VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

Vietnam Journal of MECHANICS

Volume 35 Number 4

ISSN 0866-7136

VN INDEX 12.666

4
2013

35th Anniversary

Vietnam Journal of Mechanics, VAST, Vol. 35, No. 4 (2013), pp. 313 – 323

DETERMINATION OF MODE SHAPES OF A MULTIPLE CRACKED BEAM ELEMENT AND ITS APPLICATION FOR FREE VIBRATION ANALYSIS OF A MULTI-SPAN CONTINUOUS BEAM

Tran Van Lien*, **Trinh Anh Hao**
University of Civil Engineering, Hanoi, Vietnam
*E-mail: lientv@hotmail.com

Abstract. This article presents some results on the determination of the vibration shape function of a multiple cracked elastic beam element, which is modeled as an assembly of intact sub-segments connected by massless rotational springs. Algorithms and computer programs to analyze changes of natural mode shapes of multiple cracked beams have been carried out. Numerical analysis of natural mode shapes of cracked simple support beams using the obtained expression shows a good agreement in comparison with the well-known analytical methods. The methodology approach and results presented in this article are new and basic for building an efficient method to identify cracks in beam structures using wavelet analysis of mode shapes.

Keywords: Shape function, cracked beam, transfer matrix, natural frequency, mode shape.

1. INTRODUCTION

Current researches on the identification of cracks or damages in a structure using non-destructive test method have been developed primarily based on the dynamic characteristics of the structure such as natural frequency, mode shapes, response spectrum function [1–11]. These dynamic characteristics are normally determined by analytical method, semi-analytical method, finite element method (FEM), and dynamic stiffness method (DSM). The analytical and semi-analytical methods are limited in a simple beam [1, 5, 6] and not applicable for a complex structure such as multi-span continuous beam or frame structure. Therefore, identification of dynamic characteristics of a structure is mainly based on the FEM and the DSM:

- In the FEM, a multiple-cracked beam element has been modeled as an assembly of intact sub-segments connected by massless rotational springs at the locations of cracks or damages. Sato [7] has developed a combination of the transfer matrix method (TMM) and the FEM for vibration analysis of a beam with abrupt changes of cross-section. Gounaris and Dimarogonas [8] have conjugated the FEM with the idea of a compliance matrix to develop a specific technique for vibration analysis of a cracked beam. Zheng and Kessissoglou [10] has used the overall additional flexibility matrix instead of the local additional flexibility matrix to obtain the total flexibility matrix of a cracked beam. FEM is an approximate method in comparison with the analytical method, especially with high natural frequencies and mode shapes [2, 12].

- The DSM is used to overcome the above limitations of the FEM and extend the advantages of the analytical method to a more complex structure other than beams [12]. In the DSM, a multiple-cracked beam element is also divided into intact sub-segments and joined at crack locations [13]. Combining the DSM and the TMM with the references of [2, 14], the authors have built straight 3D multiple cracked bar element under axial compression, tension, bending and twisting. The authors have analyzed the change of natural frequencies of structures depending of quantities, locations and depths of cracks [2, 9]. The authors also determined the quantities, locations and depths of cracks in a multiple-cracked beam based on natural frequencies measured from experiments [2, 15]. One advantage of the method is that the numbers of input parameters may be less than the number of parameters found out on solving extreme value problem. However, mode shapes corresponding to the measured natural frequencies have not been found yet. To determine the mode shapes, it is necessary to find the shape functions for the multiple-cracked beam with arbitrary depths and positions of cracks with an arbitrary coefficient of damping. This issue is quite complex and has not been published.

This paper presents some results on the determination of shape functions of a multiple cracked elastic beam subjected to bending, which is modeled as intact sub-segments joined together at the crack positions by rotation spring. Numerical analysis of the shape function of the cracked simple support beams using the obtained expression shows a good agreement in comparison with the well-known analytical methods. A computer program for analyzing changes in mode shapes of a beam structure with multiple-cracked elements is built. The proposed method and the obtained results are new and they are the basis for determination of cracks in the structures using wavelet analysis of mode shapes.

2. SHAPE FUNCTION OF INTACT LATERAL BEAM

According to FEM as described in [16], the shape function of intact lateral beam (Fig. 1) is the solution of static equilibrium differential equation without external forces as follows

$$EI_z \frac{d^4 w}{dx^4} = 0 \tag{1}$$

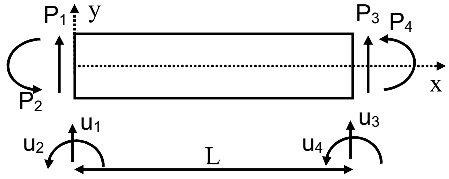

Fig. 1. Modeling of intact lateral beam

The boundary conditions are

$$w\,(0) = 1\ ;\ w'\,(0) = 0\ ;\ w\,(L) = 0\ ;\ w'\,(L) = 0 \tag{2-a}$$

$$w\,(0) = 0\ ;\ w'\,(0) = 1\ ;\ w\,(L) = 0\ ;\ w'\,(L) = 0 \tag{2-b}$$

$$w\,(0) = 0\ ;\ w'\,(0) = 0\ ;\ w\,(L) = 1\ ;\ w'\,(L) = 0 \tag{2-c}$$

$$w\,(0) = 0\ ;\ w'\,(0) = 0\ ;\ w\,(L) = 0\ ;\ w'\,(L) = 1 \tag{2-d}$$

The solutions of Eq. (1) with boundary conditions (2a-d) are the shape functions $N_1 - N_4$, called Hermit functions

$$\begin{aligned} &N_1\,(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3;\ N_2\,(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2};\\ &N_3\,(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3; \qquad N_4\,(x) = -\frac{x^2}{L} + \frac{x^3}{L^2} \end{aligned} \tag{3}$$

Let boundary values be u_1; u_2; u_3; u_4 at both end nodes, the transverse displacement of beam at the section x is

$$w\,(x) = N_1\,(x)\,u_1 + N_2\,(x)\,u_2 + N_3\,(x)\,u_3 + N_4\,(x)\,u_4 \tag{4}$$

According to DSM developed by Leung [12], shape functions of the intact beam are solutions of the undamping free vibration equation

$$EI_z\frac{d^4\Phi(x,\omega)}{dx^4} - \rho A\omega^2\Phi(x,\omega) = 0 \tag{5}$$

Solutions of Eq. (5) with boundary conditions (2a-d) are shape functions $N_1 - N_4$ as follows

$$\begin{pmatrix} N_1\\ N_2\\ N_3\\ N_4 \end{pmatrix}^T = \begin{pmatrix} \cos\lambda x/L\\ \sin\lambda x/L\\ \cosh\lambda x/L\\ \sinh\lambda x/L \end{pmatrix}^T \begin{pmatrix} 1/2 - F_4/2\lambda^2 & F_2L/2\lambda^2 & -F_3/2\lambda^2 & F_1/2\lambda^2\\ -F_6/2\lambda^3 & L/2\lambda + F_4L/2\lambda^3 & -F_5/2\lambda^3 & -F_3L/2\lambda^3\\ 1/2 + F_4/2\lambda^2 & -F_2L/2\lambda^2 & F_3/2\lambda^2 & -F_1/2\lambda^2\\ F_6/2\lambda^3 & L/2\lambda - F_4L/2\lambda^3 & F_5/2\lambda^3 & F_3L/2\lambda^3 \end{pmatrix} \tag{6}$$

Where $\lambda = \sqrt[4]{\omega^2\dfrac{\rho AL^4}{EI}}$ is the dynamic parameter; ω is the circular frequency (rad/s); $F_i(i = 1, ..., 6)$ are trigonometric functions

$$\begin{aligned} &F_1 = -\lambda(\sinh\lambda - \sin\lambda)/\delta\ ;\ F_2 = -\lambda(\cosh\lambda\sin\lambda - \sinh\lambda\cos\lambda)/\delta\ ;\\ &F_3 = -\lambda^2(\cosh\lambda - \cos\lambda)/\delta;\ F_4 = \lambda^2(\sinh\lambda\sin\lambda)/\delta\ ;\ F_5 = \lambda^3(\sinh\lambda + \sin\lambda)/\delta\ ;\\ &F_6 = -\lambda^3(\cosh\lambda\sin\lambda + \sinh\lambda\cos\lambda)\big/\delta\ ;\ \delta = \cosh\lambda\cos\lambda - 1 \end{aligned}$$

When $\omega = 0$ corresponding to the static problem, we receive the Hermit shape functions (3) from the shape functions (6).

In the case of intact beam, the determination of shape functions is the first step to obtain the dynamic stiffness matrix of beam subject to bending. In the case of multiple cracked beam, it will be modeled by rotation spring in the position of the crack, the determination of shape function is more complicated problem. In this case, it is necessary to base on the dynamic stiffness matrix obtained by the TMM.

3. DYNAMIC STIFFNESS MATRIX OF A MULTIPLE CRACKED BEAM

Let consider a beam of length L subjected to bending on surface $0xy$, cross-section area $A = b\times h$, moment of inertia I and Young's modulus E, mass density ρ. Free vibration of the beam is described by the following equation [9]

$$\frac{d^4\Phi(x,\omega)}{dx^4} - \lambda^4\Phi(x,\omega) = 0 \tag{7}$$

Where $\Phi(x,\omega)$ denotes complex amplitude of vibration; $\lambda = \sqrt[4]{\omega^2 \dfrac{\rho A}{\hat{E}I_z}\left(1 - \dfrac{i\mu_2}{\omega}\right)}$; $i = \sqrt{-1}$ is dynamic parameter; $\hat{E} = E\left(1 + i\mu_1\omega\right)$ is complex modulus; μ_1, μ_2 are the material and viscous damping coefficients, respectively; ω is circular frequency (rad/s). When $\lambda = 0$ corresponding to $\omega = 0$ it is a static deformation.

Suppose that the beam has cracks at positions x_j with the depths of $a_j, j = 1, 2, \ldots, n$, where $x_0 = 0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = L$ (Fig. 2). The cracks are modeled as rotational springs with the stiffness k_j^z calculated by converting formulas [17, 18].

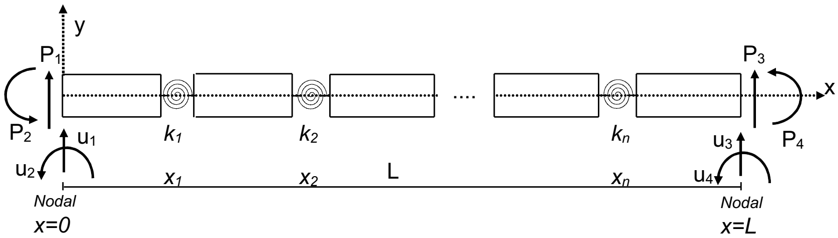

Fig. 2. Modeling of a multiple-cracked beam element

The general solution of Eq. (7) for the sub-segment $j = 1, 2, \ldots, n+1$ with $x \in (x_{j-1}, x_j)$ has the form of

$$\Phi_j(x) = K_1(\lambda\overline{x})Z^+_{j-1,1} + \frac{K_2(\lambda\overline{x})}{\lambda}Z^+_{j-1,2} + \frac{K_4(\lambda\overline{x})}{EI_z\lambda^3}Z^+_{j-1,3} - \frac{K_3(\lambda\overline{x})}{EI_z\lambda^2}Z^+_{j-1,4}\ ;\ \overline{x} = x - x_{j-1} \tag{8}$$

Where $K_i(x)$ are Krylov functions and $Z^+_{j-1,i}$ are initial parameters of this sub-segment

$$K_1(x) = \frac{\cosh x + \cos x}{2};\ K_3(x) = \frac{\cosh x - \cos x}{2};\ K_2(x) = \frac{\sinh x + \sin x}{2};\ K_4(x) = \frac{\sinh x - \sin x}{2}$$

$$\left\{Z^+_{j-1,1}, Z^+_{j-1,2}, Z^+_{j-1,3}, Z^+_{j-1,4}\right\}^T = \left(\Phi(x_{j-1}+0); \Phi'(x_{j-1}+0); \hat{E}I_z\Phi'''(x_{j-1}+0);\ -\hat{E}I_z\Phi''(x_{j-1}+0)\right)^T$$

By using the combination of dynamic stiffness and transfer matrix methods, we yield [9]

$$\left[P_1\ P_2\ P_3\ P_4\right]^T = \left[K_e\right].\left[u_1\ u_2\ u_3\ u_4\right]^T \tag{9}$$

The matrix $[K_e]_{4\times 4}$ is called dynamic stiffness matrix for the multiple cracked beam.

4. DETERMINATION OF SHAPE FUNCTIONS AND MODE SHAPES OF MULTIPLE CRACKED BEAM ELEMENTS

4.1. Shape functions

Using the general solution (8) and Eq. (9), we can determine the free vibration shape function of multiple cracked beam elements as following:

a) In order to find the shape function N_1, we determine the nodal forces P_1 and P_2 based on Eq. (9) with the boundary condition (2-a)

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix} \tag{10}$$

The initial parameters of first sub-segment ($j = 1$) are

$$Z_0^+ = \left\{ Z_1^+(0) = 1; \quad Z_2^+(0) = 0; \quad Z_3^+(0) = k_{11}; \quad Z_4^+(0) = k_{21} \right\} \tag{11}$$

Therefore, the shape function N_1 for the first sub-segment is

$$N_1^{(1)} = K_1(\lambda \overline{x}) + \frac{K_4(\lambda x)}{\hat{E} I_z \lambda^3} k_{11} - \frac{K_3(\lambda x)}{\hat{E} I_z \lambda^2} k_{21} \tag{12}$$

By using TMM, we get the shape function N_1 for the next sub-segment [9]. In the case of intact beam, the shape function has the formula

$$N_1 = K_1(\lambda x) + \frac{\tilde{K}_1 \tilde{K}_2 - \tilde{K}_3 \tilde{K}_4}{\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4} K_4(\lambda x) - \frac{\tilde{K}_2^2 - \tilde{K}_1 \tilde{K}_3}{\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4} K_3(\lambda x)$$

where $\tilde{K}_i = K_i(\lambda L)$. When damping coefficients are zero $\mu_1 = \mu_2 = 0$, one will get the shape function N_1 following the expression (6).

b) To find the shape function N_2, we determine the nodal forces P_1 and P_2 based on Eq. (9) with the boundary condition (2-b)

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} k_{12} \\ k_{22} \end{pmatrix} \tag{13}$$

Also, the initial parameters of the first sub-segment ($j = 1$) are

$$Z_0^+ = \left\{ Z_1^+(0) = 0; \quad Z_2^+(0) = 1; \quad Z_3^+(0) = k_{12}; \quad Z_4^+(0) = k_{22} \right\} \tag{14}$$

Then the shape function N_2 for the first sub-segment is

$$N_2^{(1)} = \frac{K_2(\lambda \overline{x})}{\lambda} + \frac{K_4(\lambda x)}{\hat{E} I_z \lambda^3} k_{12} - \frac{K_3(\lambda x)}{\hat{E} I_z \lambda^2} k_{22} \tag{15}$$

Similarly, we get the shape function N_2 for the next sub-segment. In the case of intact beam, the shape function has the formula

$$N_2 = \frac{K_2(\lambda x)}{\lambda} + \frac{\tilde{K}_2^2 - \tilde{K}_1 \tilde{K}_3}{\lambda(\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4)} K_4(\lambda x) - \frac{\tilde{K}_2 \tilde{K}_3 - \tilde{K}_1 \tilde{K}_4}{\lambda(\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4)} K_3(\lambda x)$$

When damping coefficients are zero $\mu_1 = \mu_2 = 0$, one will get the shape function N_2 following the expression (6).

c) To find the shape function N_3, we determine the nodal forces P_1 and P_2 based on Eq. (9) with the boundary condition (2-c)

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} k_{13} \\ k_{23} \end{pmatrix} \tag{16}$$

The initial parameters of the first sub-segment ($j = 1$) are

$$Z_0^+ = \left\{ Z_1^+(0) = 0; \quad Z_2^+(0) = 0; \quad Z_3^+(0) = k_{13}; \quad Z_4^+(0) = k_{23} \right\} \tag{17}$$

So, the shape function N_3 for the first sub-segment is

$$N_3^{(1)} = \frac{K_4(\lambda x)}{\hat{E} I_z \lambda^3} k_{13} - \frac{K_3(\lambda x)}{\hat{E} I_z \lambda^2} k_{23} \tag{18}$$

Similarly, we get the shape function N_3 for the next sub-segment. In case of the intact beam, the shape function has the formula

$$N_3 = -\frac{\tilde{K}_2}{\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4} K_4(\lambda x) + \frac{\tilde{K}_3}{\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4} K_3(\lambda x)$$

When damping coefficients are zero $\mu_1 = \mu_2 = 0$, one will get the shape function N_3 following the expression (6).

d) To find the shape function N_4, we determine the nodal forces P_1 and P_2 based on Eq. (14) with the boundary condition (2-d)

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} k_{14} \\ k_{24} \end{pmatrix} \tag{19}$$

The initial parameters of the first sub-segment ($j = 1$) are

$$Z_0^+ = \left\{ Z_1^+(0) = 0; \quad Z_2^+(0) = 0; \quad Z_3^+(0) = k_{14}; \quad Z_4^+(0) = k_{24} \right\} \tag{20}$$

The shape function N_4 for the first sub-segment is

$$N_4^{(1)} = \frac{K_4(\lambda x)}{\hat{E} I_z \lambda^3} k_{14} - \frac{K_3(\lambda x)}{\hat{E} I_z \lambda^2} k_{24} \tag{21}$$

Similarly, we get the shape function N_4 for the next sub-segment. In case of the intact beam, the shape function has the formula

$$N_4 = \frac{\tilde{K}_3}{\lambda(\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4)} K_4(\lambda x) - \frac{\tilde{K}_4}{\lambda(\tilde{K}_3^2 - \tilde{K}_2 \tilde{K}_4)} K_3(\lambda x)$$

When damping coefficients are zero $\mu_1 = \mu_2 = 0$, one will get the shape function N_4 following expression (6).

4.2. Mode shapes

The dynamic stiffness matrices of structure $\hat{K}(\omega)$ are assembled similarly as FEM from the dynamic stiffness matrices $[K_e]$ of each beam elements. Thus, the problem of free vibration of structures with the multiple-cracked bar elements leads to the determination of natural frequencies and mode shapes from the following equation

$$\hat{K}(\omega) U = 0 \tag{22}$$

Where the natural frequencies ω_j are determined from the equation:

$$\det \hat{K}\left(\omega\right) = 0 \tag{23}$$

The nodal displacements U corresponding to the natural frequencies ω_j will be determined by equations (22). Having obtained the nodal displacements u_1; u_2; u_3; u_4, we receive the mode shapes of the structure with the multiple-cracked beam elements from the expression (4).

5. NUMERICAL RESULTS AND DISCUSSION

5.1. Simple support beam

Let us consider a simple support beam with the following parameters: beam length $L = 0.235$ m; cross-section area $A = b \times h = 0.006 \times 0.0254$ m^2; Young's modulus $E = 2.06 \times 10^{11}$ N/m^2, the Poisson coefficient $\nu = 0.35$ and material mass density $\rho = 7800$ kg/m^3 [19].

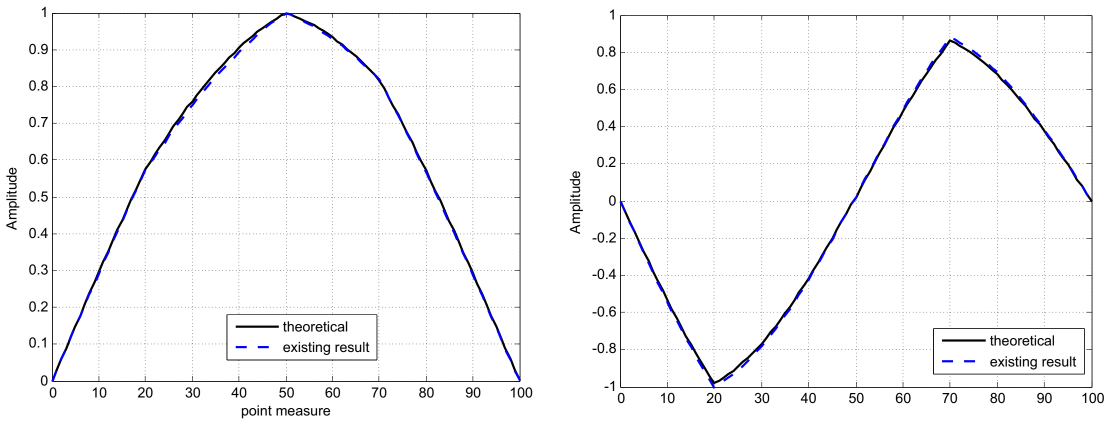

Fig. 3. The comparison of the first two mode shapes of the simple support beam with three cracks at positions $x = 0.2L, 0.5L, 0.7L$ from the left node

Fig. 3a-b show two first mode shapes of the simple support beam with three cracks at positions: $x = 0.2L, 0.5L, 0.7L$ from the left node end of the beam with the same relative crack depth of 50% calculated by the analytical method (line - - -, [19]) and the proposed method (line —). These figures present a good agreement between the numerical results obtained by the proposed method and the well-known analytical methods.

5.2. Multiple-span continuous beam

Let consider the multiple-span continuous beam with the following parameters: $L_1 = 0.8$ m, $L_2 = 1.1$ m, $L_3 = 0.6$ m, cross-section area $b \times h = 0.04 \times 0.02$ m^2, Young's modulus $E = 2.1 \times 10^{11}$ (N/m^2), the Poisson coefficient $\nu = 0.3$ and material mass density $\rho = 7800$ kg/m^3 in Fig. 4.

Figs. 5, 6, 7 show the changes in the first three mode shapes (the differences between the mode shapes obtained from cracked and uncracked beams) of the structure with one

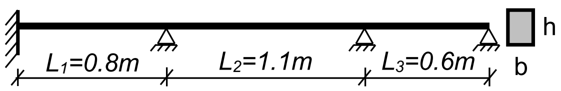

Fig. 4. Multiple-span continuous cracked beam

crack, the depth of crack ranges from 10% to 60% and the positions of cracks are as followings:

- At 0.2 m (Figs. 5a, 6a, 7a), 0.4 m (Figs. 5b, 6b, 7b), 0.6 m (Figs. 5c, 6c, 7c) from the left node of the first span;

- At 0.2 m (Figs. 5d, 6d, 7d), 0.5 m (Figs. 5e, 6e, 7e), 0.8 m (Figs. 5f, 6f, 7f) from the left node of the second span;

- At 0.1 m (Figs. 5g, 6g, 7g), 0.3 m (Figs. 5h, 6h, 7h), 0.5 m (Figs. 5i, 6i, 7i) from the left node of the third span.

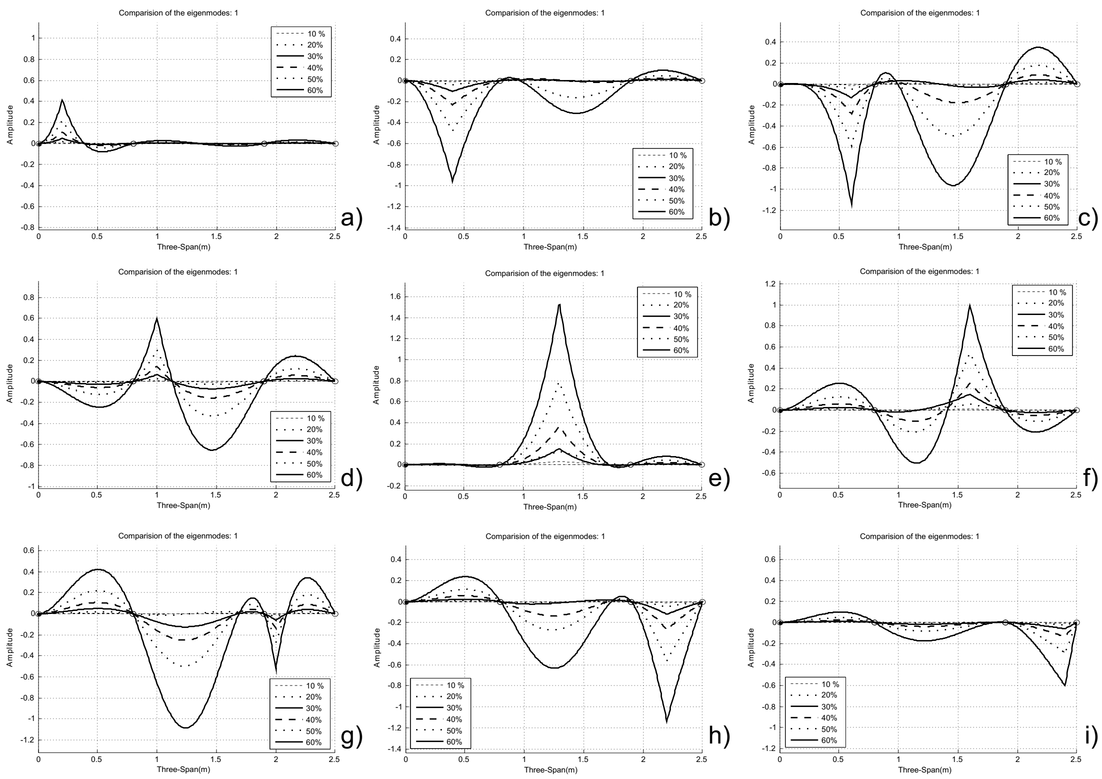

Fig. 5. The comparison of first mode shapes of the multi-span continuous beam with one crack and variable position

Fig. 8 shows the changes in the first three mode shapes of multiple cracked multiple-span continuous beam when the number of cracks increases from 1 to 6 with an equidistance 0.15 m on the second span and with invariable crack depth of 30%.

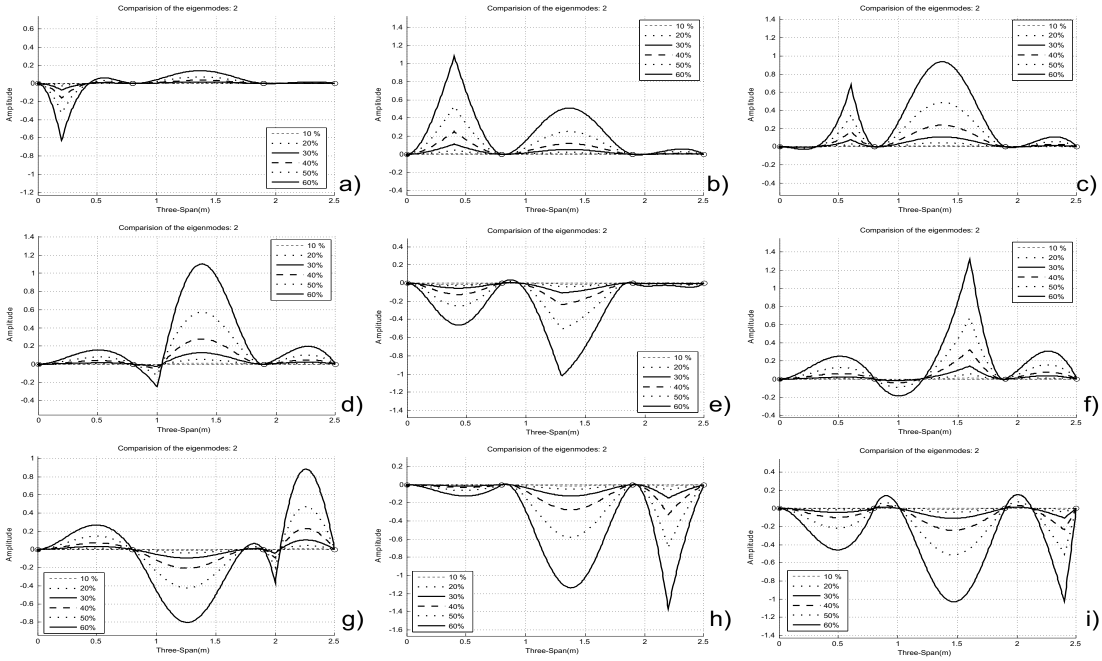

Fig. 6. The comparison of second mode shapes of the multi-span continuous beam with one crack and variable position

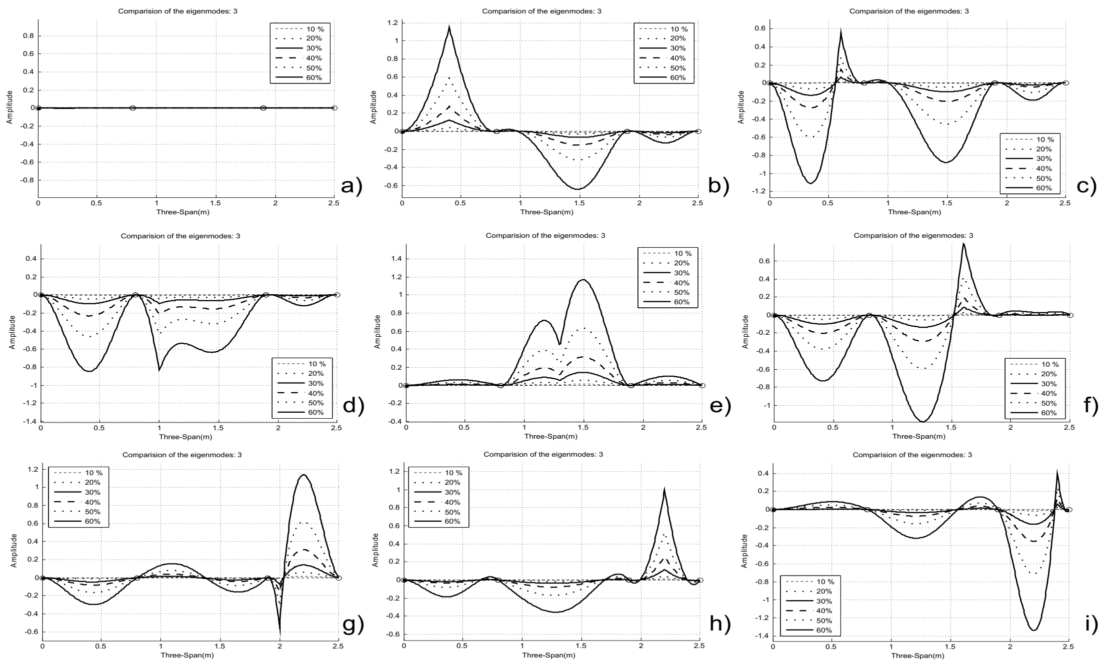

Fig. 7. The comparison of third mode shapes of the multi-span continuous beam with one crack and variable position

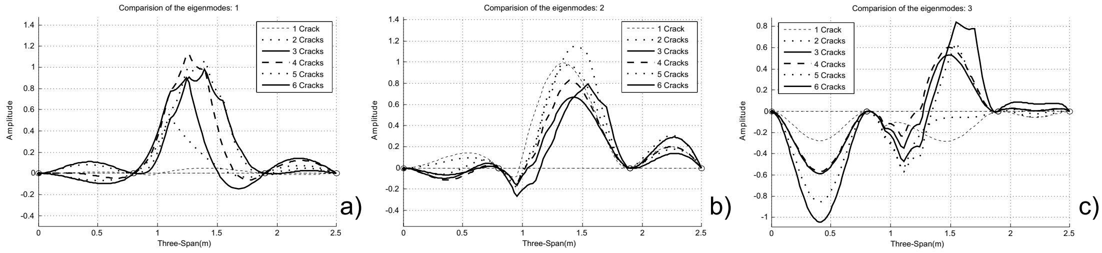

Fig. 8. The comparison of the first three mode shapes of multi-span continuous beam with the number of cracks increases from 1 to 6 with equidistance 0.15 m on the second span and with invariable crack depth of 30%

We have remarks:

a) At crack positions, the differences of the mode shape have bumpy line with the peak on the same crack position, but it is not maximum value (Figs. 5g, 6c, 6d, 6g, 7c, 7e, 7f, 7g, 7i).

b) The differences of the mode shapes increase with the increase of the crack depth.

c) At the cracked span, the mode shapes change suddenly, but on the other uncracked spans the mode shapes change smoothly, it is also relative to change large scale on adjacent spans.

d) There are some positions in which the cracks do not influence the mode shapes, for example: the crack at position of 0.2 m causes the change in the first two mode shapes but does not cause the change in the third mode shape (Fig. 7a). Thus, these positions of the cracks are called the invariable point of mode shape to discriminate between them with the fixed position in which the mode shapes have a zero amplitude value.

e) When gradually increasing the number of cracks with equidistance, then the change amplitude of the mode shape also increases but the magnitude are not necessarily the largest (Fig. 8).

6. CONCLUSION

The article presents some results on the determination of vibration shape functions of a multiple cracked elastic beam element, where the multiple cracked beam element is modeled as an assembly of intact sub-segments connected by massless rotational springs. Also, algorithms and computer programs are established for analyzing changes in mode shapes of multiple cracked multi-span continuous beams. Numerical analysis of mode shapes of the cracked simple support beams using the proposed method shows a good agreement in comparison with the well-known analytical methods. The results received are new, reliable and can be used as a basis for building an efficient method to identify cracks in beam structure using wavelet analysis of mode shapes.

REFERENCES

[1] Tran Thanh Hai, *Diagnosis of cracked beams by using vibration methods*, PhD thesis, Institute of Mechanics, in Vietnamese, (2011).

[2] Tran Van Lien, *The inverse problems in mechanics and its application*, PhD thesis, National University of Civil Engineering, in Vietnamese, (2003).
[3] Adams R.D., Cawley P., Pie C.J. and Stone B.J.A., A vibration technique for non-destructively assessing the integrity of structures, *Journal of Mechanical Engineering Science*, **20**, (1978), pp. 93–100.
[4] Cawley P., Adams R.D., The location of defects in structures from measurements of natural frequencies, *Journal of strain analysis*, **14**(2), (1979), pp. 49–57.
[5] Charti M., Rand R. and Mukherjee S., Modal analysis of cracked beam, *Journal of Sound and Vibration*, **207**, (1997), pp. 249–270.
[6] Nguyen Tien Khiem and Dao Nhu Mai, Natural frequency analysis of cracked beam, *Vietnam Journal of Mechanics*, NCNST of Vietnam, **19**(2), (1997), pp. 28–38.
[7] Sato H., Free vibration of beams with abrupt changes of cross-section, *Journal of Sound and Vibration*, **89**, (1983), pp. 59–64.
[8] Gounaris G., Dimarogonas A., A finite element of a cracked prismatic beam for structural analysis, *Computers and Structures*, **28**, (1988), pp. 309–313.
[9] Khiem N.T., Lien T.V., The dynamic stiffness matrix method in forced vibration analysis of multiple cracked beam, *Journal of Sound and Vibration*, **254**(3), (2002), pp. 541–555.
[10] Zheng D.Y., Kessissoglou N.J., Free vibration analysis of a cracked beam by finite element method, *Journal of Sound and Vibration*, **273**(3), (2004), pp. 457–475.
[11] Baris Binici,Vibration of beam with multiple open cracks subjected to axial force, *Journal of Sound and Vibration*, **287**(1-2), (2005), pp. 277–295.
[12] Leung A.Y.T., *Dynamic Stiffness and Substructures*, Springer-Verlag, London, (1993).
[13] Nguyen Xuan Hung, *Dynamics of structures and its application in structural identification*, Institute of Applied Mechanics, National Center for Natural Science and Technology, (1999).
[14] Khiem N.T., Lien T.V., A simplified method for frequency analysis of multiple cracked beam, *Journal of Sound and Vibration*, **245**(4), (2001), pp. 737–751.
[15] Khiem N.T., Lien T.V., Multi-crack detection for beam by the natural frequencies, *Journal of Sound and Vibration*, **273**(1-2), (2004), pp. 175–184.
[16] Bathe K.J., *Finite Element Procedures,* Prence – Hall, (1996).
[17] Chondros T.G., Dimarogonas A.D. and Yao J., A continuous cracked beam vibration theory, *Journal of Sound and Vibration*, **215**(1), (1998), 17–34.
[18] Haisty B.S. and Springer W.T., A general beam element for use in damage assessment of complex structures, *Journal of Vibration, Acoustics, Stress and Reliability in Design*, **110**, (1988), pp. 389–394.
[19] Nguyen Tien Khiem, Hai Tran Thanh, A procedure for multiple crack identification in beam-like structures from natural vibration mode, *Journal of Sound and Control*, DOI: 10.1177/1077546312470478, first published on April 10, (2013), jvc.sagepub.com.

Received October 29, 2012

VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY

VIETNAM JOURNAL OF MECHANICS VOLUME 35, N. 4, 2013

CONTENTS

Pages

1. Nguyen Manh Cuong, Tran Ich Thinh, Ta Thi Hien, Dinh Gia Ninh, Free vibration of thick composite plates on non-homogeneous elastic foundations by dynamic stiffness method. 257
2. Vu Lam Dong, Pham Duc Chinh, Construction of bounds on the effective shear modulus of isotropic multicomponent materials. 275
3. Dao Van Dung, Nguyen Thi Nga, Nonlinear buckling and post-buckling of eccentrically stiffened functionally graded cylindrical shells surrounded by an elastic medium based on the first order shear deformation theory. 285
4. N. T. Khiem, L. K. Toan, N. T. L. Khue, Change in mode shape nodes of multiple cracked bar: II. The numerical analysis. 299
5. Tran Van Lien, Trinh Anh Hao, Determination of mode shapes of a multiple cracked beam element and its application for free vibration analysis of a multi-span continuous beam. 313
6. Phan Anh Tuan, Pham Thi Thanh Huong, Vu Duy Quang, A method of skin frictional resistant reduction by creating small bubbles at bottom of ships. 325
7. Nguyen Thoi Trung, Bui Xuan Thang, Ho Huu Vinh, Lam Phat Thuan, Ngo Thanh Phong, An effective algorithm for reliability-based optimization of stiffened Mindlin plate. 335