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FREE VIBRATION OF THICK COMPOSITE PLATES ON NON-HOMOGENEOUS ELASTIC FOUNDATIONS BY DYNAMIC STIFFNESS METHOD

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Abstract. This research presents a new model for vibration analysis of thick laminated plates on a non-homogeneous elastic foundation by the Dynamic Stiffness Method (DSM). The non-homogeneous foundation consists of multi-segment Winkler-type and Pasternak-type elastic foundation. The Dynamic Stiffness Matrices using the First Shear Deformation Theory (FSDT) are constructed for cross-ply thick composite plates. A computer program is written using the present formulation for calculating natural frequencies and harmonic response of composite plates subjected to various types of boundary conditions. Numerical results are validated by comparison with available results in the literature and with Finite Element Method (FEM). Different test cases make evidence the advantages of the present model: higher precision, less data storage, less computing time and studied frequency range extended.

Keywords: Vibration of thick composite plate, continuous element method, dynamic stiffness matrix, Pasternak foundation, non-homogenous foundation.

1. INTRODUCTION

Laminated composite plates on elastic foundations have been widely used in many engineering application such as aerospace, automotive, marine and other structural applications because of advantageous features. Many researches are carried out in order to design safer and more economic thick laminated composite plate structures supported by non-homogenous elastic foundations which represent a vast amount of usages in industry. The elastic foundation can be modeled by different models: Winkler model and two-parameter Pasternak model. If the second parameter vanished, the Pasternak foundation is reduced to the Winkler type. There have been a considerable number of studies on the plates resting on elastic foundation. Thambiratnam [1] solved the vibration of a stepped beam supported on a stepped elastic foundation by a finite-element method. Omurtag [2] investigated the vibration of Kirchoff plates on Winkler and Pasternak foundations using a mix finite element method. Buckling and vibrations of unsymmetric laminates resting on elastic foundations under in-plane and shear forces are studied by Aiello [3] by use of Rayleigh-Ritz method. Recently, Hui-Shen [4] examined the dynamic response of laminated plates under thermo-mechanical loading and resting on a two-parameter elastic foundation by the analytical method based on Reddy's higher order shear deformable plate theory. More advance study on buckling and free vibration analysis of symmetric and anti-symmetric laminated composite plates on an elastic foundation was conducted by Akavci [5] using Navier technique and a new hyperbolic displacement model. Ugurlu et al. [6] investigated the effects of elastic foundation and fluid on the dynamic response characteristics of rectangular Kirchhoff plates using a mixed-type finite element formulation. Malekzadeh [7] analysed the vibration of non-ideal simply supported laminated plates on an elastic foundation subject to in-plane stresses with the Lindstedt–Poincare perturbation technique. The vibration of isotropic Mindlin plate on non-homogeneous Winkler foundation has been considered by Xiang [8] using Levy solutions.

In the effort to further advance finite element technologies, Liu et al. have formulated a cell/element-based smoothed FEM (SFEM or CS-FEM) for mechanics problems [9]. Recently, C. Thai-Hoang et al. [10] have presented an alternative alpha finite element method (AαFEM) using triangular meshes for static, free vibration and buckling analyses of laminated composite plates.

The vibration analysis of structures in medium and high frequency ranges plays an important role in sound transmission, sound isolation problems, fast spinning shafts as well as in the detection of defects by wave propagation or in avoiding possible resonance. Actually, only few approaches such as Statistical Energy Analysis [11] can be used efficiently for high frequencies range but there is no adequate method suitable for predicting the vibration of structure in medium frequencies. Both FEM and Boundary Element Method (BEM) are widely used for analyzing the vibration of structure in low frequencies but they meet difficulty when dealing with the computation in medium and high frequencies. In those frequency ran, a very fine mesh of FEM or BEM model is required which can exceed the storage capacity of computer, reduce the computing time and accumulate the numerical computing errors.

The Dynamic Stiffness Method (DSM) or Continuous Element Method (CEM) [12] has been developed in order to overcome these difficulties of dynamic problems. The CEM is based on the exact closed form solution of the governing differential equations of motion which lead to the dynamic stiffness matrix relating a state vector of loads to the corresponding state vector of displacements at the edges of the structure. By using CE model, one or three continuous elements are enough to compute any range of frequencies with any desired accuracy. In addition, continuous elements can also be assembled together in order to model more complex structures by using the same principle of assembly in FEM. The use of minimum of continuous elements allows a fast acquisition of harmonic response thus it reduces the computing time compared to FEM. Numerous researches have been carried out for DSM of metal and composite beams [13-15] as well as for plate structures [16-18]. CE model for vibration of isotropic shells of revolution and shells subjected to symmetrical load have also deeply been examined [19, 20]. Several industrial computer codes using DSM such as BUNVIS-RG [21], PFVIBAT [22] or ETAPE [20] have been developed which confirmed the performance of CEM.

Recently, our previous study [23] focusing on DSM of composite cylindrical shells has been presented. At the same time, a new study on DSM of ring structures was introduced as well [24]. Concerning composite plates, Boscolo [25-27] has proposed the DSM and the assembly of DSMs for the vibration analysis of composite plates and plates with stiffeners but in those researches only cross-ply composite plates without contact with elastic foundations are investigated.

Despite of abundant researches on CEM for isotropic and anisotropic beam, shell and plate structures, there is a lack of studies on DSM of thick composite plates including both cross-ply and angle-ply laminates on Pasternak foundation or non-homogenous elastic foundation. This paper aims to fill the apparent gap in this area by providing the powerful dynamic stiffness matrices for the vibration of thick cross-ply laminated plates without contact with elastic foundation and plates resting on a Pasternak or a non-homogenous elastic foundation. The effects of shear forces and rotational inertia have also been taken into account. The effects of the foundation stiffness parameter, the foundation length ratio and the plate thickness ratio on the frequency parameters of cross-ply composite plates are examined.

Our study is validated by comparing with different international researches and with FEM (Ansys) solutions. CEM has proved excellent accuracy, especially in the range of medium and high frequencies where FEM and BEM give unreliable solutions due to errors of discretization. Results on natural frequencies and harmonic responses of composite plates without or on non-homogenous elastic foundation make evidence the advantages of the present model: better precision of solution, size of model and computing time reduced. The proposed CEM results serve as a benchmark for FEM and other semi-analytical methods.

2. THEORETICAL FORMULATIONS OF LAMINATE COMPOSITE

Consider a thick composite laminated rectangular plate of dimensions $a \times b$ resting on an elastic foundation as shown in Fig. 1. The plate is supported at two opposite edges $y = 0$ and $y = b$ and the boundary conditions at the two remaining edges can be any combination of free, clamped or supported types. The plate has a uniform thickness h, and in general is made up of a number of laminate layers; each consists of unidirectional fiber reinforced composite material. The fiber angle θ is measured from the x-axis in the counterclockwise direction. Based on the FSDT, displacement field at a point M_0 in the middle plane of the plate is express as [28]

$$u = u_0(x, y) + z\varphi_x(x, y)\ ,\ v = v_0(x, y) + z\varphi_y(x, y),\ w = w_0(x, y), \tag{1}$$

where: u_0, v_0 are the in-plane displacements and w_0 is the transverse displacement of a point (x, y) on the middle plane. φ_x, φ_y are rotations of the normal to the middle plane about y, x axes respectively. The strains are related to the displacements by the following expressions

$$\begin{gathered}\varepsilon_x = \frac{\partial u_0}{\partial u} + z\frac{\partial \varphi_x}{\partial x}, \quad \varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \varphi_y}{\partial y}, \\ \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right), \quad \gamma_{xz} = \frac{\partial w_0}{\partial x} + \varphi_x,\ \gamma_{yz} = \frac{\partial w_0}{\partial y} + \varphi_y\end{gathered} \tag{2}$$

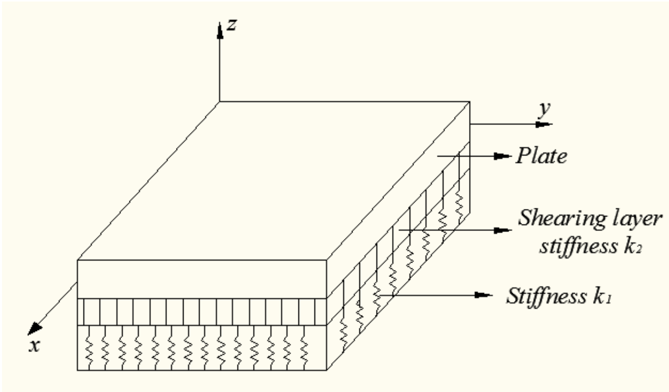


Fig. 1. A composite plate on Pasternak elastic foundation

The stress-strain relationships, accounting for transverse shear deformation, in the plate coordinate for the k^{th} layer can be expressed as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix}_k = \begin{pmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{pmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}_k \tag{3}$$

where Q_{ij} are elastic coefficients in material principal directions [28].

The stress and moment resultants of laminated composite plates can be obtained by integrating (3) over the thickness. The constitutive equations for the composite laminate plate are determined by

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ k_x \\ k_y \\ k_{xy} \end{bmatrix} \quad \begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} A_{55} & 0 \\ 0 & A_{44} \end{bmatrix} \cdot \begin{bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{bmatrix} \tag{4}$$

The laminate stiffness coefficients in above equations are estimated as

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k [z_k - z_{k-1}], B_{ij} = \frac{1}{2}\sum_{k=1}^{N} (\bar{Q}_{ij})_k [z_k^2 - z_{k-1}^2], D_{ij} = \frac{1}{3}\sum_{k=1}^{N} (\bar{Q}_{ij})_k [z_k^3 - z_{k-1}^3], (i, j = 1, 2, 6)$$

$$A_{ij} = K \sum_{k=1}^{N} (\bar{Q}_{ij})_k [z_k - z_{k-1}] (i, j = 4, 5) \tag{5}$$

with $K = 5/6$: the shear correction factor, N: number of layers, z_{k-1}, z_k: the coordinates of the top and bottom faces of the k^{th} layer.

3. DIFFERENTIAL EQUATIONS OF COMPOSITE PLATE ON PASTERNAK FOUNDATION

3.1. Force-strain relationship of cross-ply composite plates

This research investigates cross-ply composite plates only. For general cross-ply composite laminated plates, forces and moment resultants are determined by [28]

$$
\begin{aligned}
&N_{xx} = A_{11}\frac{\partial u_0}{\partial x} + A_{12}\frac{\partial v_0}{\partial y} + B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y}, M_{yy} = B_{12}\frac{\partial u_0}{\partial x} + B_{22}\frac{\partial v_0}{\partial y} + D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y},\\
&N_{yy} = A_{12}\frac{\partial u_0}{\partial x} + A_{22}\frac{\partial v_0}{\partial y} + B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y}, M_{xy} = B_{66}(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}) + D_{66}(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y}),\\
&N_{xy} = A_{66}(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}) + B_{66}(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y}), Q_x = A_{55}(\phi_x + \frac{\partial w_0}{\partial x}),\\
&M_{xx} = B_{11}\frac{\partial u_0}{\partial x} + B_{12}\frac{\partial v_0}{\partial y} + D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}, Q_y = A_{44}(\phi_y + \frac{\partial w_0}{\partial y}),
\end{aligned}
\tag{6}
$$

3.2. Equation of motion of composite plate on Pasternak foundation

The governing equations of equilibrium can be derived using the principle of virtual displacements. The equation of motion for cross-ply composite plate resting on Pasternak foundation using the FSDT is written by

$$
\begin{aligned}
&\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \varphi_x}{\partial t^2}\\
&\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \varphi_y}{\partial t^2}\\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k_1 w_0 + k_2\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) = I_0\frac{\partial^2 w_0}{\partial t^2}\\
&\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\frac{\partial^2 \varphi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2}\\
&\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2\frac{\partial^2 \varphi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2}
\end{aligned}
\tag{7}
$$

where: $I_i = \sum\limits_{k=1}^{N}\int\limits_{z_k}^{z_{k+1}} \rho^{(k)} z^i dz, (i = 0, 1, 2), \rho^{(k)}$ is the material mass density of the k^{th} layer, k_1: linear stiffness of foundation, k_2: the shear modulus of the sub-grade .

Here, the Winkler foundation will be obtained by setting k_2 to zero, otherwise the Pasternak foundation corresponds to the case where $k_1 \neq 0$ and $k_2 \neq 0$.

4. DYNAMIC STIFFNESS MATRICES FOR CROSS-PLY COMPOSITE PLATES ON A PASTERNAK FOUNDATION

The DSM for composite plate will be derived from the exact closed form solutions of the Eqs. (7) and (8). Following the procedure in [19], [23], a state-vector of solution must be chosen. Then, all variables will be expressed in series of Levy in order to obtain the derivation of the state-vector with respect to x. The DSM will be easily obtained from this derivation after some matrix manipulations.

4.1. Strong formulation of cross-ply composite plate on Pasternak foundation

For natural vibration mode of the cross-ply plates, displacements, forces and moment resultants can be expressed by series of Levy as

$$\begin{aligned}
&u_0(x,y,t)=\sum_{m=1}^{\infty}u_m^C(x)\sin\alpha y.e^{i\omega t},\ N_{xx}(x,y,t)=\sum_{m=1}^{\infty}N_{xm}^C(x)\sin\alpha y.e^{i\omega t}\\
&v_0(x,y,t)=\sum_{m=1}^{\infty}v_m^C(x)\cos\alpha y.e^{i\omega t},\ N_{xy}(x,y,t)=\sum_{m=1}^{\infty}N_{xym}^C(x)\cos\alpha y.e^{i\omega t}\\
&w_0(x,y,t)=\sum_{m=1}^{\infty}w_m^C(x)\sin\alpha y.e^{i\omega t},\ Q_x(x,y,t)=\sum_{m=1}^{\infty}Q_{xm}^C(x)\sin\alpha y.e^{i\omega t}\\
&\phi_x(x,y,t)=\sum_{m=1}^{\infty}\phi_{xm}^C(x)\sin\alpha y.e^{i\omega t},\ M_{xx}(x,y,t)=\sum_{m=1}^{\infty}M_{xm}^C(x)\sin\alpha y.e^{i\omega t}\\
&\phi_y(x,y,t)=\sum_{m=1}^{\infty}\phi_{ym}^C(x)\cos\alpha y.e^{i\omega t},\ M_{xy}(x,y,t)=\sum_{m=1}^{\infty}M_{xym}^C(x)\cos\alpha y.e^{i\omega t}
\end{aligned}\tag{8}$$

where: ω is the circular frequency; the superscript " $-^C$ " denotes the cross-ply laminates and $\alpha=\dfrac{m.\pi}{b}$. In this case, the vector $\mathbf{y}_m^C=\{u_m^C, v_m^C, w_m^C, \varphi_{xm}^C, \varphi_{ym}^C, N_{xm}^C, N_{xym,}^C Q_{xm}^C, M_{xm}^C, M_{xym}^C\}^T$ is called state vector for cross-ply composite plate on Pasternak foundation. It is important to note that the components N_{yy}(x,y,t), M_{yy}(x,y,t), and Q_y(x,y,t) are calculated from other components of $\mathbf{y}_m^C$ and the chosen vector $\mathbf{y}_m^C$ is sufficient for determining the static and dynamic behaviors of composite plates.

Next, the derivation of state vector of solutions with respect to x will be evaluated from Eqs. (6), (7) and (8) as follows

$$\begin{aligned}
&\frac{du_m^C}{dx}=f_1^C\left(v_m^C,\varphi_{ym}^C,N_{xm}^C,M_{xym}^C\right), && \frac{dv_m^C}{dx}=f_2^C\left(u_m^C,N_{xym}^C,M_{xym}^C\right),\\
&\frac{dw_m^C}{dx}=f_3^C\left(Q_{xm}^C,\varphi_{xm}^C\right), && \frac{d\varphi_{xm}^C}{dx}=f_4^C\left(v_m^C,\varphi_{ym}^C,N_{xm}^C,M_{xm}^C\right),\\
&\frac{d\varphi_{ym}^C}{dx}=f_5^C\left(\varphi_{xm}^C,N_{xym}^C,M_{xym}^C\right), && \frac{dN_{xm}^C}{dx}=f_6^C\left(\omega,u_m^C,N_{xym}^C\right),\\
&\frac{dN_{xym}^C}{dx}=f_7^C\left(\omega,v_m^C,\ \varphi_{ym}^C,N_{xm}^C,M_{xm}^C\right), && \frac{dQ_{xm}^C}{dx}=f_8^C\left(\omega,v_m^C,w_m^C,\varphi_{ym}^C,N_{xm}^C,M_{xm}^C,k_1,k_2\right),\\
&\frac{dM_{xm}^C}{dx}=f_9^C\left(\omega,\varphi_{xm}^C,M_{xym}^C,Q_{xm}^C\right), && \frac{dM_{xym}^C}{dx}=f_{10}^C\left(\omega,v_m^C,w_m^C,\varphi_{ym}^C,N_{xm}^C,M_{xm}^C\right)
\end{aligned}\tag{9}$$

where functions $f_i^C(i=1\div 10)$ are found in Appendix A.

Eqs. (9) can be written in the matrix form for each vibration mode m as

$$\frac{d\mathbf{y}_m^C}{dx}=\mathbf{A}_m^C\left(\omega,k_1,k_2\right)\mathbf{y}_m^C\tag{10}$$

4.2. Dynamic stiffness matrix for composite plate on Pasternak foundation

The matrices $\mathbf{A}_m^C\left(\omega,k_1,k_2\right)$ in (10) for cross-ply plate on Pasternak foundation is presented in Appendix B. Next, the dynamic transfer matrix $\mathbf{T}_m\left(\omega,k_1,k_2\right)$ is calculated by

$$\mathbf{T}_m(\omega, k_1, k_2) = e^{\mathbf{A}_m^C(\omega,k_1,k_2)a} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix}_m \tag{11}$$

Therefore, the dynamic stiffness matrix $\mathbf{K}(\omega)_m$ is determined by [23]

$$\mathbf{K_m}(\omega, k_1, k_2) = \begin{bmatrix} \mathbf{T}_{12}^{-1}\mathbf{T}_{11} & -\mathbf{T}_{12}^{-1} \\ \mathbf{T}_{21} - \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11} & \mathbf{T}_{22}\mathbf{T}_{12}^{-1} \end{bmatrix}_m \tag{12}$$

By applying boundary conditions on the Dynamics Stiffness Matrix, natural frequencies of the structure will be easily determined.

For example,

- Free-free boundary condition: $det(\mathbf{K}) = 0$.
- Clamped-clamped boundary condition: $det(\mathbf{T}_{12}) = 0$.

A computer program based on Matlab code has been written in order to compute the DSM for composite plates on elastic foundation. It is important to note that these DSMs can be used for vibration analysis of composite plates without contact with elastic foundation (by setting $k_1 = k_2 = 0$) as well as for composite plates resting on Winkler foundation ($k_1 \neq 0, k_2 = 0$) and plates on Pasternak foundation ($k_1 \neq 0, k_2 \neq 0$).

5. CEM FOR VIBRATION ANALYSIS OF COMPOSITE PLATE ON NON-HOMOGENOUS FOUNDATIONS

In this study, the solution for a composite plate resting on a non-homogenous foundation is established by using the DSM of the composite plate on a Pasternak foundation. First, the procedure of assembly of DSM will be shortly presented. Then it will be used to model and solve the problem of composite plates on non-homogenous foundations.

5.1. Assembly of dynamic stiffness matrices

The dynamic stiffness matrix constitutes a continuous element of composite plates on elastic foundation. Such element can be easily assembled with other continuous elements in order to model a long plate structures, plates made off sections with different properties or to avoid the numerical instability problem due to the limit length of the element [18, 19, 23]. The presented assembly procedure takes the same principle as those of FEM which is very easy to be implemented.

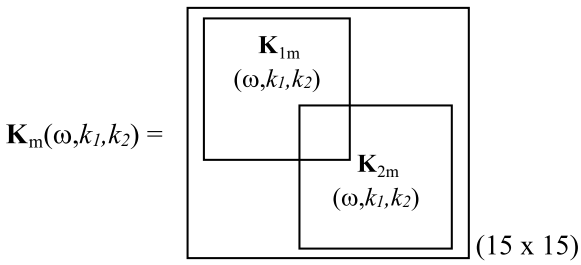


Fig. 2. Assembly of two continuous elements of plates on Pasternak foundation

The construction of the DSM $\mathbf{K}_m(\omega,k_1,k_2)$ of the whole structure from dynamics stiffness matrices of two continuous elements $\mathbf{K}_{1m}(\omega,k_1,k_2)$ and $\mathbf{K}_{2m}(\omega,k_1,k_2)$ assembled along a common edge is shown in Fig. 2.

5.2. DSM for composite plate on non-homogenous foundations

Fig. 3 illustrates a thick composite plate resting on a non-homogenous foundation made by N different sections having different values of $a_i, k_1^{(i)}$ and $k_2^{(i)} (i = 1 \div N)$ representing all types of foundations: no foundation ($k_1^{(i)} = k_2^{(i)} = 0$), Winkler foundation ($k_1^{(i)} \neq 0, k_2^{(i)} = 0$) and Pasternak foundation ($k_1^{(i)} \neq 0, k_2^{(i)} \neq 0$).

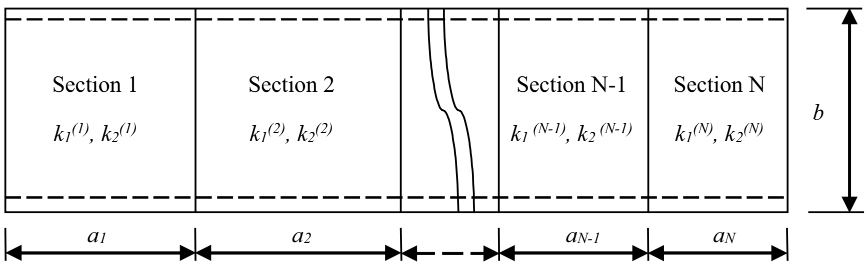


Fig. 3. A composite plate on a non-homogeneous foundation

For this complex structure, it is necessary to reconstruct another system of equations if the analytical methods are used. Moreover, the resolution by FEM meets difficulty either in modeling of the structure with complex property or in meshing of the structure because it needs a large number of elements.

In this case, the CEM demonstrates considerable advantages vis-à-vis other analytical or FE methods. The vibration analysis of the investigated structure is easy to carry out by an assembly of only N continuous element (or a little more to overcome the numerical errors of computation) having different properties of $k_1^{(i)}$ and $k_2^{(i)}$. By using both the exact closed form solution of differential equations and a minimum of elements for meshing, CE model not only saves the computing data storage and the calculating time but also provides high precision results in all low, medium and high range of studied frequencies. These features are the mains advantages of DSM compared to FEM or BEM which give unreliable responses of structure in medium and high frequencies even though an important number of meshing elements is used.

Let's examine an example of a composite plate composed by three sections resting on three different elastic foundations, i.e. the first section has no foundation ($k_1^{(1)} = k_2^{(1)}$=0), the second one rests on a Winkler foundation ($k_1^{(2)}$ ≠0, $k_2^{(2)}$=0) and the last one is in contact with a Pasternak foundation ($k_1^{(3)}$ ≠0, $k_2^{(3)}$ ≠0) (see Fig. 4).

The CEM solution for this problem is easy constructed. The first step consists to calculate separately tree dynamic stiffness matrices denoted $\mathbf{K}_{1m}(\omega), \mathbf{K}_{2m}(\omega, k_1^{(2)})$ and $\mathbf{K}_{3m}(\omega, k_1^{(3)}, k_2^{(3)})$ for tree sections having different properties of lengths and foundation's stiffness.

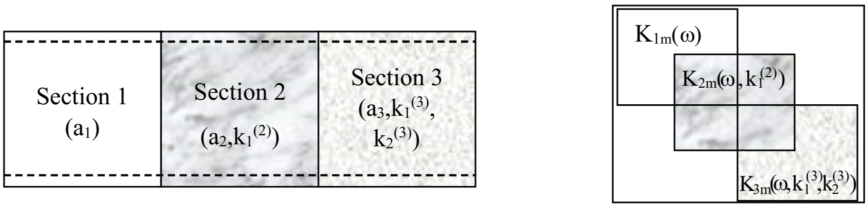


Fig. 4. A composite plate composed by tree sections on non-homogeneous and the construction of the DSM $\mathbf{K}_m(\omega, k_1^{(2)}, k_2^{(3)}, k_3^{(3)})$ for the whole plate elastic foundation

Secondly, the dynamic stiffness matrix of whole composite plate $\mathbf{K}_m(\omega, k_1^{(2)}, k_2^{(3)}, k_3^{(3)})$ is calculated by the assembly of above tree DSM using the mentioned procedure as in Fig. 4. The natural frequencies and harmonic responses of the composite plate on non-homogenous foundation will be computed from $\mathbf{K}_m(\omega, k_1^{(2)}, k_2^{(3)}, k_3^{(3)})$.

By this way, CE model allows using a minimum of elements in order to model the structure and avoiding discretization errors occurring in FEM and in BEM when the number of meshing elements becomes important. Thus CEM saves the data storage of computers and accelerates the computing time. In addition, the DSM gives more exact results which are valid in all low, medium or high frequency range.

5.3. Harmonic response and natural frequencies detection

As known, the special formulation of the CE model allows a fast and direct computation of structure response with a given excitation [23, 29]. The composite plate is subjected to a unity point load at the point M located in the middle of the edge $x = a$. The harmonic response will be extracted from the same point M (see Fig. 5). Harmonic responses of various types of boundary conditions will be examined.

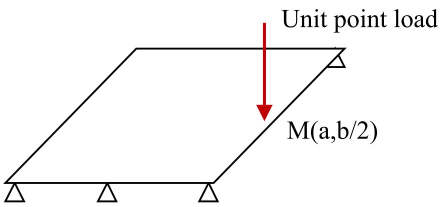


Fig. 5. Types of load and response point M of composite plates on elastic foundation

The detection of natural frequencies of the structure from the DSM is generally carried out by the Williams-Wittrick algorithm [16, 17]. Nevertheless, it is more interesting to obtain directly natural frequencies profiting the plotted harmonic response [23, 29]. As discussed in our previous research on DSM for composite cylindrical shells [23], the speed of computation by Williams-Wittrick method is rather slow due to the complicated interaction of bending-tension-twisting between composite layers. Thus, once the harmonic response curve has been drawn, the natural frequencies will correspond to the peaks of this diagram. This method is easily and rapidly applied and can reach any desired precision [19, 23, 29].

6. NUMERICAL RESULTS AND DISCUSSION

In this study, free vibration analysis of symmetric, anti-symmetric cross-ply laminated composite plates on elastic foundation by our model is investigated. Comparisons are made with available solutions in literature and with FEM results (Ansys) using SHELL99 elements.

6.1. Free vibration of laminated composite plates on elastic foundation

Example 1. Vibration of cross-ply composite plates on elastic foundation

The first step of our research is to validate the present formulation for composite plates on elastic foundation including Winkler and Pasternak foundations. Here, the frequencies of a cross-ply simply supported composite square plate on elastic foundation are compared with results of Hui-Shen [4] and Akavci [5] in Tab. 1. The material properties for the considered layers are $E_1/E_2 = 40, G_{12}/E_2 = 0.6, G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, \upsilon_{12} = 0.25$. The following dimensionless foundation parameters, r_1 and r_2 are used in this study: $r_1 = \dfrac{k_1 a^4}{E_2 h^3}$; $r_2 = \dfrac{k_2 a^2}{E_2 h^3}$.

Table 1. Fundamental frequency parameter $\Omega = (\omega \times a^2/h)\sqrt{\rho/E_2}$ for cross-ply $(0°/90°/0°)$ simply supported composite plates on elastic foundations $(a/b = 1, E_1/E_2 = 40, G_{12}/E_2 = 0.6, G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, \upsilon_{12} = 0.25)$

r_1	r_2	Source	a/h			
			5	10	20	50
0	0	[4]	10.263	14.702	17.483	18.649
		[5]	10.265	14.700	17.481	18.640
		Present	10.288	14.764	17.527	18.658
100	0	[4]	14.244	17.753	20.132	21.152
		[5]	14.246	17.751	20.131	21.152
		Present	14.262	17.803	20.167	21.158
100	10	[4]	19.879	22.596	24.536	25.390
		[5]	19.880	22.595	24.535	25.390
		Present	19.890	22.638	24.556	25.396

Tab. 1 demonstrates the comparison of the present results with respect to other shear deformation theories for symmetrically laminated $(0°/90°/0°)$ cross-ply square plate on elastic foundation, with varying a/h ratios. It can be seen from the Tab. 4 that, the CEM yields results very close to those of Hui-Shen [4] and Akavci [5] (based on higher order shear deformation theories) for various values of aspect ratios a/h, even with the presence of the elastic foundation.

From Tab. 1, it can be seen that the natural frequencies increase when the thickness and the elastic parameters increase. These results confirm that the developed continuous elements are exact and they can efficiently be used to analyse the vibration of composite plates on elastic foundations.

Example 2. Vibration of anti-symmetric cross-ply plate on elastic foundation

The comparison of natural frequencies computed by CEM and by FEM (Ansys) for an anti-symmetric cross-ply plate resting on Winkler foundation is conducted here. An analysis on the convergence of FEM is done beforehand for avoiding the errors relating to the size of mesh. Different tests of meshing show that a mesh of 576 (24×24) SHEL99 elements is good enough for this study and a finer mesh does almost not improve the results. The first five frequency parameters for angle-ply SFSF composite plates with different composition of layers calculated by CEM and by FEM (576 elements) are presented in Tab. 2.

Table 2. Natural frequency (Hz) for cross-ply SFSF composite plates
($a/h = 10; a = b = 1$ m; $E_2 = 6.92 \times 10^9; E_1 = 276.8 \times 10^9, G_{12} = 4.152 \times 10^9,$
$G_{13} = 4.152 \times 10^9, G_{23} = 3.46 \times 10^9, \upsilon_{12} = 0.25$)

Foundation stiffness	Layer	Source	Mode sequences(Hz)				
			1	2	3	4	5
$k_1 = 100, k_2 = 0$	$[0/90]_1$	CEM	239.0831	347.0449	410.1248	526.4210	753.8996
		FEM	240.0330	349.3917	412.0975	532.1768	764.7990
		Error(%)	0.3973	0.6762	0.4810	1.0934	1.4457
	$[0/90]_2$	CEM	348.2034	448.8670	493.0010	687.7581	782.8677
		FEM	349.2692	450.8496	495.4371	693.5769	782.8975
		Error(%)	0.3060	0.4417	0.4941	0.8460	0.0038
	$[0/90]_3$	CEM	362.1877	461.7689	505.5819	709.7887	782.8909
		FEM	363.2468	463.6589	508.0908	715.5280	782.9107
		Error(%)	0.2924	0.4093	0.4962	0.8086	0.0025

The small errors (<1.5%) between two models with various numbers of layers represent the excellent exactness of our formulation. In conclusion, presented continuous elements are validated for cross-ply anti-symmetric composite plates and in all low, medium and high frequency ranges.

6.2. Harmonic response of laminated composite plates on elastic foundation

As mentioned in [23], the direct and fast comparison of harmonic response curves drawn by CEM and those obtained by experiment devices is an incontestable advantage of DSM compared to other approaches. Now a study on harmonic response of cross-ply (0°/90°/0°/90°) composite plates on Winkler foundation calculated by FEM and by CEM is carried out in order to emphasize the advantages of DSM in terms of the computing time and the precision of results in medium and high frequencies

The investigated plate has the following properties: $E_1 = 40E_2, E_2 = 6.92$ GPa, $G_{12} = G_{13} = 0.6E_2, G_{23} = 0.5E_{2,}\upsilon_{12} = 0.25, \rho = 1600$ kg/m^3).

The harmonic response curves obtained with 3 continuous elements are used for comparing with curves obtained with 100 (10×10) and with 576 (24×24) plate finite elements (see Fig. 6). Fig. 6 demonstrates that in low frequencies, the three curves coincide totally up to 2835 Hz. Beyond this limit, an important discrepancy is noticed between CE and 10×10 FEM curve because the meshing in FE model is not fine enough. Effectively a mesh with more elements (24×24) gives a better result when FE curve converges towards CE harmonic response. Nevertheless, differences between these two curves still occur from 7083 Hz which confirms the difficulty of FEM in dynamic analysis, especially in medium and high frequencies. Meanwhile, by using analytical solution, only 3 continuous elements are sufficient to determine with high precision all frequencies in the studied range (0-1200 Hz).

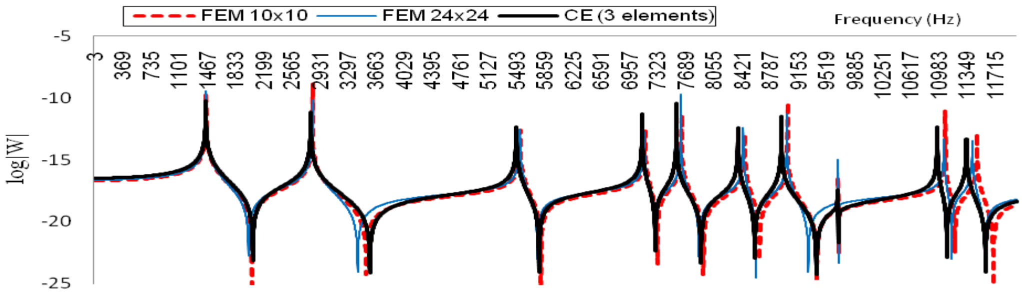


Fig. 6. Comparison of harmonic responses by different methods for a SCSF composite cross-ply (0°/90°/0°/90°) plate ($a = b = 0.254$ m, $h/b = 0.1, k_1 = 15\times10^4$ N/m^3)

Fig. 6 illustrates the excellent convergence of CE and FE models when increasing the number of elements from 100 (10×10) to 576 (24×24) for a SCSF composite cross-ply (0°/90°/0°/90°) plate. These graphics prove that the accuracy of FEM decreases for higher natural frequencies but CE model works well in all frequency range.

7. VIBRATION OF LAMINATED COMPOSITE PLATES ON NON-HOMOGENEOUS ELASTIC FOUNDATION

In this section, the presented model is applied to obtain solutions for vibration of rectangular composite plates resting on non-homogeneous elastic foundation that consists of multi-segment Winkler and Pasternak-type of elastic foundations.

Harmonic responses computed by different methods for a SFSF composite cross-ply (0°/90°/0°/90°) plate composed by two sections resting on a non-homogenous foundation are presented in Fig. 7.

All remarks in Section 6.2 are also valid for composite plates resting on non-homogenous elastic foundation. CE and FE models provide the same results in low frequency range, i.e. from 0 to 1380.8 Hz for cross-ply SFSF plate. Important discrepancies

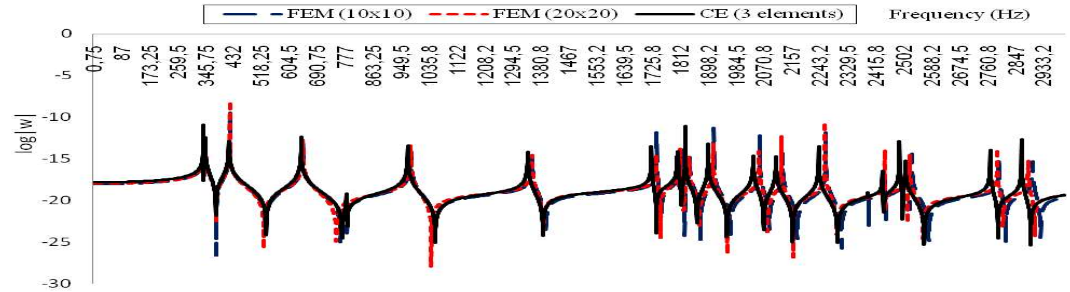


Fig. 7. Harmonic responses by different methods for a SFSF composite cross-ply ($0°/90°/0°/90°$) plate composed by two sections resting on a non-homogenous foundation ($a_1 = a_2 = b = 1$ m, $h/b = 0.1, k_1^{(1)} = 15 \times 10t^4$ N/m^3, $k_2^{(1)} = k_1^{(2)} = k_2^{(2)} = 0$)

are revealed between three harmonic responses (frequencies greater than 1380.8 Hz) which encore confirms the difficulty of FEM in medium and high frequency analysis.

It is clear that the present model based on exact closed form solutions of the problem allows obtaining high precision results in all low, medium and high frequency ranges. In addition, by using a minimum number of continuous elements, the DSM saves the data storage volume and thus economizes the computing time [23].

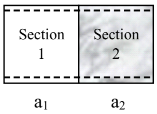
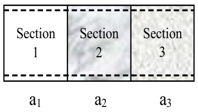
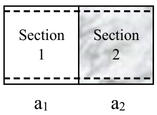
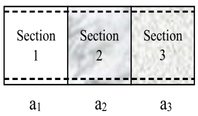
Concerning the comparison of calculating time, the same computer (CPU Q6600, RAM 4GB, 1GB Graphic Card) is adopted for all computations of harmonic responses. To obtain harmonic responses in Fig. 7, the 10×10 elements FE model needed 3 times of the CEM computing time whereas the 24×24 FE model required 8 times compared to the calculating time of DSM. These data illustrate the strength of our continuous elements and of our assembly procedure for studying vibration of composite plate on non-homogenous foundations.

8. EFFECTS OF ELASTIC FOUNDATIONS AND VARIOUS BOUNDARY CONDITIONS ON FREE VIBRATION OF COMPOSITE PLATES ON NON-HOMOGENOUS FOUNDATIONS

Tab. 3 presents 6 first frequencies for cross-ply, $[0°/90°]_2$, $[0°/90°]_S$, simply supported composite plates resting on multi-segment Winkler-type and Pasternak-type elastic foundations. The foundation segments are of equal length and aspect ratios of the segment are set to be $a_i/b = 1 (i = 1, 2, 3)$. The plate thickness ratio b/h is fixed at 10. Stiffness parameters are $(r_1, r_2) = (100, 10)$ for a Pasternak-type elastic foundation and $(r_1, r_2) = (100, 0)$ for a Winkler elastic foundation, and $(r_1, r_2) = (0, 0)$ for the foundationless plate.

Fundamental frequency parameter $\Omega = (\omega \times a^2/h)\sqrt{\rho/E_2}$ for cross-ply composite plates resting on non-homogenous foundation with different boundary conditions is computed and presented in Tab. 4. Two x-parallel edges of the plate are assumed to be simply supported and the two remaining edges may have any combinations of free, simply supported or clamped conditions.

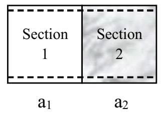
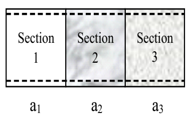
Table 3. Frequency parameter $\Omega = (\omega \times a^2/h)\sqrt{\rho/E_2}$ for cross-ply simply supported composite plates $(a_1/b = a_2/b = a_3/b = 1, b/h = 10, E_1/E_2 = 40, G_{12}/E_2 = 0.6, G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, \upsilon_{12} = 0.25)$

Case	Layer	Foundation stiffness	Mode sequences					
			1	2	3	4	5	6
	$[0°/90°]_2$	$(r_1^{(1)} = 0$	13.747	23.717	28.460	32.871	37.403	38.219
		$r_2^{(1)} = 0$	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(1, 4)	(2, 2)
	$[0°/90°]_S$	$r_1^{(2)} = 500$	12.538	23.958	25.620	30.092	32.811	37.705
		$r_2^{(2)} = 0)$	(1, 1)	(1, 2)	(2, 1)	(1, 3)	(2, 2)	(2, 3)
	$[0°/90°]_2$	$(r_1^{(1)}=0$	16.012	21.148	27.130	32.206	39.608	42.962
		$r_2^{(1)} = 0$	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 1)
		$r_1^{(2)} = 100$						
	$[0°/90°]_S$	$r_2^{(2)} = 0$	15.106	21.934	28.339	32.961	34.472	40.092
		$r_1^{(3)} = 100$	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(1, 4)	(2, 2)
		$r_2^{(3)}=0)$						
	$[0°/90°]_2$	$(r_1^{(1)} = 0$	14.260	26.949	33.113	34.865	40.364	41.300
		$r_2^{(1)} = 0$	(1, 1)	(1, 2)	(2, 1)	(1, 3)	(2, 2)	(1, 4)
	$[0°/90°]_S$	$r_1^{(2)} = 1000$	13.414	13.414	28.399	35.409	36.285	41.934
		$r_2^{(2)} = 0)$	(1, 1)	(2, 1)	(1, 2)	(1, 3)	(2, 2)	(2, 3)
	$[0°/90°]_2$	$(r_1^{(1)}=0$	16.133	21.601	29.004	35.167	41.240	49.850
		$r_2^{(1)} = 0$	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		$r_1^{(2)} = 100$						
	$[0°/90°]_S$	$r_2^{(2)} = 0$	15.348	22.750	30.756	36.829	40.847	44.231
		$r_1^{(3)} = 100$	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(2, 1)	(1, 5)
		$r_2^{(3)}=50)$						

The values in brackets (m, n) denote numbers of halfwaves in the y-direction and x-direction respectively.

The presented assembly procedure of plate continuous elements represents a powerful approach with high precision of results, simplicity of application and suitable for medium and high frequencies for studying composite plates without or on non-homogenous elastic foundation. The exact frequency parameters in Tabs. 3 and 4 are very valuable as benchmarks for verifying approximate numerical solutions for such a composite plate vibration problem.

Table 4. Fundamental frequency parameter $\Omega = (\omega \times a^2/h)\sqrt{\rho/E_2}$ for cross-ply composite plates on non-homogenous foundation with different boundary conditions. $(a_1/b = a_2/b = a_3/b = 1, b/h = 10, E_1/E_2 = 40, G_{12}/E_2 = 0.6, G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, \upsilon_{12} = 0.25)$

Case	Layer	Foundation stiffness	Boundary conditions					
			SS	FF	SF	CF	SC	CC
	$[0°/90°]_2$	$(r_1^{(1)} = 0$ $r_2^{(1)} = 0$	12.357	10.514	12.304	14.683	12.440	13.414
	$[0°/90°]_S$	$r_1^{(2)} = 100$ $r_2^{(2)} = 0)$	10.091	7.0244	10.083	11.194	10.400	11.760
	$[0°/90°]_2$	$(r_1^{(1)} = r_2^{(1)} = 0$ $r_1^{(2)} = 0$ $r_2^{(2)} = 0$	15.710	13.595	15.702	15.702	15.725	16.111
	$[0°/90°]_S$	$r_1^{(3)} = 100$ $r_2^{(3)} = 10)$	14.411	11.783	14.366	14.774	14.569	15.091

9. CONCLUSIONS

This paper has succeeded in constructing new powerful continuous elements based on the Dynamic Stiffness Method for analyzing vibration of general cross-ply plates without or on non-homogeneous foundation that consists of multi-segment Winkler-type and Pasternak-type elastic foundations. The effects of shear force and rotational inertia has been taken into account in the models. These composite plate continuous elements complete the library of existing continuous elements for isotropic and anisotropic beam, shell and plate structures.

An interesting and strong procedure of assembly of continuous elements has also been presented which has then successfully been applied to solve the vibration problem of composite plates on non-homogenous elastic foundation.

Obtained results using the present formulation have been evaluated against those available in the literature and those from finite element method and excellent agreements have been found. It is concluded, through the numerical examples, that the present Continuous Element based on the First Shear Deformation Theory can provide accurate solutions for the vibration analysis of general cross-ply laminated thick composite plates resting on elastic foundation. Results on natural frequencies and harmonic responses of composite plates without or on non-homogenous elastic foundation make evidence the advantages of the present model: better precision of solution, size of model and computing time reduced, efficient for analyzing medium and high frequencies.

The exact vibration solutions of composite plates on non-homogenous foundation presented in this paper should be useful in providing benchmark values for testing the accuracy of numerical results for this class of problem. The results are also important for engineers to design laminated composite plates supported by non-homogenous elastic foundations.

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REFERENCES

[1] Thambiratnam D, Zhuge Y. Free vibration analysis of beams on elastic foundation, *Computer and Structure*, **60**, (1996), pp. 971–980.

[2] Omurtag MH, Kadioglu F, Free vibration analysis of orthotropic plates resting on Pasternak foundation by mixed finite element formulation, *Comput Struct*, **67**, (1998), pp. 253–265.

[3] M.A. Aiello, L. Ombres, Buckling and vibrations of unsymmetric laminates resting on elastic foundations under in-plane and shear forces, *Composite structure*, **44**, (1999), pp. 31–41.

[4] Hui-Shen Shen, J.-J. Zheng, X.-L. Huang, Dynamic response of shear deformable laminated plates under thermomechanical loading and resting on elastic foundations, *Composite structure*, **60**, (2003), pp. 57–66.

[5] S.S. Akavci, Buckling and Free Vibration Analysis of Symmetric and Antisymmetric Laminated Composite Plates on an Elastic Foundation, *Journal of Reinforced Plastics and Composites*, **26**, (2007), pp. 1907–1919.

[6] B. Ugurlu, A. Kutlu, A. Ergina, M.H. Omurtag, Dynamics of a rectangular plate resting on an elastic foundation and partially in contact with a quiescent fluid, *Journal of Sound and Vibration*, **317**, (2008), pp. 308–328.

[7] K. Malekzadeh, S.M.R. Khalili, P. Abbaspour, Vibration of non-ideal simply supported laminated plate on an elastic foundation subjected to in-plane stresses, *Composite Structure*, **92**, (2010), pp. 1478–1484.

[8] Y. Xiang, Vibration of rectangular Mindlin plates resting on non-homogenous elastic foundations, *International Journal of Mechanical Sciences*, **45**, (2003), pp. 1229–1244.

[9] G.R. Liu, K.Y. Dai, T.T. Nguyen, A smoothed finite element method for mechanics problems, *Computational Mechanics*, **39**, (2007), pp. 859–877.

[10] C. Thai-Hoang, N. Nguyen-Thanh, H. Nguyen-Xuan, T. Rabczuk, An alternative alpha finite element method with discrete shear gap technique for analysis of laminated composite plates, *Applied Mathematics and Computation*, **217**, (2011), pp. 7324–7348.

[11] Lyon RH, Dejong RG, *Theory and application of Statistical Energy Analysis*, 2nd ed. Butterworth-Heinemann: Boston, (1995).

[12] Leung AYT., *Dynamic stiffness and substructures*, New-York: Springer; (1993).

[13] Banerjee JR, Sobey AJ, Dynamic stiffness formulation and free vibration analysis of a three-layered sandwich beam, *Int J Solids Struct*, **42**, (2005), pp. 2181–2197.

[14] Banerjee JR, Williams FW, Coupled bending-torsional dynamic stiffness matrix for Timoshenko beam elements, *Comput Struct*, **42**, (1992), pp. 301–310.

[15] Howson WP, Zare A, Exact dynamic stiffness matrix for flexural vibration of three-layered sandwich beams. *J Sound Vib*, **282**, (2005), pp. 753–767.

[16] W. H. Wittrick, F. W. Williams, Buckling and vibration of anisotropic or isotropic plate assemblies under combined loadings, *International Journal of Mechanical Sciences*, **16** (4), (1974), pp. 209–239.

[17] W. H. Wittrick, A unified approach to initial buckling of stiffened panels, *Journal of Numerical Methods in Engineering*, **11**, (1968), pp. 1067–1081.

[18] Nguyen M.C., *Eléments Continus de plaques et coques avec prise en compte du cisaillement transverse*, Application à l'interaction fluide-structure. PhD Thesis, University Paris 6, (2003).

[19] Casimir JB, Nguyen MC, Tawfiq I, Thick shells of revolution: derivation of the dynamic stiffness matrix of continuous elements and application to a tested cylinder, *Comput Struct*, **85**, (2007), pp. 1845–1857.

[20] M. A. Khadimallah, J. B. Casimir, M. Chafra, H. Smaoui, Dynamic stiffness matrix of an axisymmetric shell and response to harmonic distributed loads. *Computers and Structures*, **89** (5), (2011), pp. 467–475.

[21] Anderson MS, Williams FW, Banerjee JR, Durling BJ, Herstorm CL, KennedyD, et al., *User manual for BUNVIS-RG: an exact buckling and vibration program for lattice structures with repetitive geometry and substructuring options*, NASA Technical Memorandum 87669, (1986).

[22] Akeson BA, PFVIBAT, A computer program for plane frame vibration analysis by an exact method, *Int J Numer Methods Eng*, **10**, (1976), pp. 1221–1231.

[23] Tran Ich Thinh, M.C. Nguyen, Dynamic stiffness matrix of continuous element for vibration of thick cross-ply laminated composite cylindrical shells, *Composite Structures*, **98**, (2013), pp. 93–102.

[24] D. Tounsi, J. B. Casimir, M. Haddar, Dynamic stiffness formulation for circular rings, *Computers and Structures*, **112-113**, (2012), pp. 258–265.

[25] M. Boscolo, J.R. Banerjee, Dynamic stiffness elements and their applications for plates using first order shear deformation theory, *Journal Computers and Structures*, **89** (3-4), (2011), pp. 395–410.

[26] M. Boscolo, R. J. Banerjee, Dynamic stiffness formulation for composite Mindlin plates for exact modal analysis of structures. Part I: Theory, *Computers and Structures*, **96-97**, (2012), pp. 61–73.

[27] M. Boscolo, R. J, Banerjee, Dynamic stiffness formulation for composite Mindlin plates for exact modal analysis of structures. Part II: Results and applications, *Computers and Structures*, **96-97**, (2012), pp. 74–83.

[28] Reddy J.N, *Mechanics of laminated composite plates and shells*, Theory and Analysis, CRC Press, (2004).

[29] Juan Jose Lopez Cela, Consuelo Huerta, Enrique Alarcon, Numerical analysis of axisymmetric shells by one-dimensional continuum elements suitable for high frequency excitations, *Computers and Structures*, **78**, (2000), pp. 711-724.

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APPENDIX A

Functions of forces-displacement relationships for cross-ply plates on Pasternak foundation

$$
\begin{aligned}
f_1^C &= -\alpha c_4 v_m^C - \alpha c_5 \varphi_{ym}^C + \frac{D_{11}}{c_1} N_{xm}^C - \frac{B_{11}}{c_1} M_{xm}^C \\
f_2^C &= \alpha u_m^C - \frac{D_{66}}{c_{10}} N_{xym}^C - \frac{B_{66}}{c_{10}} M_{xym}^C \\
f_3^C &= \frac{1}{KA_{55}} Q_{xm}^C - \varphi_{xm}^C \\
f_4^C &= -\alpha c_2 v_m^C - \alpha c_3 \varphi_{ym}^C - \frac{B_{11}}{c_1} N_{xm}^C + \frac{A_{11}}{c_1} M_{xm}^C \\
f_5^C &= -\alpha \varphi_{xm}^C + \frac{B_{66}}{c_{10}} N_{xym}^C - \frac{A_{11}}{c_{10}} M_{xym}^C \\
f_6^C &= -I_0 \omega^2 u_m + \alpha N_{xym} \\
f_7^C &= \left(-I_0\omega^2 + \alpha^2 c_6\right) v_m^C + \alpha^2 c_7 \varphi_{ym}^C + \alpha c_4 N_{xm}^C + \alpha c_2 M_{xm}^C \\
f_8^C &= -\frac{c_2 \alpha k_2}{c_0} v_m^C + \frac{\left(-I_0\omega^2 + (k_2 + KA_{44})\alpha^2 + k_1\right)}{c_0} w_m^C + \\
&+ \frac{\alpha K A_{44} - k_2 c_3 \alpha}{c_0} \varphi_{ym}^C - \frac{B_{11} k_2}{c_1 c_0} N_{xm}^C + \frac{A_{11} k_2}{c_1 c_0} M_{xm}^C \\
f_9^C &= -I_2 \omega^2 \varphi_{xm}^C + \alpha M_{xym}^C + Q_{xm}^C, \\
f_{10}^C &= \alpha^2 c_8 v_m^C + KA_{44}\alpha w_m^C + (-I_2\omega^2 + \alpha^2 c_9 + KA_{44})\varphi_{ym}^C + \alpha c_5 N_{xm}^C + \alpha c_3 M_{xm}^C
\end{aligned}
$$

with

$$
\begin{aligned}
c_1 &= A_{11}D_{11} - B_{11}B_{11}, c_2 = \frac{A_{12}B_{11} - A_{11}B_{12}}{c_1} \\
c_3 &= \frac{B_{11}B_{12} - A_{11}D_{12}}{c_1}, c_4 = \frac{B_{11}B_{12} - A_{12}D_{11}}{c_1} \\
c_5 &= \frac{B_{11}D_{12} - B_{12}D_{11}}{c_1}, c_6 = A_{12}c_4 + A_{22} + B_{12}c_2 \\
c_7 &= A_{12}c_5 + B_{22} + B_{12}c_3 \\
c_8 &= B_{12}c_4 + B_{22} + D_{12}c_2 \\
c_9 &= B_{12}c_5 + D_{22} + D_{12}c_3 \\
c_{10} &= B_{66}B_{66} - A_{66}D_{66} \\
c_0 &= 1 + \frac{k_2}{KA_{55}}
\end{aligned}
$$

APPENDIX B

Matrices $\mathbf{A}_m^C(\omega, k_1, k_2)$:

$$
\begin{bmatrix}
0 & -\alpha c_4 & 0 & 0 & -\alpha c_5 & \frac{D_{11}}{c_1} & 0 & 0 & \frac{-B_{11}}{c_1} & 0 \\
-\alpha & 0 & 0 & 0 & 0 & 0 & \frac{-D_{66}}{c_{10}} & 0 & 0 & \frac{B_{66}}{c_{10}} \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{KA_{55}} & 0 & 0 \\
0 & -\alpha c_2 & 0 & 0 & -\alpha c_3 & -\frac{B_{11}}{c_1} & 0 & 0 & \frac{A_{11}}{c_1} & 0 \\
0 & 0 & 0 & -\alpha & 0 & 0 & \frac{B_{66}}{c_{10}} & 0 & 0 & -\frac{A_{66}}{c_{10}} \\
-\mathrm{I}_0\omega^2 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
0 & -\mathrm{I}_0\omega^2 + \alpha^2 c_6 & 0 & 0 & \alpha^2 c_7 & \alpha c_4 & 0 & 0 & \alpha c_2 & 0 \\
0 & -\frac{c_2\alpha k_2}{c_0} & \frac{-\mathrm{I}_0\omega^2 + \alpha^2(KA_{44} + k_2) + k_1}{c_0} & 0 & \frac{\alpha KA_{44} - k_2 c_3 \alpha}{c_0} & \frac{B_{11}k_2}{c_1 c_0} & 0 & 0 & \frac{A_{11}k_2}{c_1 c_0} & 0 \\
0 & 0 & 0 & -I_2\omega^2 & 0 & 0 & 0 & 1 & 0 & \alpha \\
0 & \alpha^2 c_8 & KA_{44}\alpha & 0 & -\mathrm{I}_0\omega^2 + \alpha^2 c_9 + KA_{44} & \alpha c_5 & 0 & 0 & \alpha c_3 & 0
\end{bmatrix}
$$

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