n_{26}(\eta - X) + r_2 r_6 \end{aligned} \tag{29}$$

From $(26)_1$, (27), (28) it can be shown that either:

$$U_1(0) = t_2(0) = 0 \tag{30}$$

or

$$U_2(0) = t_1(0) = 0 \tag{31}$$

Following Mozhaev et al. [18], the Stoneley wave satisfying (30) (resp. (31)) is called IAW1 (resp. IAW2). Thus, the amplitude vector ξ of the IAW1 satisfies Eq. (8) and the boundary condition:

$$U_1(0) = t_2(0) = U_1(+\infty) = t_2(+\infty) = 0 \tag{32}$$

For the the IAW1 the vector ξ is a solution of Eq. (8) with the following boundary condition:

$$U_2(0) = t_1(0) = U_2(+\infty) = t_1(+\infty) = 0 \tag{33}$$

3.1. The MFI for displacement-traction equations and explicit secular equation of the IAW1

In view of (32), it suggests that in order to use directly the boundary condition (32) we should establish the MFI that is based on the equations for U_1 and t_2. To this end, we introduce the displacement-traction vectors: $\Sigma_1 = [U_1 \;\; t_2]^T$ and $\Sigma_2 = [t_1 \;\; U_2]^T$, and find the (ordinary differential) equation for Σ_1. It is follows from (8) that:

$$\zeta' = iM\zeta \tag{34}$$

where $\zeta = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix}$

$$\begin{aligned} M &= \begin{bmatrix} M_1 & M_2 \\ Q & M_1^T \end{bmatrix}; \;\; M_1 = \begin{bmatrix} -r_6 & n_{26} \\ 0 & 0 \end{bmatrix} \\ M_2 &= \begin{bmatrix} n_{66} & -1 \\ -1 & X \end{bmatrix}; \;\; Q = \begin{bmatrix} X-\eta & -r_2 \\ -r_2 & n_{22} \end{bmatrix} \end{aligned} \tag{35}$$

Eliminating Σ_2 from (34), we have:

$$\widehat{\alpha}\Sigma_1'' - i\widehat{\beta}\Sigma_1' - \widehat{\gamma}\Sigma_1 = 0 \tag{36}$$

where:

$$\widehat{\alpha}=\begin{bmatrix}\dfrac{X}{n_{66}X-1} & \dfrac{1}{n_{66}X-1}\\ \dfrac{1}{n_{66}X-1} & \dfrac{n_{66}}{n_{66}X-1}\end{bmatrix},\ \widehat{\beta}=\begin{bmatrix}\dfrac{-2r_6X}{n_{66}X-1} & \dfrac{n_{26}X-r_6}{n_{66}X-1}\\ \dfrac{n_{26}X-r_6}{n_{66}X-1} & \dfrac{2n_{26}}{n_{66}X-1}\end{bmatrix}$$
$$\widehat{\gamma}=\begin{bmatrix}\dfrac{r_6^2X}{n_{66}X-1}+\eta-X & \dfrac{-r_6n_{26}X}{n_{66}X-1}+r_2\\ \dfrac{-r_6n_{26}X}{n_{66}X-1}+r_2 & \dfrac{n_{26}^2X}{n_{66}X-1}-n_{22}\end{bmatrix} \tag{37}$$

Equation (36) is desired equation for Σ_1. Note that $\widehat{\alpha},\widehat{\beta},\widehat{\gamma}$ are symmetric real matrices. Following the same procedure as done in Section 2, from (36) and (32) (i. e. $\Sigma_1(0)=\Sigma_1(+\infty)=0$) we arrive at (19), in which D, E, F are antisymmetric matrices being of the form (18). Note that in order to prove the antisymmetry of D, E, F, the fact $\Sigma_1(0)=\Sigma_1(+\infty)=0$ is taken into account. In component form (19) is of the form (20), and vanishing the determinant of this homogeneous linear system gives the secular equation of the IAW1, namely:

$$\begin{vmatrix}\widehat{\alpha}_{11} & \widehat{\beta}_{11} & \widehat{\gamma}_{11}\\ \widehat{\alpha}_{12} & \widehat{\beta}_{12} & \widehat{\gamma}_{12}\\ \widehat{\alpha}_{22} & \widehat{\beta}_{22} & \widehat{\gamma}_{22}\end{vmatrix}=0 \tag{38}$$

Expansion of the left-hand side of (38) leads to:

$$e_3X^3+e_2X^2+e_1X+e_0=0 \tag{39}$$

where:

$$\begin{aligned}
&e_3=n_{26}(n_{26}^2+n_{66}^2-n_{22}n_{66})\\
&e_2=n_{22}n_{26}-3n_{26}n_{66}-\eta n_{26}n_{66}^2-2n_{26}n_{66}r_2\\
&\quad+3n_{26}^2r_6-n_{22}n_{66}r_6-n_{66}^2r_6-2n_{66}^2r_2r_6+n_{26}n_{66}r_6^2\\
&e_1=2n_{26}+3\eta n_{26}n_{66}+2n_{26}r_2+n_{22}r_6+n_{66}r_6+\eta n_{66}^2r_6+2n_{66}r_2r_6+2n_{26}r_6^2+n_{66}r_6^3\\
&e_0=-2n_{26}\eta-n_{66}r_6\eta
\end{aligned} \tag{40}$$

It is clear that Eq. (40) is fully explicit, and it coincides with the result found by Destrade [19]. However, it should be stressed that the deriving of the explicit secular equation in this paper is more simple than that in [14] because that here we are involved in only 2×2-matrices, while in [14] it was concerned with 4×4-matrices.

3.2. Explicit secular equation of the IAW2

Since the IAW2 is required to satisfy (33), i. e. $\Sigma_2(0)=\Sigma_2(+\infty)=0$, analogously as above we need an (differential) equation for Σ_2. The explicit secular equation of the IAW2 will be then obtained implementing the MFI on this equation. The equation for Σ_2 is derived eliminating Σ_1 from (34), and it is:

$$\widehat{\alpha}\Sigma_2''-i\widehat{\beta}\Sigma_2'-\widehat{\gamma}\Sigma_2=0 \tag{41}$$

where:

$$\widehat{\alpha} = \frac{1}{\delta}\begin{bmatrix} n_{22} & r_2 \\ r_2 & X-\eta \end{bmatrix}, \quad \widehat{\beta} = \frac{1}{\delta}\begin{bmatrix} 2(r_2 n_{26} - r_6 n_{22}) & n_{26}(X-\eta) - r_2 r_6 \\ n_{26}(X-\eta) - r_2 r_6 & 0 \end{bmatrix}$$

$$\widehat{\gamma} = \frac{1}{\delta}\begin{bmatrix} r_6^2 n_{22} - 2r_2 r_6 n_{26} + n_{26}^2 (X-\eta) - n_{66}\delta & \delta \\ \delta & -X\delta \end{bmatrix} \tag{42}$$

and $\delta = n_{22}(X-\eta) - r_2^2$. Applying the MFI to Eq. (41) and taking into account the fact $\Sigma_2(0) = \Sigma_2(+\infty) = 0$ yield Eq. (21) in which the elements $\widehat{\alpha}_{ij}$, $\widehat{\beta}_{ij}$, $\widehat{\gamma}_{ij}$ are defined by (42). Introducing (42) into (21) we have:

$$\bar{e}_3 X^3 + \bar{e}_2 X^2 + \bar{e}_1 X + \bar{e}_0 = 0 \tag{43}$$

where:

$$\begin{aligned}
\bar{e}_3 &= -n_{22}^2 n_{26} - n_{26}^3 + n_{22} n_{26} n_{66} \\
\bar{e}_2 &= 2\eta n_{22}^2 n_{26} + 3\eta n_{26}^3 - 3\eta n_{22} n_{26} n_{66} + 2n_{22} n_{26} r_2 + 3n_{22} n_{26} r_2^2 \\
&- n_{26} n_{66} r_2^2 - 2n_{22}^2 r_6 - n_{22}^2 r_2 r_6 + 3n_{26}^2 r_2 r_6 - n_{22} n_{66} r_2 r_6 - n_{22} n_{26} r_6^2 \\
\bar{e}_1 &= -\eta^2 n_{22}^2 n_{26} - 3\eta^2 n_{26}^3 + 3\eta^2 n_{22} n_{26} n_{66} - 4\eta n_{22} n_{26} r_2 - 3\eta n_{22} n_{26} r_2^2 + 2\eta n_{26} n_{66} r_2^2 \\
&- 2n_{26} r_2^3 - 2n_{26} r_2^4 + 4\eta n_{22}^2 r_6 + \eta n_{22}^2 r_2 r_6 - 6\eta n_{26}^2 r_2 r_6 + 2\eta n_{22} n_{66} r_2 r_6 + 2n_{22} r_2^2 r_6 \\
&+ n_{22} r_2^3 r_6 + n_{66} r_2^3 r_6 + 2\eta n_{22} n_{26} r_6^2 - 2n_{26} r_2^2 r_6^2 + n_{22} r_2 r_6^3 \\
\bar{e}_0 &= \eta^3 n_{26}^3 - 3\eta^3 n_{22} n_{26} n_{66} + 2\eta^2 n_{22} n_{26} r_2 - \eta^2 n_{26} n_{66} r_2^2 + 2\eta n_{26} r_2^3 \\
&- 2\eta^2 n_{22}^2 r_6 + 3\eta^2 n_{26}^2 r_2 r_6 - \eta^2 n_{22} n_{66} r_2 r_6 - 2\eta n_{22} r_2^2 r_6 \\
&- \eta n_{66} r_2^3 r_6 - \eta^2 n_{22} n_{26} r_6^2 + 2\eta n_{26} r_2^2 r_6^2 - \eta n_{22} r_2 r_6^3
\end{aligned} \tag{44}$$

Equation (43) is the explicit dispersion equation of the IAW2. It is the same as the result found by Destrade [19], but the procedure leading to it is more simple than that in [14], as noted above.

4. PIEZOACOUSTIC SHEAR - HORIZONTAL WAVES

Consider a half space $x_2 \geq 0$ of piezoelectric crystals with mass density ρ, possessing at most the tetragonal $\bar{4}$ symmetry (this symmetry includes the tetragonal $\overline{42}$m, cubic $\overline{43}$m, and cubic 23 cases) (see [8]), and its crystallographic axes are OX, OY, OZ where OZ coincides wth Ox_3, OX is inclined at an angle θ to Ox_1. Let $\hat{c}_{ijkl}$, $\hat{e}_{ijk}$, $\hat{\epsilon}_{ij}$ be its respective elastic, piezoelectric, and dielectric constants with respect to the coordinate system $OXYZ$. In the $Ox_1x_2x_3$ system, under the electrostatic approximation for the electric field, the stress tensor components σ_{ij} and the electric induction components D_i are related to the gradients of the mechanical displacement vector $\mathbf{u}$ and of the electrical potential ϕ by:

$$\sigma_{ij} = c_{ijkl} u_{l,k} + e_{ijk}\phi_{,k}, \;\; D_i = e_{ikl} u_{l,k} - \epsilon_{ik}\phi_{,k} \tag{45}$$

Using the Voigt constracted notation, these relations are written in matrix form as (see [8]):

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} & 0 & 0 & e_{31} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & -c_{16} & 0 & 0 & -e_{31} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 & e_{14} & -e_{15} & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 & e_{15} & e_{14} & 0 \\ c_{16} & -c_{16} & 0 & 0 & 0 & c_{66} & 0 & 0 & e_{36} \\ 0 & 0 & 0 & e_{14} & e_{15} & 0 & -\epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & -e_{15} & e_{14} & 0 & 0 & -\epsilon_{11} & 0 \\ e_{31} & -e_{31} & 0 & 0 & 0 & e_{36} & 0 & 0 & -\epsilon_{33} \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{2,3}+u_{3,2} \\ \phi_{,1} \\ \phi_{,2} \\ \phi_{,3} \end{bmatrix} \quad (46)$$

Explicitly, the c_{ij}, e_{ij} and ϵ_{ij} are deduced from $\hat{c}_{ij}$, $\hat{e}_{ij}$ and $\hat{\epsilon}_{ij}$ in $OXYZ$ by the well-known relationships. In particular (see also [8]):

$$c_{44} = \hat{c}_{44},\ \epsilon_{11} = \hat{\epsilon}_{11},\ e_{14} = \hat{e}_{14}\cos 2\theta - \hat{e}_{15}\sin 2\theta,\ e_{15} = \hat{e}_{15}\cos 2\theta + \hat{e}_{14}\sin 2\theta \quad (47)$$

Now we consider an anti-plane SH wave propagating in the x_1-direction with the speed c and the wave number k, and attenuating in the x_2-direction. It is known [20] that for the crystals under consideration this waves decouples entirely from its in-plane counterpart, a purely elastic two-component Rayleigh wave. Therefore, the wave is represented as $u_1 = u_2 = 0$ and:

$$\{u_3,\ \phi\} = \{U_3(y),\ \varphi(y)\}\exp[ik(x_1 - ct)] \quad (48)$$

where $y = kx_2$. From (46), (47) it follows σ_{11}=σ_{22}=σ_{33}=σ_{12}=D_3=0, and:

$$\{\sigma_{31},\ \sigma_{32},\ D_1,\ D_2\} = ik\{t_1(y),\ t_2(y),\ d_1(y),\ d_2(y)\}\exp[ik(x_1 - ct)] \quad (49)$$

where:

$$\begin{aligned} &t_1 = c_{44}U_3 + e_{15}\varphi - ie_{14}\varphi',\ t_2 = -ic_{44}U_3' + e_{14}\varphi + ie_{15}\varphi' \\ &d_1 = e_{15}U_3 - ie_{14}U_3' - \epsilon_{11}\varphi,\ d_2 = e_{14}U_3 + ie_{15}U_3' + i\epsilon_{11}\varphi' \end{aligned} \quad (50)$$

the prime signifies the derivative with respect to $y = kx_2$. Using the above-introduced functions $U_3(y)$, $\varphi(y)$, $t_i(y)$, $d_i(y)$ (i=1, 2), the classical equations of piezoelasticity:

$$\sigma_{ij,j} = \rho\ddot{u}_i,\ D_{i,i} = 0 \quad (51)$$

reduce to:

$$-t_1 + it_2' = -XU_3,\ -d_1 + id_2' = 0 \quad (52)$$

In addition to six equations (50) and (52) six unknown $U_3(y)$, $\varphi(y)$, $t_i(y)$, $d_i(y)$ $(i = 1, 2)$ are required to satisfy the boundary condions:

i) Electrical boundary conditions at the surface $x_2 = 0$ (see [8]):

$$d_2(0) = 0,\ \text{or}\ \varphi(0) = 0 \quad (53)$$

ii) Traction-free condition at $x_2 = 0$:

$$t_2(0) = 0 \quad (54)$$

and must vanish at infinity:

$$U_3(+\infty) = \varphi(+\infty) = t_1(+\infty) = t_2(+\infty) = d_1(+\infty) = d_2(+\infty) = 0 \quad (55)$$

The condtions $(53)_1$ and $(53)_2$ are called the elecrically open boundary condition and the metallized boundary condition, respectively. Now we consider separately these conditions and derive corresponding explicit secular equations using the MFI.

4.1. Electrically open boundary condition

Let $V = [t_2\ d_2]^T$, a traction-electric induction vector. From $(53)_1$, (54), (55) it follows:

$$V(0) = V(+\infty) = 0 \tag{56}$$

Thus, we need an equation for V on which we can run the MFI. Let $\xi = \begin{bmatrix} U \\ V \end{bmatrix}$, where $U = [U_3\ \varphi]^T$. It is not difficult to verify, from (50), (52), that the vector ξ satisfies the following equation:

$$\xi' = iN\xi \tag{57}$$

where:

$$N = \begin{bmatrix} N_1 & N_2 \\ K & N_1^T \end{bmatrix} \tag{58}$$

with N_1, N_2, K are defined as ($p^2 = \dfrac{e_{14}^2}{c_{11}\epsilon_{11} + e_{15}^2}$):

$$N_1 = \begin{bmatrix} \dfrac{p^2 e_{15}}{e_{14}} & -\dfrac{p^2 \epsilon_{11}}{e_{14}} \\ \dfrac{p^2 c_{44}}{e_{14}} & \dfrac{p^2 e_{15}}{e_{14}} \end{bmatrix}; \ N_2 = \begin{bmatrix} \dfrac{p^2 \epsilon_{11}}{e_{14}^2} & -\dfrac{p^2 e_{15}}{e_{14}^2} \\ -\dfrac{p^2 e_{15}}{e_{14}^2} & -\dfrac{p^2 c_{44}}{e_{14}^2} \end{bmatrix}$$

$$K = \begin{bmatrix} X - c_{44}(1+p^2) & -e_{15}(1+p^2) \\ -e_{15}(1+p^2) & \epsilon_{11}(1+p^2) \end{bmatrix} \tag{59}$$

Eliminating U from (57), we have:

$$\widehat{\alpha}V'' - i\widehat{\beta}V' - \widehat{\gamma}V = 0 \tag{60}$$

where:

$$\widehat{\alpha} = \frac{1}{\Delta}\begin{bmatrix} \epsilon_{11} & e_{15} \\ e_{15} & \frac{X}{1+p^2} - c_{44} \end{bmatrix}, \ \widehat{\beta} = \frac{p^2}{\Delta e_{14}(1+p^2)}\begin{bmatrix} 0 & \epsilon_{11}X - 2\Delta \\ \epsilon_{11}X - 2\Delta & 2e_{15}X \end{bmatrix}$$

$$\widehat{\gamma} = \frac{p^2}{e_{14}^2(1+p^2)}\begin{bmatrix} -\epsilon_{11} & e_{15} \\ e_{15} & \text{check} \end{bmatrix} \tag{61}$$

Note that $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ are symmetric real matrices. From (58) and (59) it is readily to see that the characteristic equation $|N - \lambda I| = 0$ of Eq. (57) is fully quartic, so that we should employ the MFI for getting the explicit secular of the wave. Applying the MFI to Eq. (60) and taking into account (56) yield Eq. (21) in which $\widehat{\alpha}_{ij}$, $\widehat{\beta}_{ij}$, $\widehat{\gamma}_{ij}$ are defined by (61). Introducing (61) into (21) and expanding its left-hand side lead to:

$$\frac{\epsilon_{11}^2 p^2}{e_{14}^2}X^2 - \epsilon_{11}(p^2+1)[3 - p^2(1 + 4\frac{e_{15}^2}{e_{14}^2})]X + 2(p^2+1)^2(\epsilon_{11}c_{44} - e_{14}^2 - e_{14}^2) = 0 \tag{62}$$

This result is the same as Eq. (30) in [8] that was found by Collet and Destrade by using the method of polarization vector that is not simple as here due to the same reason mentioned above.

4.2. Metallized boundary condition

In this case we also have (56), but now $V = [t_2\ \varphi]^T$, called the traction-electrical potential vector. Let $\xi = \begin{bmatrix} U \\ V \end{bmatrix}$, where $U = [U_3\ d_2]^T$, then one can see, from (50), (52), that the vector ξ satisfies the following equation:

$$\xi' = iN\xi \tag{63}$$

where:

$$N = \begin{bmatrix} N_1 & N_2 \\ K & N_1^T \end{bmatrix}$$

in which:

$$N_1 = \begin{bmatrix} p^2 \dfrac{e_{15}}{e_{14}} & -p^2 \dfrac{e_{15}}{e_{14}} \\ -e_{15}(1+p^2) & p^2 \dfrac{e_{15}}{e_{14}} \end{bmatrix}; \quad N_2 = \begin{bmatrix} p^2 \dfrac{\epsilon_{11}}{e_{14}^2} & -p^2 \dfrac{\epsilon_{11}}{e_{14}} \\ -p^2 \dfrac{\epsilon_{11}}{e_{14}} & \epsilon_{11}(1+p^2) \end{bmatrix}$$
$$K = \begin{bmatrix} X - e_{44}(1+p^2) & p^2 \dfrac{c_{44}}{e_{14}} \\ p^2 \dfrac{c_{44}}{e_{14}} & -p^2 \dfrac{c_{44}}{e_{14}^2} \end{bmatrix} \tag{64}$$

By runing the MFI on Eqs. (63) and (56) we derive the explicit dispersion equation of the wave, namely:

$$X^2 - c_{44} \frac{3c_{44}\epsilon_{11} + 4e_{14}^2 + 4e_{15}^2}{c_{44}\epsilon_{11} + e_{15}^2} X + 2c_{44}^2 \frac{c_{44}\epsilon_{11} + 2e_{14}^2 + 2e_{15}^2}{c_{44}\epsilon_{11} + e_{15}^2} = 0 \tag{65}$$

Equation (65) coinsides with Eq. (24) in [8] found by using the method of polarization vector, but the deriving of it here is more simple.

5. CONCLUSIONS

In this paper we examine the Stoneley wave in twinned crystals and the SH-waves in piezoelastic media. The main aim is to find explicit secular equations of these waves. Since both of them have a fully quartic characteristic equation, so that the traditional approach is not applicable. We have used the MFI in order to get explicit secular equations of these waves running it on mixed vectors such as the displacement-traction vector, traction-electric induction vector, the traction-electrical potential vector, not on the displacement-displacement vector or the traction-traction vector as previously. This shows that, not only to Rayleigh waves, the method of first integrals can be applied also to other waves by applying it to equations for mixed vectors. The obtained dispersion equations are the same as the ones previously derived using the method of polarization vector, but the procedure of deriving them is more simple. It is noted that there really exist problems other than the ones mentioned in this paper to which the MFI based mixed vectors can be applicable.

REFERENCES

[1] L. Rayleigh, On waves propagated along the plane surface of an elastic solid, *Proc. R. Soc. Lond. A* **17** (1885) 4-11.

[2] R. Stoneley, Elastic waves at the surface of separation of two solids, *Proc. Roy. Soc. Lond. A.* **106** (1924) 416-428.

[3] S. M. D. Adams, et al., Rayleigh waves guided by topography, *Proc. R. Soc. Lond. A* **463** (2007) 531-550.

[4] J. D. Achenbach, Wave propagation in Elastic Solids, *North-Holland, Amsterdam* (1973).

[5] A. K. Head, The Galois unsolvability of the sextic equation of anisotropic elasticity, *J. Elast.* **9** (1979) 9-20.

[6] P. K. Curie, The secular equation for Rayleigh waves on elastic crystals, *Q. J. Appl. Math. Mech.* **32** (1979) 163-173.

[7] R. M. Taziev, Dispersion relation for acoustic waves in an anisotropic elastic half-space, *Sov. Phys. Acoust.* **35 (5)** (1989) 535-538.

[8] B. Collet, M. Destrade, Explicit secular equation for piezoacoustic surface waves: Shear - horizontal modes, *J. Acoust. Soc. Am.* **116(6)** (2004) 3432-3442.

[9] T. C. T. Ting, Explicit secular equations for surface waves in an anisotropic elastic half-space from Rayleigh to today, Surface waves in anisotropic and laminated bodies and defects detection, *NATO Sci. Ser. II Math. Phys. Chem. 163, Kluwer Acad. Publ. Dordrecht* (2004).

[10] T. C. T. Ting, The polarization vectors at the interface and the secular equation for stoneley waves in monoclinic bimaterials, *Proc. R. Soc. Lond. A.* **461** (2005) 711-731.

[11] T. C. T. Ting, Explicit secular equations for surface waves in monoclinic materials with the symmetry plane at $x_1 = 0$, $x_2 = 0$ or $x_3 = 0$, *Proc. Roy. Soc. Lond. A* **458** (2002) 1017-1031.

[12] T. C. T. Ting, Explicit secular equations for surface waves in an elastic material of general anisotropy, *Q. J. Appl. Math. Mech.* **55** (2002), 297-311.

[13] V. G. Mozhaev, Some new ideas in the theory of surface acoustic waves in anisotropic media, IUTAM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solid Mechanics, edited by D. F. Parker and A. H. England, Kluwer Acad. Publ. (1995) 455-462.

[14] M. Destrade, The explicit secular equation for surface acoustic waves in monoclinic elastic crystals, *J. Acoust. Soc. Am.* **109(4)** (2001) 1398-1402.

[15] T. C.T. Ting, Anisotropic elasticity: theory and applications, *Oxford university press* (1996) 509-511.

[16] M. Destrade, Rayleigh waves in symmetry planes of crystals: explicit secular equations and some explicit wave speeds, *Mech. Mater.* **35** (2003) 931-939.

[17] T. C. T. Ting, Anisotropic elastic constants that are structurally invariant, *Q. J. Mech. Appl. Math.* **53** (2000) 511-523.

[18] V. G. Mozhaev, S. P. Tokmakova, M. Weihnacht, Interface acoustic modes of twisted Si(001) wafers, *J. Appl. Phys.* **83** (1998) 3057-3060.

[19] M. Destrade, Elastic interface acoustic waves in twinned crystals, *Int. J. Solids Struct.* **40** (2003), 7375-7383.

[20] G. W. Farnell and E. L. Adler, in Physical Acoustics, Vol. 9, edited by W. P. Mason and R. N. Thurston, Acad. Press, New York (1972) 35-127.

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PHƯƠNG PHÁP TÍCH PHÂN ĐẦU VÀ SÓNG MẶT

Khi phương trình đặc trưng của sóng Rayleigh (phương trình xác định hệ số tắt dần) là bậc bốn đầy đủ hay có bậc cao hơn, để tìm phương trình tán sắc dạng hiện của nó, thì phương pháp truyền thống không còn hiệu lực. Để vượt qua khó khăn này, phương pháp tích phân đầu (PPTPD) (the method of first integrals) dựa trên các phương trình chuyển động đối với véctơ chuyển dịch hay véctơ ứng suất đã được đưa ra gần đây. Bài báo này khẳng định rằng, không chỉ có hiệu lực đối với sóng Rayleigh, PPTPD còn có thể áp dụng đối với các sóng khác bằng cách thực hiện nó trên các phương trình đối với các véctơ hỗn hợp. Cụ thể: (i) Bằng cách áp dụng PPTPD trên các phương trình đối với véctơ chuyển dịch-ứng suất hay véctơ ứng suất-chuyển dịch, các tác giả đã thu được phương trình tán sắc dạng hiện của sóng Stoneley trong các tinh thể xoắn (ii) Thực hiên PPTPD trên trên các phương trình đối với véctơ ứng suất-cảm ứng điện hay véctơ ứng suất-thế điên đưa đến phương trình tán sắc dạng hiện của sóng SH trong môi trường đàn-điện. Các phương trình tán sắc thu được trùng với các kết quả tìm ra trước đây bằng phương pháp véctơ phân cực, nhưng quá trình tìm ra chúng thì đơn giản hơn.