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MODAL ANALYSIS OF PLATES RESTING ON ELASTIC FOUNDATION BASED ON THE FIRST-ORDER SHEAR DEFORMATION THEORY AND FINITE ELEMENT METHOD

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Abstract. This paper investigates the free vibration characteristics of plate structures supported by a Pasternak elastic foundation, utilizing the first-order shear deformation theory (FSDT). FSDT simplifies the plate theory by considering only first-order shear deformation, enhancing formulation simplicity. Additionally, employing plate theory reduces computational complexity, as 2D models entail fewer degrees of freedom compared to their 3D counterparts. The finite element method (FEM) with 8-node quadrilateral element is employed for computational analysis, implemented using MATLAB. First, a comparison is made with some existing data to show the accuracy and reliability of the research. Numerical examples are then presented of the influence of the effects of thickness variation, foundation parameters and boundary conditions on frequency are investigated. The results show that the method converges very fast and reliability when compared to other research findings. The results of the research can be applied to many different engineering applications related to plates resting on elastic foundation.

Keywords: free vibration, FSDT plate, Pasternak foundation, finite element method.

1. INTRODUCTION

Free vibration behavior of plates resting on elastic foundations is an important topic in the design of various engineering applications, including road pavements, and machine bases. There are several theories used to analyze plate problems, such as the classical plate theory (CPT) [1], the first-order shear deformation theory (FSDT) [2] and the high-order shear deformation theory (HSDT) [3]. The thickness and foundation parameters are used to change resonant frequency and to reduce the weight and size of the structures. Winkler [4] type elastic foundation is the simplest model to describe the interaction between the plate and the foundations, in which the foundation is modeled as a series of separated springs without coupling effects between each other. Building upon this model, Pasternak [5] made significant improvements by incorporating a shear spring to simulate the interaction between the individual springs in the Winkler model. The two-parameter Pasternak model is employed in this study to characterize the interaction between the plate and its foundation. Zhou et al. [6], and Xiang et al. [7] conducted the free vibration analysis of clamped and simply supported thick rectangular plates resting on Pasternak elastic foundation. Omurtag et al. [8] have studied the free vibration of thin plates resting on Pasternak foundation. Li et al. [9] have studied the free vibration analysis of functionally graded plates on Pasternak foundation based on a simple quasi-3D HSDT. Bahmyari et al. [10] used element free Galerkin method to study vibration analysis of thin plates resting on Pasternak foundations. Park and Choi [11] used a simplified first-order shear deformation theory for free vibration analysis of isotropic plates on elastic foundations.

In this study, the first-order shear deformation theory is used to form the plate and the finite element method is the chosen numerical computation method. A two-parameter Pasternak elastic foundation model is employed for the investigation of natural frequencies of plates. The material is assumed to be homogeneous, isotropic, and linearly elastic. Numerical examples for thickness variation, foundation parameters and boundary conditions are presented to verify the validity of the present theory and these results are valuable benchmarks for researchers to verify their numerical methods and for engineers to use in future structural applications.

2. FORMULATION OF PLATES RESTING ON AN ELASTIC FOUNDATION

2.1. Kinematic of first-order shear deformation theory

Fig. 1 shows the five degrees of freedom and their sign convention. In the FSDT plate, the displacement field at an arbitrary point with the coordinate (x, y, z) is represented as the following equations

$$\begin{aligned}
\bar{u}\left(x,y,z\right) &= z\theta_x\left(x,y\right), \\
\bar{v}\left(x,y,z\right) &= z\theta_y\left(x,y\right), \\
\bar{w}\left(x,y,z\right) &= w\left(x,y\right),
\end{aligned}$$

where w is the deflection, θ_x and θ_y are the section rotations about the y and x axes, respectively.

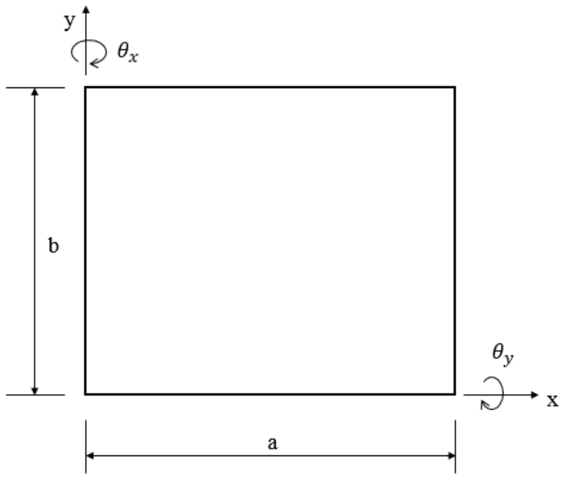

Fig. 1. A 3-DOF plate and the sign convention

2.2. The Pasternak foundation model

The Pasternak foundation model is an extension of the Winkler foundation model that takes into account the shear interaction between the spring elements. The Pasternak theory examines a plate consisting of incompressible vertical elements, deformed by transverse shear factors. This plate is placed on top of the springs to connect the spring ends together (see Fig. 2).

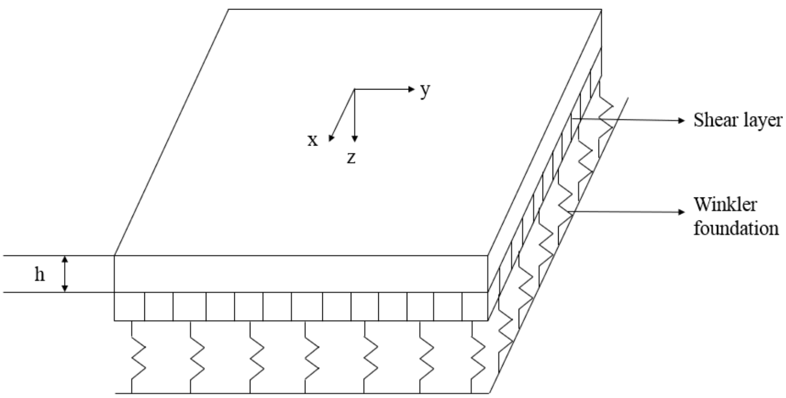

Fig. 2. The plate resting on two-parameter Pasternak elastic foundation

According to reference [12], the Pasternak model is described by the following differential equation

$$p = kw - G_p \nabla^2 w,$$

where k is the first foundation parameter or Winkler parameter, G_p is the second foundation parameter or shear foundation parameter, and ∇^2 is the Laplace operator.

In the study, the non-dimensional Winkler foundation coefficient $\bar{k}$ is introduced

$$\bar{k} = \frac{kL^4}{D},$$

and the non-dimensional shear foundation coefficient $\bar{G}_p$

$$\bar{G}_p = \frac{G_p L^2}{D},$$

where L is the size length of the plate and $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ is the flexural rigidity of the plate.

3. FINITE ELEMENT METHOD FOR FREE VIBRATION OF FSDT PLATE RESTING ON PASTERNAK FOUNDATION

Since the Laplace operator requires a second-order derivative, an 8-node quadrilateral element (Q8) with is used for the finite element analysis (see Fig. 3). The interpolation of DOFs of the Q8 element is given by the following equation

$$w = \sum_{i=1}^{8} N_i\left(\xi,\eta\right) w_i\,, \quad \theta_x = \sum_{i=1}^{8} N_i\left(\xi,\eta\right)\left(\theta_x\right)_i\,, \quad \theta_y = \sum_{i=1}^{8} N_i\left(\xi,\eta\right)\left(\theta_y\right)_i\,,$$

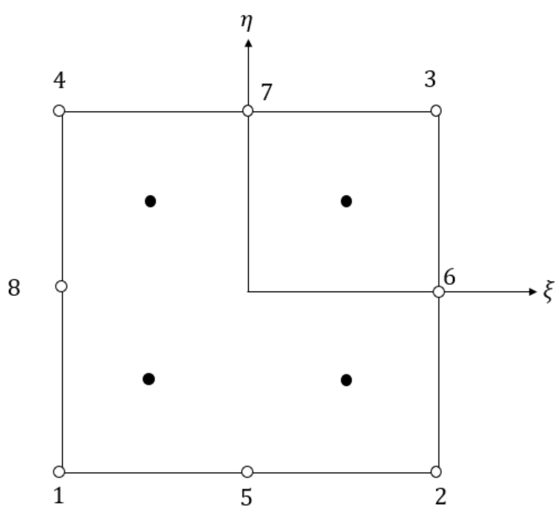

Fig. 3. An 8-node quadrilateral element with 4 integration points

For the analysis of free vibration, the equations of motion of the FSDT plate are expressed as follows

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0},$$

where $\mathbf{M}$, $\mathbf{K}$ are the mass and stiffness matrices of the system, and $\ddot{\mathbf{u}}$, $\mathbf{u}$ are the acceleration and displacement.

The stiffness matrix of the FSDT plate resting on the Pasternak foundation is defined as

$$\mathbf{K} = \frac{h^3}{12}\int_A \mathbf{B}_b^T \mathbf{D}_b \mathbf{B}_b \mathrm{d}A + \int_{\Omega^e} \mathbf{B}_s^T \mathbf{A}_s \mathbf{B}_s \mathrm{d}\Omega^e + \int_A \mathbf{N}_k^T k \mathbf{N}_k \mathrm{d}A - \int_A \mathbf{N}_k^T G_p \mathbf{B}_{G_f} \mathrm{d}A, \tag{1}$$

in which the bending and shear part are

$$\mathbf{B}_b = \left[\begin{array}{ccc|c|ccc} 0 & \dfrac{\partial N_1}{\partial x} & 0 & & 0 & \dfrac{\partial N_n}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial N_1}{\partial y} & \cdots & 0 & 0 & \dfrac{\partial N_n}{\partial y} \\ 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & & 0 & \dfrac{\partial N_n}{\partial y} & \dfrac{\partial N_n}{\partial x} \end{array}\right],$$

$$\mathbf{B}_s = \left[\begin{array}{ccc|c|ccc} \dfrac{\partial N_1}{\partial x} & N_1 & 0 & \cdots & \dfrac{\partial N_n}{\partial x} & N_n & 0 \\ \dfrac{\partial N_1}{\partial y} & 0 & N_1 & & \dfrac{\partial N_n}{\partial y} & 0 & N_n \end{array}\right],$$

$$\mathbf{D}_b = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix},$$

$$\mathbf{A}_s = k_s G h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The shear correction factor k_s can be taken as $5/6$ for simply supported on four edges (SSSS) case and 0.8601 for fully clamped on four edges (CCCC) case.

The two last terms in Eq. (1) are related to the Pasternak foundation which contains the second-order derivatives of the shape functions

$$\mathbf{B}_{G_f} = \left[\begin{array}{ccc} \left(\dfrac{\partial^2 N_1}{\partial x^2} + \dfrac{\partial^2 N_1}{\partial y^2}\right) & 0 & 0 \end{array} |\ldots| \begin{array}{ccc} \left(\dfrac{\partial^2 N_n}{\partial x^2} + \dfrac{\partial^2 N_n}{\partial y^2}\right) & 0 & 0 \end{array}\right],$$

$$\mathbf{N}_k = \left[\begin{array}{ccc} N_1 & 0 & 0 \end{array} |\ldots| \begin{array}{ccc} N_n & 0 & 0 \end{array}\right],$$

The mass matrix is given by

$$\mathbf{M}^e = \int_{\Omega^e} \mathbf{N}^T \mathbf{I} \mathbf{N} \mathrm{d}\Omega^e,$$

where **I** is the inertia matrix and is given by

$$\mathbf{I} = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix},$$

in which m_2 represents rotational inertia which for thin plates is usually negligible.

Assuming a harmonic motion, the natural frequencies and mode shapes are obtained by solving the eigenvalue equation

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{X} = 0,$$

where ω is the natural frequency and $\mathbf{X}$ is the vibration mode. By using the previously determined mass and stiffness matrices, the problem of free vibration can be solved after assembly.

4. NUMERICAL EXAMPLES

4.1. Convergence and validation

In this first example, the accuracy of the finite element MATLAB code is examined. A square plate is considered with the dimensions: side length $a = 1$ m and plate's thickness $h = 0.1$ m. The material properties are as follows: Young's modulus E = 10920 Pa, Poisson's ratio ν = 0.3. Two cases of boundary conditions are considered: simply supported on four edges (SSSS) and fully clamped on four edges (CCCC). Various mesh sizes are examined in the convergence study to find an appropriate mesh for the remaining study in the project. The non-dimensional natural frequency is defined as $\bar{\omega} = \omega a\sqrt{\dfrac{\rho}{G}}$. The obtained results are compared with the analytical solution [13]. The finite element model is discretized by the Q8 element.

Table 1 shows the convergence rate of the simply supported plate. One can see from the figure that the obtained result quickly converges to the analytical result. Similarly, Table 2 also shows that the results converge very quickly to the analytical solution.

Table 1. Convergence of natural frequency $\bar{\omega}$ for SSSS plate

Mesh	Present study	Analytical solution [13]
6 × 6	0.9304	0.930
10 × 10	0.9303	
14 × 14	0.9303	
20 × 20	0.9303	

Table 2. Convergence of natural frequency $\bar{\omega}$ for CCCC plate

Mesh	Present study	Analytical solution [13]
6 × 6	1.5919	1.5940
10 × 10	1.5911	
14 × 14	1.5911	
20 × 20	1.5910	

From this convergence study, it has been determined that the mesh with 14 × 14 Q8 element is sufficient to ensure rapid and accurate convergence for analyzing FSDT plates. Therefore, in the following examples, the Q8 mesh with 14 × 14 element is used.

4.2. Free vibration of the thick square plate resting on Pasternak foundations

In this example, a comparison of non-dimensional fundamental frequencies for a plate resting on elastic foundation, thickness ratios is presented. The boundary condition are considered: simply supported on four edges (SSSS). As stated in the convergence study, the Q8 mesh with 14 × 14 mesh is used. Material properties used in this analysis are Young's modulus: $E = 380$ GPa, density $\rho_c = 3800$ kg/m^3 and Poisson's ratio $\nu = 0.3$.

In Table 3, a comparison of non-dimensional fundamental frequencies for a plate with various elastic boundary conditions and thickness ratios is presented. In general, it can be seen from the table that the results of the present numerical results have better agreement with the quasi-3D results [9]. The nondimensional fundamental frequency increases with the increase of k, G_p and h/a. Moreover, the variation of G_p has far more effect on the fundamental frequency than k does.

Table 3. Non-dimensional fundamental frequencies $\tilde{\omega} = \omega h\sqrt{\rho_m/E_m}$, $E_m = 70$ GPa, $\rho_m = 2702$ kg/m^3 of plate with various elastic boundary conditions and thickness ratios

$\bar{k}$	$\bar{G}_p$	h/a	Model	Results	Difference (%)
0	0	0.05	Quasi-3D [9]	0.0291	-
			Present study	0.0291	0
		0.1	Quasi-3D [9]	0.1135	-
			Present study	0.1133	0.176
		0.15	Quasi-3D [9]	0.2459	-
			Present study	0.2452	0.285
		0.2	Quasi-3D [9]	0.4168	-
			Present study	0.4150	0.432

$\bar{k}$	$\bar{G}_p$	h/a	Model	Results	Difference (%)
100	0	0.05	Quasi-3D [9]	0.0298	-
			Present study	0.0298	0
		0.1	Quasi-3D [9]	0.1163	-
			Present study	0.1161	0.172
		0.15	Quasi-3D [9]	0.2521	-
			Present study	0.2516	0.198
		0.2	Quasi-3D [9]	0.4281	-
			Present study	0.4268	0.304
100	100	0.05	Quasi-3D [9]	0.0410	-
			Present study	0.0410	0
		0.1	Quasi-3D [9]	0.1613	-
			Present study	0.1615	0.124
		0.15	Quasi-3D [9]	0.3535	-
			Present study	0.3553	0.509
		0.2	Quasi-3D [9]	0.6085	-
			Present study	0.6149	1.052

4.3. Free vibration of the thin square plate resting on Pasternak foundations

In this example, a thin plate resting on elastic foundation is examined. The SSSS and CCCC boundary conditions are considered. As stated in the convergence study, the Q8 mesh with 14×14 mesh is used.

Since the FSDT is used to formulate thick plates, a numerical issue called "shear locking" is observed when using it for thin plates. To avoid shear locking, a simple treatment is applied which is lower order integration for shear part. For the shear stiffness matrix, only 2×2 Gauss point is used. For the remaining stiffness matrices and the load vector, a full 3×3 Gauss-quadrature integration is used.

In Tables 4 and 5 the first four natural frequencies for an SSSS and CCCC plate with different elastic foundation parameters are calculated and compared with available results. As can be seen very good agreements is achieved in all cases. Because of the high accuracy results, one can say that using a 2×2 integration point for the shear stiffness matrix does not cause the "hourglass mode" phenomenon.

To compare the natural frequency changes of Pasternak's elastic foundation types by varying the elastic foundation coefficient and boundary conditions, in Figs. 4 and 5 represent some examples to show the first ten natural frequencies and mode shapes of these plates are display. It can be seen that when increasing the number of modes and

Table 4. Dimensionless parameter of natural frequencies, $\tilde{\omega} = \dfrac{\omega a^2}{\pi^2}\sqrt{\dfrac{\rho h}{D}}$, of a square plate on Pasternak's type elastic foundation ($\bar{G}_p = 10$, $h/a = 0.01$, $\nu = 0.3$)

	$\bar{k}$		Mode			
			1st	2nd	3rd	4th
SSSS	100	Ref. [10]	2.6560	5.5817	5.5817	8.5572
		Present study	2.6535	5.5657	5.5657	8.5334
	500	Ref. [10]	3.3407	5.9380	5.9380	8.7937
		Present study	3.3387	5.9230	5.9230	8.7705
CCCC	100	Ref. [10]	4.1067	7.9246	7.9246	11.4696
		Present study	4.0867	7.8612	7.8612	11.3830
	500	Ref. [10]	4.5793	8.1795	8.1795	11.6471
		Present study	4.5614	8.1181	8.1181	11.5618
	1000	Ref. [10]	5.1090	8.4873	8.4873	11.8652
		Present study	5.0930	8.4281	8.4281	11.7815

Table 5. Dimensionless parameter of natural frequencies, $\tilde{\omega} = \dfrac{\omega a^2}{\pi^2}\sqrt{\dfrac{\rho h}{D}}$, of a square plate on Pasternak's type elastic foundation ($\bar{G}_p = 50$, $h/a = 0.01$, $\nu = 0.3$)

	$\bar{k}$		Mode			
			1st	2nd	3rd	4th
SSSS	100	Ref. [10]	3.8935	7.1703	7.1703	10.2777
		Present study	3.8874	7.1379	7.1379	10.2316
	500	Ref. [10]	4.3892	7.4510	7.4510	10.4754
		Present study	4.3838	7.4198	7.4198	10.4302
	1000	Ref. [10]	4.9393	7.7877	7.7877	10.7175
		Present study	4.9345	7.7579	7.7579	10.6733
CCCC	100	Ref. [10]	5.1501	9.2762	9.2762	12.9650
		Present study	5.1135	9.1771	9.1771	12.8307
	500	Ref. [10]	5.5344	9.4949	9.4949	13.1223
		Present study	5.5003	9.3981	9.3981	12.9896
	1000	Ref. [10]	5.9801	9.7613	9.7613	13.3163
		Present study	5.9486	9.6672	9.6672	13.1856

shear stiffness parameters of the elastic foundation G_p, the difference between these two foundations increases. The natural frequency value of the CCCC plate is higher than the SSSS plate.

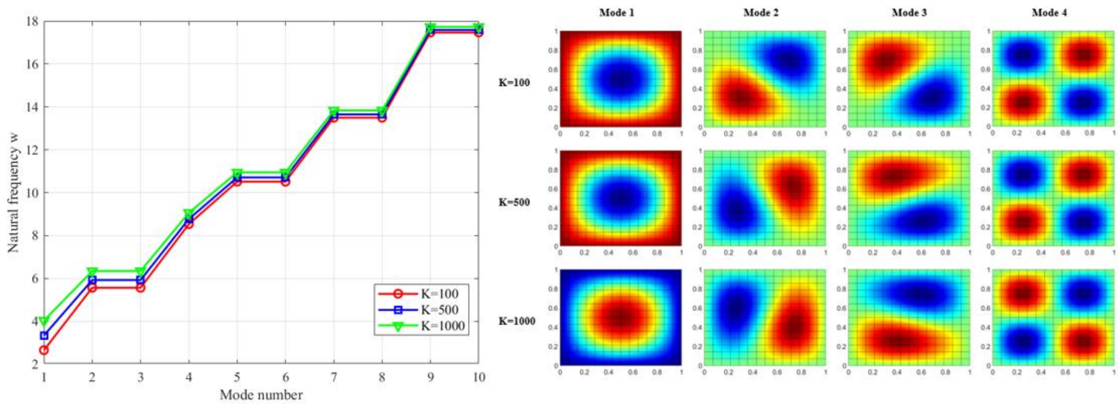

Fig. 4. Frequency and mode shape of vibration for a SSSS plate with $\bar{G}_p = 10$, $h/a = 0.01$, $\nu = 0.3$

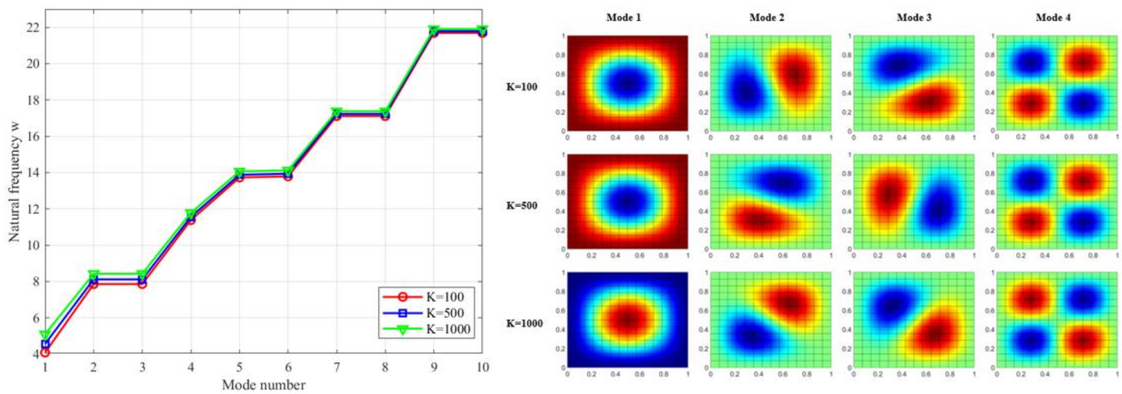

Fig. 5. Frequency and mode shape of vibration for a CCCC plate with $\bar{G}_p = 10$, $h/a = 0.01$, $\nu = 0.3$

4.4. Free vibration of thick plate with a complicated shape resting on Pasternak foundations

In this example, a plate with a complicated shape resting on Pasternak foundation is investigated (see Fig. 6). The material properties are as follows: Young's modulus E = 10920 Pa, Poisson's ratio $\nu = 0.3$. Two boundary conditions are considered SSSS and CCCC. The dimensionless frequencies are shown in Table 6. The mode shape in the case of CCCC plate is shown in Fig. 7.

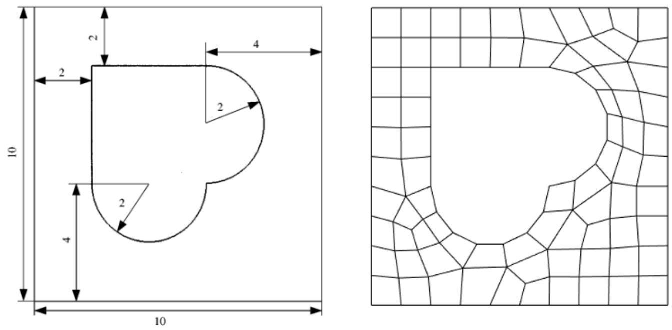

Fig. 6. Geometry of the plate with a hole of complicated shape and its discrete model

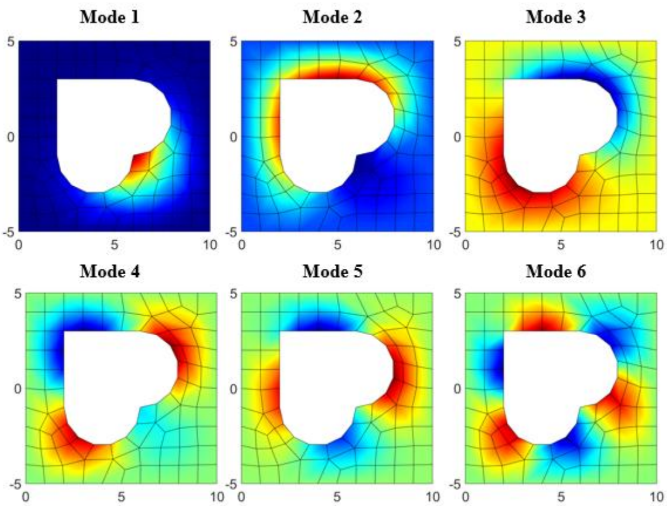

Fig. 7. Mode shape of the CCCC plate with a heart-shape hole ($\bar{G}_p = 10$, $\bar{k} = 100$, $h/a = 0.1$)

Table 6. Dimensionless parameter of natural frequencies, $\bar{\omega} = \omega a\sqrt{\dfrac{\rho}{G}}$, of a plate with a hole of complicated shape resting on Pasternak's type elastic foundation ($h/a = 0.1$)

	$\bar{G}_p$	$\bar{k}$	Mode					
			1st	2nd	3rd	4th	5th	6th
SSSS	0	100	0.7756	1.3166	1.6983	2.2979	2.6424	3.3400
		500	1.2413	1.6293	1.9483	2.4877	2.7975	3.4654
		1000	1.6476	1.9508	2.2216	2.7061	2.9798	3.6162
	10	100	0.8999	1.5269	1.8724	2.5089	2.8111	3.5116
		500	1.3225	1.8027	2.1010	2.6833	2.9556	3.6321
		1000	1.7096	2.0970	2.3556	2.8863	3.1266	3.7774

	$\bar{G}_p$	$\bar{k}$	Mode					
			1st	2nd	3rd	4th	5th	6th
CCCC	0	100	2.3521	3.8512	3.8699	4.5606	4.6752	5.5193
		500	2.5404	3.9669	3.9848	4.6583	4.7703	5.6001
		1000	2.7577	4.1068	4.1239	4.7776	4.8864	5.6994
	10	100	2.4961	3.9365	3.9705	4.7197	4.8067	5.7582
		500	2.6741	4.0492	4.0825	4.8141	4.8989	5.8355
		1000	2.8811	4.1857	4.2184	4.9296	5.0119	5.9308

5. CONCLUSION

In this study, a finite element method for free vibration analysis of plates resting on Pasternak foundation based on the first-order shear deformation theory (FSDT) is presented. The obtained result from the MATLAB code is good agreement with the result from the references. The method was developed to also take into account the surrounding effect outside the plate. A significant influence of the elastic foundation on the natural frequency was observed and the analysed system exhibited the nonlinear trend when considering the variation of eigenfrequency for the smaller values of the Winkler coefficient, while the eigenfrequency increased almost linearly for the greater values. The influence of the second coefficient for the investigated range of coefficients was linear. The present results agree quite well with the theoretical and numerical results given by other authors. The results are valuable benchmarks for researchers to verify their numerical methods and for engineers to use in future structural applications.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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