FREQUENCY RESPONSE SENSITIVITY TO CRACK FOR PIEZOELECTRIC FGM BEAM SUBJECTED TO MOVING LOAD

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Abstract. Since functionally graded material (FGM) is increasingly used in high-tech engineering, free and forced vibrations of FGM structures become an important issue. This report addresses the analysis of frequency response sensitivity to crack for piezoelectric FGM beams subjected to moving load. First, a frequency domain model of a cracked FGM beam with a piezoelectric layer is conducted to derive an explicit expression of the electrical charge produced in the piezoelectric layer under the moving load. It was shown in the previous works of the authors that the electrical charge is a reliable representation of the beam frequency response to moving load and can be efficiently employed as a measured diagnostic signal for structural health monitoring. Then, a damage indicator acknowledged as a spectral damage index (SDI) calculated from the electrical frequency response is introduced and used for sensitivity analysis of the response to crack. Under the sensitivity analysis the effect also of FGM and moving load parameters on the sensitivity is examined and illustrated by numerical results.

Keywords: FGM beam, piezoelectric layer, frequency response, moving load, sensitivity analysis.

1. INTRODUCTION

Damage detection in general and crack identification in particular are essential problem in the structural health monitoring that has been intensively studied through several latest decades and it was reviewed by numerous authors, for instance, Sohn et al. [1]; Fan and Qiao [2] and Hou and Xia [3]. Most of researchers in the field of structural health

monitoring have agreed to that dynamic behavior or vibration circumstance of a structure provide the most useful tool for diagnosis of potential damages in the structure. Hence, the important issue in structural damage detection is to gather and examine the structural dynamic features that have been chosen as indicators for the detection. Conventional approach to assessing structure integrity is the dynamic testing technique that proposes to measure response of a structure under given external excitations. This technique is bulky and expensive for huge structures because it requires a large number of sensors and actuators to obtain truthful signature of potential damages. Moreover, the conventional dynamic testing method does not allow direct identification of damage and it is difficult to perform in the real time mode.

Alternately, many authors [4–7] have demonstrated that using smart material such as piezoelectric one the structural health monitoring becomes much more advantaged in its implementation and improved in the results obtained. This is because of the smart material could be used not only for transmitting load to structure (as actuator) but also for sensing signal of the structure response (as sensor). The smart sensors are distributed [8] and may be permanently installed as components of a structure of interest [9]. Recent progress in structural health monitoring by the use of distributed piezoelectric transducers was reported in [10–14].

Particularly, Wang and Quek [15] used the sandwich beam model for modal analysis of a Euler-Bernoulli beam embedded with piezoelectric layers and they found that natural frequency of the sandwich beam is function of stiffness and thickness of the piezoelectric layers. Wang and Quek [16] showed that the buckling and flutter capacities of an elastic column could be enhanced by using piezoelectric patches bonded to both sides of the column as actuators with an applied voltage. Wang et al. [17] revealed an effect of a piezoelectric patch bonded to a beam on natural frequency of the beam and demonstrated an interesting fact that piezoelectric patch used as an actuator could restore the healthy condition of a cracked beam. Zhao et al. [18] proposed a procedure for crack identification in beam based on the crack-induced frequency change that is amplified by applying a feedback voltage output from piezoelectric sensor through collocated actuator. The so-called Electro-Mechanical Impedance (EMI) method was developed in [19–21] for crack identification in beam using piezoelectric transducers. The authors have concluded that the EMI is sensitive to local damage such as crack only at the high frequency range and when sensor patch is positioned near the damage location. Therefore, using a piezoelectric layer bonded to a beam structures as full-length distributed sensor is promising idea that is investigated in the present study for functionally graded beam with crack.

Various problems in dynamics of functionally graded beams were studied in the widespread literature, some of which are, for instance, Li [22], Sina et al. [23], Larbi et al. [24], Su and Banerjee [25], Wang et al. [26]. A number of works is devoted also to

study vibrations of the beams with localized damages such as cracks, for example, Yang and Chen [27], Akbas [28], Aydin [29], Khiem et al. [30]. Some procedures were proposed by Yu and Chu [31]; Banerjee et al. [32] and Khiem and Huyen [33] to detect cracks in functionally graded beams with natural frequencies measured by the conventional technique of modal testing. Stability of FGM Timoshenko beam embedded by the top and bottom piezoelectric layers has been investigated by Kharramabadi and Nezamabadi [34] and it is found a significant effect of both the piezoelectric actuators and FGM parameters on the critical buckling loads. Li et al. [35] even proposed a model of functionally graded piezoelectric beam for its vibration analysis and revealed the increase of natural frequency and decrease of electric potential with increasing gradient index of the material. Bendine et al. [36] studied the problem for active vibration control of functionally graded beams with upper and lower surface-bonded piezoelectric layers by the finite element method. Khiem et al. [37] examined the effect of piezoelectric patches on natural frequencies of undamaged functionally graded beam.

The present paper addresses the analysis of frequency response sensitivity to crack for piezoelectric FGM beams subjected to moving load. First, a frequency domain model of a cracked FGM beam with a piezoelectric layer is conducted to derive an explicit expression of the electrical charge produced in the piezoelectric layer under the moving load. It was shown in the previous works of the authors that the electrical charge is a reliable representation of the beam frequency response to moving load and can be efficiently employed as a measured diagnostic signal for structural health monitoring. Then, a damage indicator acknowledged as a spectral damage index (SDI) calculated from the electrical frequency response is introduced and used for sensitivity analysis of the response to crack. Under the sensitivity analysis the effect also of FGM and moving load parameters on the sensitivity is examined and illustrated by numerical results.

2. GOVERNING EQUATIONS

Consider an FGM beam of length L, cross sectional area $A_b = b \times h_b$ bonded with a piezoelectric layer and subjected to a moving force as shown in Fig. 1 [38]. It is assumed also that the beam is made of functionally graded material with properties varying along the thickness direction by the power law

$$\mathfrak{R}(z) = \mathfrak{R}_b + (\mathfrak{R}_t - \mathfrak{R}_b)(z/h + 0.5)^n, \quad -h_b/2 \leq z \leq h_b/2, \tag{1}$$

where $\mathfrak{R}$ stands for Young's, shear modulus and material density E, G, ρ; subscripts t and b denote the top and bottom material respectively; n is power law exponent or material distribution index; z is ordinate of point along the beam height from the mid plane. According to thee Timoshenko beam theory with the constituting equations

$$u\left(x,z,t\right)=u_0\left(x,t\right)-\left(z-h_0\right)\theta\left(x,t\right),\qquad w\left(x,z,t\right)=w_0\left(x,t\right),$$
$$\varepsilon_x=\partial u_0/\partial x-\left(z-h_0\right)\partial\theta/\partial x,\qquad\qquad\gamma_{xz}=\partial w_0/\partial x-\theta,$$

(2)

where $u(x,z,t),w(x,z,t)$ are axial and transverse displacements in cross-section at x; $u_0\left(x,t\right),w_0\left(x,t\right)$ are the displacements on the neutral plane and θ is rotation of the cross-section; $\varepsilon_x,\gamma_{xz},\sigma_x,\tau$ are deformation and strain components; κ is geometry correction factor; h_0 is acknowledged as exact position of neutral plane measured from the beam midplane.
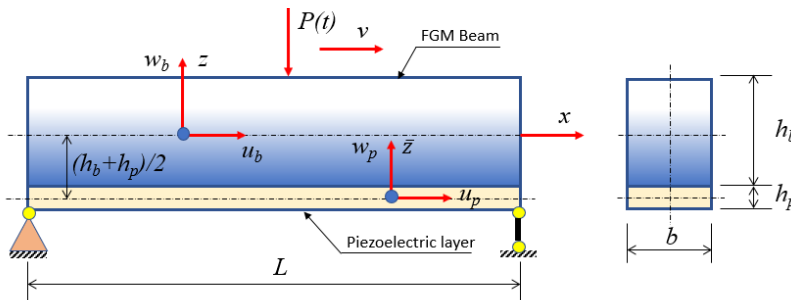

Fig. 1. Model of piezoelectric FGM beam under moving force

Let the piezoelectric layer be considered as a homogeneous Timoshenko beam element, so that constitutive equations can be expressed as

$$u_p\left(x,\bar{z},t\right)=u_{p0}\left(x,t\right)-\bar{z}\theta_p\left(x,t\right),\qquad w_p\left(x,\bar{z},t\right)=w_{p0}\left(x,t\right),$$
$$\varepsilon_{px}=u'_{p0}-\bar{z}\theta'_p,\qquad\gamma_p=w'_{p0}-\theta_p,$$
$$\sigma_{px}=C^p_{11}\varepsilon_{px}-h_{13}D,\qquad\tau_p=C^p_{55}\gamma_p,\qquad\in=-h_{13}\varepsilon_{px}+\beta^p_{33}D,$$

(3)

where $C^p_{11},h_{13},\beta^p_{33}$ are elastic modulus, piezoelectric and dielectric constants respectively, $\in$ and D are electric field and displacement of the piezoelectric layer. Hence, conditions of perfect bonding between the base beam and piezoelectric layer can be represented as

$$u\left(x,\frac{h_b}{2},t\right)=u_p\left(x,-\frac{h_p}{2},t\right),\qquad w\left(x,h_b/2,t\right)=w_p\left(x,-h_p/2,t\right),$$

that yield

$$u_{p0}=u_0-\theta h/2,\qquad h=h_b+h_p,\qquad w_{p0}=w_0,\qquad\theta=\theta_p,$$
$$\varepsilon_{px}=u'_0-\left(\bar{z}+\frac{h}{2}\right)\theta',\qquad\gamma_p=w'_0-\theta.$$

(4)

Using the constituting equations (2)–(3)–(4) and Hamilton's principle, one gets the following equations of motion [38]

$$\left(I_{11}^* \ddot{u}_0 - B_{11}^* u_0''\right) + \left(I_{12}^* \ddot{\theta} - B_{12}^* \theta''\right) = 0,$$
$$\left(I_{12}^* \ddot{u}_0 - B_{12}^* u_0''\right) + \left(I_{22}^* \ddot{\theta} - B_{22}^* \theta''\right) - A_{33}^* \left(w_0' - \theta\right) = 0, \tag{5}$$
$$I_{11}^* \ddot{w}_0 - A_{33}^* \left(w_0'' - \theta'\right) = P\left(t\right) \delta\left(x - vt\right),$$

and

$$D\left(x,t\right) = h_{13} \left(u_0' + h\theta'\right) / \beta_{33}^p, \tag{6}$$

where

$$B_{11}^* = A_{11} + E_p A_p, \quad B_{12}^* = E_p A_p h, \quad B_{22}^* = A_{22} + C_{11}^p I_p + E_p A_p h^2, \quad E_p = C_{11}^p - h_{13}^2 / \beta_{33}^p,$$
$$I_{11}^* = I_{11} + \rho_p A_p, \quad I_{12}^* = I_{12} + \rho_p A_p h, \quad I_{22}^* = I_{22} + \rho_p I_p + \rho_p A_p h^2, \quad A_{33}^* = \kappa A_{33} + C_{55}^p A_p,$$
$$A_{11} = bh_b E_b \varphi_1\left(r_e, n\right), \quad A_{22} = bh_b^3 E_b \varphi_3\left(r_e, n\right), \quad A_{33} = bh_b G_b \varphi_1\left(r_g, n\right),$$
$$I_{11} = bh_b \rho_b \varphi_1\left(r_\rho, n\right), \quad I_{12} = bh_b^3 \rho_b \varphi_2\left(r_\rho, n\right), \quad I_{22} = bh_b^3 \rho_b \varphi_3\left(r_\rho, n\right),$$
$$\varphi_1\left(r, n\right) = \left(r + n\right) / \left(1 + n\right), \quad \varphi_2\left(r, n\right) = \left(2r + n\right) / 2\left(2 + n\right) - \alpha\left(r + n\right) / \left(1 + n\right),$$
$$\varphi_3\left(r, n\right) = \left(3r + n\right) / 3\left(3 + n\right) - \alpha\left(2r + n\right) / \left(2 + n\right) - \alpha^2\left(r + n\right) / \left(1 + n\right),$$
$$\alpha = 1/2 + h_0/h_b, \quad r_e = E_t/E_b, \quad r_\rho = \rho_t/\rho_b, \quad r_g = G_t/G_b. \tag{7}$$

Transferring equations (5) and (6) into the frequency domain, one gets

$$[\mathbf{A}] \left\{\mathbf{Z}''\left(x,\omega\right)\right\} + [\mathbf{B}] \left\{\mathbf{Z}'\left(x,\omega\right)\right\} + [\mathbf{\Omega}] \left\{\mathbf{Z}\left(x,\omega\right)\right\} = -\left\{\mathbf{P}\left(x,\omega\right)\right\}, \tag{8}$$

$$\left\{\mathbf{Z}\left(x,\omega\right)\right\} = \int_{-\infty}^{\infty} \left\{u_0\left(x,t\right), \theta\left(x,t\right), w_0\left(x,t\right)\right\} e^{-i\omega t} dt, \quad \mathbf{Z}' = d\mathbf{Z}/dx, \quad \mathbf{Z}'' = d^2\mathbf{Z}/dx^2,$$

$$\mathbf{P}\left(x,\omega\right) = \left\{0, 0, Q\left(x,\omega\right)\right\}^T, \quad Q\left(x,\omega\right) = P\left(x/v\right) \exp\left\{-i\omega x/v\right\}, \tag{9}$$

with the matrices

$$[\mathbf{A}] = \begin{bmatrix} B_{11}^* & B_{12}^* & 0 \\ B_{12}^* & B_{22}^* & 0 \\ 0 & 0 & A_{33}^* \end{bmatrix}, \quad [\mathbf{B}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{33}^* \\ 0 & -A_{33}^* & 0 \end{bmatrix},$$

$$[\mathbf{\Omega}] = \begin{bmatrix} \omega^2 I_{11}^* & \omega^2 I_{12}^* & 0 \\ \omega^2 I_{12}^* & \omega^2 I_{22}^* - A_{33}^* & 0 \\ 0 & 0 & \omega^2 I_{11}^* \end{bmatrix}, \tag{10}$$

and

$$\hat{D}\left(x,\omega\right) = h_{13} \left[Z_1'\left(x,\omega\right) + hZ_2'\left(x,\omega\right)\right] / \beta_{33}^p = h_{13} \left[U'\left(x,\omega\right) + h\Theta'\left(x,\omega\right)\right] / \beta_{33}^p. \tag{11}$$

If the piezoelectric layer is employed as a distributed sensor, the frequency domain output charge of which can be calculated as

$$\hat{Q}\left(\omega\right) = b \int_0^L D dx = \left(bh_{13}/\beta_{33}^p\right) \int_0^L \left[U'\left(x,\omega\right) - h\Theta'\left(x,\omega\right)/2\right] dx. \tag{12}$$

Furthermore, assume that the host FGM beam has been cracked at the position e measured from its left end and crack is modeled by a pair of equivalent springs of stiffness T for transnational spring and R for rotational one [39]. In this case, conditions that must be satisfied at the crack are

$$U(e+0) = U(e-0) + N(e)/T, \quad \Theta(e+0) = \Theta(e-0) + M(e)/R,$$
$$W(e+0) = W(e-0), \quad U_x'(e+0) = U_x'(e-0),$$
$$\Theta_x'(e+0) = \Theta_x'(e-0), \quad W_x'(e+0) = W_x'(e-0) + M(e)/R,$$
(13)

where $N(x) = A_{11}U_x'(x)$, $M(x) = A_{22}\Theta_x'(x)$ are respectively internal axial force and bending moment at section x. Substituting the expressions for axial force and bending moment into (13) that can be rewritten as

$$U(e+0) = U(e-0) + \gamma_1 U_x'(e), \quad \Theta(e+0) = \Theta(e-0) + \gamma_2\Theta_x'(e),$$
$$W(e+0) = W(e-0), \quad U_x'(e+0) = U_x'(e-0),$$
$$\Theta_x'(e+0) = \Theta_x'(e-0), \quad W_x'(e+0) = W_x'(e-0) + \gamma_2\Theta_x'(e),$$
$$\gamma_1 = A_{11}/T, \quad \gamma_2 = A_{22}/R.$$
(14)

The so-called crack magnitudes γ_1, γ_2 are functions of the material parameters such as elastic modulus and they should be those of homogeneous beam when $E_t = E_b = E_0$. Using expressions (7) and the latter conditions the crack magnitudes can be rewritten as

$$\gamma_1 = \gamma_a\varphi_1(r_e, n), \quad \gamma_2 = 12\gamma_b\varphi_3(r_e, n),$$
(15)

where [39]

$$\gamma_a = E_0A/T = 2\pi\left(1 - \nu_0^2\right)hf_1(z), \quad \gamma_b = E_0I_0/R = 6\pi\left(1 - \nu_0^2\right)hf_2(z), \quad z = a/h, \quad (16)$$

$$f_1(z) = z^2\big(0.6272 - 0.17248z + 5.92134z^2 - 10.7054z^3 + 31.5685z^4 - 67.47z^5$$
$$+ 139.123z^6 - 146.682z^7 + 92.3552z^8\big),$$
$$f_2(z) = z^2\big(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5$$
$$+ 47.1063z^6 - 40.7556z^7 + 19.6z^8\big).$$
(17)

Obviously, for uncraled beam when $E_t = E_b = E_0$ or $r_e = 1$, Eqs. (15) yield

$$\gamma_1 = \gamma_a\varphi_1(1, n) = \gamma_a, \quad \gamma_2 = 12\gamma_b\varphi_3(1, n) = \gamma_b.$$
(18)

3. FREQUENCY RESPONSE CRACKED PIEZOELECTRIC FGM BEAM SUBJECTED TO MOVING HARMONIC LOAD

First, seeking solutions of homogeneous equation (8) in the form: $\mathbf{Z}_0 = \boldsymbol{d}e^{\lambda x}$ one gets general solution for free vibration of the beam in the form

$$\{\mathbf{Z}_0(x, \omega)\} = [\mathbf{G}_0(x, \omega)]\{\boldsymbol{C}\},$$
(19)

where $\{C\} = (C_1, \ldots, C_6)^T$ is constant vector and $\mathbf{G}_0(x, \omega)$ is matrix

$$[G_0(x,\omega)] = \begin{bmatrix} \alpha_1 e^{k_1 x} & \alpha_2 e^{k_2 x} & \alpha_3 e^{k_3 x} & \alpha_1 e^{-k_1 x} & \alpha_2 e^{-k_2 x} & \alpha_3 e^{-k_3 x} \\ e^{k_1 x} & e^{k_2 x} & e^{k_3 x} & e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} \\ \beta_1 e^{k_1 x} & \beta_2 e^{k_2 x} & \beta_3 e^{k_3 x} & -\beta_1 e^{-k_1 x} & -\beta_2 e^{-k_2 x} & -\beta_3 e^{-k_3 x} \end{bmatrix}, \tag{20}$$

$$\alpha_j = \left(\omega^2 I_{11}^* + k_j^2 B_{11}^*\right) / \left(\omega^2 I_{12}^* + k_j^2 B_{12}^*\right), \quad \beta_j = k_j A_{33}^* / \left(\omega^2 I_{11}^* + k_j^2 A_{33}^*\right), \quad j = 1,2,3$$

and k_j $(j = 1,2,3)$ are wave numbers obtained from the roots $\lambda_{1,4} = \pm k_1$, $\lambda_{2,5} = \pm k_2$, $\lambda_{3,6} = \pm k_3$ of the characteristic equation $\det\left[\lambda^2 \mathbf{A} + \lambda \mathbf{B} + \mathbf{\Omega}\right] = 0$.

Using expression (18) we can find a particular solution $\mathbf{Z}_c(x, \omega)$ in the form

$$\{\mathbf{Z}_c(x,\omega)\} = [\mathbf{G}_c(x,\omega)]\{\mathbf{Z}_0'(e,\omega)\}, \tag{21}$$

where $\mathbf{G}(x, \omega)$ is 3×3-matrix of the form

$$[\mathbf{G}_c(x,\omega)] = \begin{bmatrix} \gamma_a \sum_{i=1}^{3} \alpha_i \delta_{i1} \cosh k_i x & \gamma_b \sum_{i=1}^{3} \alpha_i (\delta_{i2} + \delta_{i3}) \cosh k_i x & 0 \\ \gamma_a \sum_{i=1}^{3} \delta_{i1} \cosh k_i x & \gamma_b \sum_{i=1}^{3} (\delta_{i2} + \delta_{i3}) \cosh k_i x & 0 \\ \gamma_a \sum_{i=1}^{3} \beta_i \delta_{i1} \sinh k_i x & \gamma_b \sum_{i=1}^{3} \beta_i (\delta_{i2} + \delta_{i3}) \sinh k_2 x & 0 \end{bmatrix}, \tag{22}$$

and

$$\begin{aligned} \delta_{11} &= (k_3 \beta_3 - k_2 \beta_2)/\Delta, & \delta_{12} &= (\alpha_3 k_2 \beta_2 - \alpha_2 k_3 \beta_3)/\Delta, & \delta_{13} &= (\alpha_2 - \alpha_3)/\Delta, \\ \delta_{21} &= (k_1 \beta_1 - k_3 \beta_3)/\Delta, & \delta_{22} &= (\alpha_1 k_3 \beta_3 - \alpha_3 k_1 \beta_1)/\Delta, & \delta_{23} &= (\alpha_3 - \alpha_1)/\Delta, \\ \delta_{31} &= (k_2 \beta_2 - k_1 \beta_1)/\Delta, & \delta_{32} &= (\alpha_2 k_1 \beta_1 - \alpha_1 k_2 \beta_2)/\Delta, & \delta_{33} &= (\alpha_1 - \alpha_2)/\Delta, \\ & \Delta = k_1 \beta_1 (\alpha_2 - \alpha_3) + k_2 \beta_2 (\alpha_3 - \alpha_1) + k_3 \beta_3 (\alpha_1 - \alpha_2), \end{aligned}$$

that satisfies the conditions

$$\{\mathbf{Z}_c(0)\} = \left(\gamma_a U_0'(e), \gamma_b \Theta_0'(e), 0\right)^T, \quad \left\{\mathbf{Z}_c'(0)\right\} = \left(0, 0, \gamma_b \Theta_0'(e)\right)^T. \tag{23}$$

So, it is easily to verify that solution (18) for free vibration of integrated piezoelectric FGM beam satisfying the conditions at crack (14) can be represented as

$$\{\mathbf{Z}(x,\omega)\} = \begin{cases} \{\mathbf{Z}_0(x,\omega)\} & : \text{for } x < e, \\ \{\mathbf{Z}_0(x,\omega)\} + \{\mathbf{Z}_c(x - e, \omega)\} & : \text{for } e \leq x, \end{cases}$$

that is rewritten in the form

$$\{\mathbf{Z}(x,\omega)\} = \{\mathbf{Z}_0(x,\omega)\} + [\mathbf{K}(x - e)]\{\mathbf{Z}_0'(e,\omega)\} = [\mathbf{\Phi}(x,\omega)]\{C\}, \tag{24}$$

with the matrices introduced

$$[\mathbf{\Phi}(x,\omega)] = \left[\mathbf{G}_0(x,\omega) + \mathbf{K}(x - e)\,\mathbf{G}_0'(x,\omega)\right],$$

$$[\mathbf{K}(x)] = \begin{cases} [\mathbf{G}_c(x)] & : x > 0, \\ [\mathbf{0}] & : x \leq 0, \end{cases} \quad [\mathbf{K}'(x)] = \begin{cases} [\mathbf{G}_c'(x)] & : x > 0, \\ [\mathbf{0}] & : x \leq 0. \end{cases} \tag{25}$$

Thus, expression (24) is general solution for free vibration of cracked FGM piezoelectric beam that would be determined seeking constant vector $\{C\} = (C_1, \ldots, C_6)^T$ from specifically given boundary conditions. For example, in case of simply supported beam with boundary conditions

$$U(0) = W(0) = M(0) = U(L) = W(L) = M(L) = 0,$$

where $M(x) = B_{12}^* \partial_x U(x) - B_{22}^* \partial_x \Theta(x)$, one gets

$$[\mathbf{G}(\omega)]\{C\} = 0, \tag{26}$$

where

$$[\mathbf{G}(\omega)] = [\mathbf{B}_{SS}(\omega)] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & -\beta_1 & -\beta_2 & -\beta_3 \\ m_1 & m_2 & m_3 & -m_1 & -m_2 & -m_3 \\ \phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\ \phi_{31}(L) & \phi_{32}(L) & \phi_{33}(L) & \phi_{34}(L) & \phi_{35}(L) & \phi_{36}(L) \\ M_1(L) & M_2(L) & M_3(L) & M_4(L) & M_5(L) & M_6(L) \end{bmatrix},$$

$$m_j = \left(B_{12}^*\alpha_j - B_{22}^*\right)k_j, \quad j = 1,2,3, \quad M_j(L) = B_{12}^*\phi_{1j}'(L) - B_{22}^*\phi_{2j}'(L), \quad j = 1,2,\ldots,6,$$

$\phi_{ij}(x), \phi_{ij}'(x), i = 1,2,3; j = 1,2,\ldots,6$ are elements of matrices $[\mathbf{\Phi}(x,\omega)]$ and $[\mathbf{\Phi}'(x,\omega)]$ defined in (26). Therefore, frequency equation of the beam is

$$\det[\mathbf{G}(\omega)] = 0, \tag{27}$$

positive roots of which give rise desired natural frequencies $\omega_1, \omega_2, \omega_3, \ldots$ of simply supported FGM beam with piezoelectric layer and cracks. As a consequence, the natural frequencies are dependent On crack parameters such as crack location e and depth a as well as material properties and piezoelectric layer thickness. The effect of latter factors on natural frequencies was studied in [39] and [37]. In this study, the natural frequencies are computed as function of crack parameters (e, a): $\omega_k = \omega_k(e, a), k = 1, 2, \ldots$, that would be employed as a database for crack detection from measured natural frequencies.

Now, we are going to find the solution of the inhomogeneous equation (8) that is acknowledged as the frequency response of the integrated beam subjected to a moving force. Let's consider the case of moving harmonic force $P(t) = P_0 \exp\{i\Omega_m t\}$ that gives rise

$$Q(x, \omega) = (P_0/v)\exp\{-i\Omega x\}, \quad \Omega = (\omega - \Omega_m)/v. \tag{28}$$

It is not difficult to find a particular solution $\mathbf{Z}_q(x, \omega)$ of Eq. (8)–(9)–(28) in the form

$$\mathbf{Z}_q(x, \omega) = \left\{U_q^0(\omega), \Theta_q^0(\omega), W_q^0(\omega)\right\}^T \exp\{-i\Omega x\}, \tag{29}$$

where

$$U_q^0(\omega) = (i\Omega)P_0 A_{33}^*\left(\Omega^2 B_{12}^* - \omega^2 I_{12}^*\right)/v\Delta, \quad \Theta_q^0(\omega) = (i\Omega)P_0 A_{33}^*\left(\omega^2 I_{11}^* - \Omega^2 B_{11}^*\right)/v\Delta,$$

$$W_q^0(\omega) = P_0 D/\Delta, \quad \Delta = \left(\omega^2 I_{11}^* - \Omega^2 A_{33}^*\right) D + i\Omega A_{33}^{*2} \left(\omega^2 I_{11}^* - \Omega^2 B_{11}^*\right),$$

$$D = \omega^4 \left(I_{11}^* I_{22}^* - I_{12}^{*2}\right) + \Omega^4 \left(B_{11}^* B_{22}^* - B_{12}^{*2}\right) + A_{33}^* \left(\Omega^2 B_{11}^* - \omega^2 I_{11}^*\right)$$
$$+ \omega^2 \Omega^2 \left(2 I_{12}^* B_{12}^* - I_{11}^* B_{22}^* - I_{22}^* B_{11}^*\right).$$

Therefore, general solution of Eq. (8) can be expressed as

$$\{\overline{Z}(x,\omega)\} = \{U(x,\omega), \Theta(x,\omega), W(x,\omega)\}^T = [\boldsymbol{\Phi}(x,\omega)]\{C\} + Z_q(x,\omega), \qquad (30)$$

where matrix $\boldsymbol{\Phi}(x,\omega)$ is defined in Eq. (25) and constant vector C is sought by given boundary conditions. Namely, for simply supported beam one can find

$$\{C\} = -\left[\mathbf{G}(\omega)\right]^{-1} \{\hat{P}(\omega)\}, \qquad (31)$$

with matrix $\mathbf{G}(\omega)$ defined in Eq. (27) and vector $\hat{P}(\omega) = \left\{\hat{P}_1(\omega), \ldots, \hat{P}_6(\omega)\right\}^T$ where

$$\hat{P}_1(\omega) = U_q^0(\omega), \quad \hat{P}_2(\omega) = -i\Omega \left[B_{12}^* U_q^0(\omega) - B_{22}^* \Theta_q^0(\omega)\right], \quad \hat{P}_3(\omega) = W_q^0(\omega),$$

$$\hat{P}_4(\omega) = -i\Omega \left[B_{11}^* U_q^0(\omega) - B_{12}^* \Theta_q^0(\omega)\right] \exp\{-i\Omega L\},$$

$$\hat{P}_5(\omega) = -i\Omega \left[B_{12}^* U_q^0(\omega) - B_{22}^* \Theta_q^0(\omega)\right] \exp\{-i\Omega L\}, \quad \hat{P}_6(\omega) = W_q^0(\omega) \exp\{-i\Omega L\}.$$

Thus, mechanical frequency response (30) for simply supported beams gets the form

$$\{\overline{Z}(x,\omega)\} = Z_q(x,\omega) - [\boldsymbol{\Phi}(x,\omega)][\mathbf{G}(\omega)]^{-1} \{\hat{P}(\omega)\}. \qquad (32)$$

Owning mechanical frequency response $\{\overline{Z}(x,\omega)\}$, the sensor output charge $\hat{Q}(\omega)$ can be calculated by

$$\hat{Q}(\omega) = \left(bh_{13}/\beta_{33}^p\right) \int_0^L \left[U'(x,\omega) - h\Theta'(x,\omega)/2\right] \mathrm{d}x$$
$$= \frac{bh_{13}}{\beta_{33}^p} \Big\{ \left[U(L,\omega) - U(0,\omega) - \gamma_1 U_x'(e,\omega)\right]$$
$$- (h/2) \left[\Theta(L,\omega) - \Theta(0,\omega) - \gamma_2 \Theta_x'(e,\omega)\right] \Big\}, \qquad (33)$$

where $U(x,\omega), \Theta(x,\omega)$ are components of solution (30) acknowledged as mechanical frequency response of cracked FGM piezoelectric beam subjected to moving load. As consequence, the frequency domain charge generated in the piezoelectric layer is called herein electrical frequency response of the beam. This is the basics for using the piezoelectric layer as a distibuted sensor and its output charge as a diagnostic signal for crack detection in FGM beam subjected to moving load.

4. SENSITIVITY OF ELECTRICAL FREQUENCY RESPONSE TO CRACK - NUMERICAL RESULTS

For sensitivity analysis of electrical frequency response, the sensor output charge $\hat{Q}(\omega)$, by using so-called spectral damage index defined for the responses of intact and cracked beams $\hat{Q}(\omega, e, a)$, $\hat{Q}^0(\omega)$ as [40]

$$SDI(e, a) = \sum_{k=1}^{N} \hat{Q}(\omega_k, e, a) \hat{Q}^0(\omega) \left/ \left[\sum_{k=1}^{N} \hat{Q}^2(\omega_k, e, a) \sum_{k=1}^{N} \hat{Q}^{02}(\omega_k) \right]^{1/2} \right. . \qquad (34)$$

The introduced above damage index lies between 0 and 1, which equals 1 only if the two frequency-dependent functions are fully similar. Hence, its deviation of unique represents a measure of effect of crack on the index and as usual it is acknowledged as sensitivity of the frequency response to crack. Note, the sensitivity represented by the spectral damage index depends also on the material and load parameters such as the material distribution index n, frequency and speed of the moving load Ω_m, v that are all useful for us to control the success of crack detection.

Thus, the spectral damage index is numerically examined herein with the following geometry and material constants:

$$L_b = L_p = L = 1 \text{ m}; b = 0.1 \text{ m}; h_b = L/10;$$

$$E_t = 390 \text{ MPa}; \rho_t = 3960 \text{ kg/m}^3; \mu_t = 0.25; E_b = 210 \text{ MPa}; \rho_b = 7800 \text{ kg/m}^3; \mu_b = 0.31;$$

$$C_{11}^p = 69.0084 \text{ GPa}; C_{55}^p = 21.0526 \text{ GPa}; \rho_p = 7750 \text{ kg/m}^3; h_{13} = -7.70394 \times 10^8 \text{ V/m}.$$
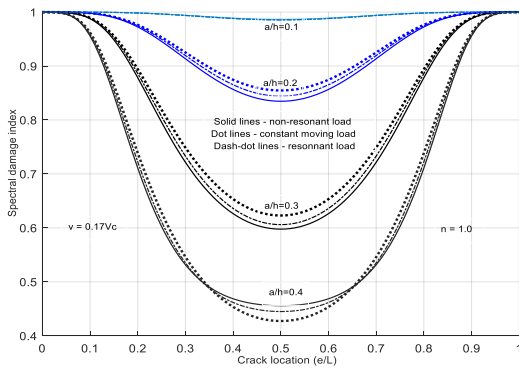

Fig. 2. Spectral damage index versus crack location in dependence on crack depth and moving load frequency
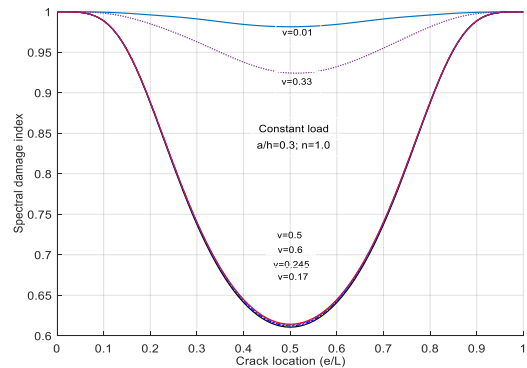

Fig. 3. Spectral damage index versus crack location in dependence on moving load speed (*v*)

In Figs. 2–5, there are depicted spectral damage index as function of crack location in dependence on crack depth and load frequency (Fig. 2), load speed (Fig. 3), material distribution index *n* (Fig. 4) and thickness of the piezoelectric layer (Fig. 5).
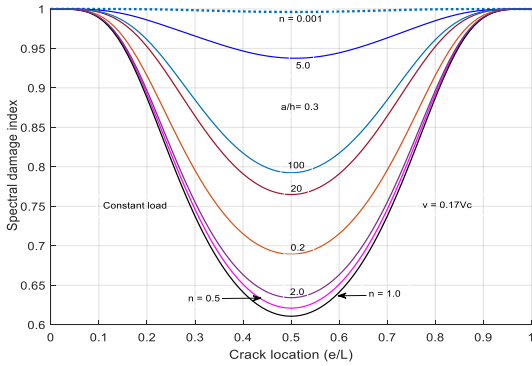

Fig. 4. Spectral damage index versus crack location in dependence on material distribution index (*n*)
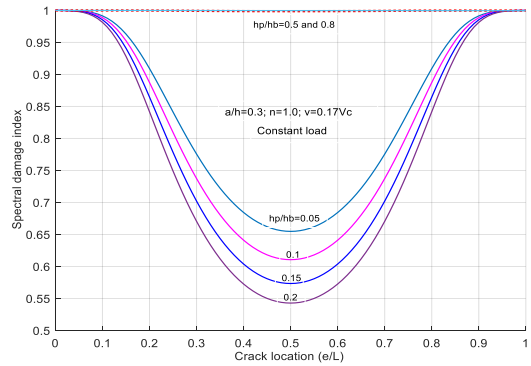
Fig. 5. Spectral damage index versus crack location in dependence on thickness of sensor layer (*hp*)

Observing graphs presented in the figures one can make the following discussion: (i) Change in the spectral damage index due to crack position and depth is similar to the crack-induced change in fundamental frequency with an exception that magnitude of the change in spectral damage index is larger than that of natural fundamental frequency. In the other words, it can be acknowledged that spectral damage index is significantly more sensitive to crack than the natural frequencies; (ii) The effect of moving load parameters such as their frequency and speed on the sensitivity of spectral damage index to crack is so slight that they may be difficult to employ for crack detection as it has done for homogeneous beams [40]; (iii) Sensitivity of SDI to crack is first increasing with material distribution index (*n*) from $n = 0$ to $n = 1.0$, then the sensitivity decreases until $n = 5$, from that it becomes again increasing. This means that dependence of the SDI sensitivity to crack on the FGM property is not monotonous, but it reaches maximum at the value $n = 1.0$; (iv) Sensitivity of SDI to crack is increasing with piezoelectric layer thickness until the thickness reaches critical value $h_p = 0.2h_b$. Further increase of the thickness from the critical value leads to decrease of SDI sensitivity to crack. This discussion is useful for designing crack detection procedure in FGM beams using moving load and piezoelectric distributed sensor.

5. CONCLUSIONS

The main results obtained in this study can be summarized as follow:

- A frequency domain model of cracked functionally graded beams bonded with a piezoelectric layer and subjected to a harmonic moving force has been established and it could be usefully employed for both analysis and identification of the composite structures.

- There has been introduced a novel damage index called spectral damage index for cracked functionally graded beams with piezoelectric layer. The index is easily calculated from the output charge produced in the piezoelectric layer acknowledged as an electrical frequency response to the moving load.

- Sensitivity of the electrical frequency response to crack has been examined in dependence On load parameters, FGM properties, and thickness of the piezoelectric layer used as a distributed sensor for measuring the frequency response of FGM beam subjected to moving load.

- The completed above theoretical development and numerical analysis provide useful information and instruction for solving the crack detection problem that is the subject of the next study for the authors.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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REFERENCES

[1] H. Sohn, C. R. Farrar, F. M. Hemez, and J. J. Czarnecki. *A review of structural health review of structural health monitoring literature 1996-2001*. Los Alamos National Laboratory Report, (2002).

[2] W. Fan and P. Qiao. Vibration-based damage identification methods: A review and comparative study. *Structural Health Monitoring*, **10**, (2010), pp. 83–111. https://doi.org/10.1177/1475921710365419.

[3] R. Hou and Y. Xia. Review on the new development of vibration-based damage identification for civil engineering structures: 2010–2019. *Journal of Sound and Vibration*, **491**, (2021). https://doi.org/10.1016/j.jsv.2020.115741.

[4] S. S. Rao and M. Sunar. Piezoelectricity and its use in disturbance sensing and control of flexible structures: A survey. *Applied Mechanics Reviews*, **47**, (1994), pp. 113–123. https://doi.org/10.1115/1.3111074.

[5] G. Park, H. Sohn, C. R. Farrar, and D. J. Inman. Overview of piezoelectric impedance-based health monitoring and path forward. *The Shock and Vibration Digest*, **35**, (2003), pp. 451–463. https://doi.org/10.1177/05831024030356001.

[6] V. Giurgiutiu. *Structural health monitoring with piezoelectric wafer active sensors*. Academic Press, Oxford, (2007).

[7] W. H. Duan, Q. Wang, and S. T. Quek. Applications of piezoelectric materials in structural health monitoring and repair: Selected research examples. *Materials*, **3**, (2010), pp. 5169–5194. https://doi.org/10.3390/ma3125169.

[8] H. S. Tzou and C. I. Tseng. Distributed piezoelectric sensor/actuator design for dynamic measurement/control of distributed parameter systems: A piezoelectric finite element approach. *Journal of Sound and Vibration*, **138**, (1990), pp. 17–34. https://doi.org/10.1016/0022-460x(90)90701-z.

[9] E. F. Crawley and J. de Luis. Use of piezoelectric actuators as elements of intelligent structures. *AIAA Journal*, **25**, (1987), pp. 1373–1385. https://doi.org/10.2514/3.9792.

[10] H. A. Winston, F. Sun, and B. S. Annigeri. Structural health monitoring with piezoelectric active sensors. In *Volume 4: Manufacturing Materials and Metallurgy; Ceramics; Structures and Dynamics; Controls, Diagnostics and Instrumentation; Education*, American Society of Mechanical Engineers, GT2000, (2000). https://doi.org/10.1115/2000-gt-0051.

[11] S. Bhalla and C.-K. Soh. Progress in structural health monitoring and non-destructive evaluation using piezo-impedance transducers. In *Smart Materials and Structures: New Research, Nova Science Publishers, Inc.*, New York, (2007), pp. 177–228.

[12] G. Huang, F. Song, and X. Wang. Quantitative modeling of coupled piezo-elastodynamic behavior of piezoelectric actuators bonded to an elastic medium for structural health monitoring: A review. *Sensors*, **10**, (2010), pp. 3681–3702. https://doi.org/10.3390/s100403681.

[13] Z. Shu. Application of piezoelectric materials in structural health monitoring of civil engineering structure. *Chemical Engineering Transactions*, **59**, (2017), pp. 523–528. https://doi.org/10.3303/CET1759088.

[14] W. S. Na and J. Baek. A review of the piezoelectric electromechanical impedance based structural health monitoring technique for engineering structures. *Sensors*, **18**, (2018). https://doi.org/10.3390/s18051307.

[15] Q. Wang and S. T. Quek. Flexural vibration analysis of sandwich beam coupled with piezoelectric actuator. *Smart Materials and Structures*, **9**, (1), (2000). https://doi.org/10.1088/0964-1726/9/1/311.

[16] Q. Wang and S. T. Quek. Enhancing flutter and buckling capacity of column by piezoelectric layers. *International Journal of Solids and Structures*, **39**, (2002), pp. 4167–4180. https://doi.org/10.1016/s0020-7683(02)00334-7.

[17] Q. Wang, W. H. Duan, and S. T. Quek. Repair of notched beam under dynamic load using piezoelectric patch. *International Journal of Mechanical Sciences*, **46**, (2004), pp. 1517–1533. https://doi.org/10.1016/j.ijmecsci.2004.09.012.

[18] S. Zhao, N. Wu, and Q. Wang. Crack identification through scan-tuning of vibration characteristics using piezoelectric materials. *Smart Materials and Structures*, **26**, (2016). https://doi.org/10.1088/1361-665x/aa520a.

[19] S. Ritdumrongkul and Y. Fujino. Identification of the location and size of cracks in beams by a piezoceramic actuator–sensor. *Structural Control and Health Monitoring*, **14**, (6), (2007), pp. 931–943. https://doi.org/10.1002/stc.204.

[20] D. Wang, H. Song, and H. Zhu. Electromechanical impedance analysis on piezoelectric smart beam with a crack based on spectral element method. *Mathematical Problems in Engineering*, **2015**, (2015), pp. 1–13. https://doi.org/10.1155/2015/713501.

[21] T. Wang, B. Tan, M. Lu, Z. Zhang, and G. Lu. Piezoelectric electro-mechanical impedance (EMI) based structural crack monitoring. *Applied Sciences*, **10**, (2020). https://doi.org/10.3390/app10134648.

[22] X.-F. Li. A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams. *Journal of Sound and Vibration*, **318**, (2008), pp. 1210–1229. https://doi.org/10.1016/j.jsv.2008.04.056.

[23] S. A. Sina, H. M. Navazi, and H. Haddadpour. An analytical method for free vibration analysis of functionally graded beams. *Materials & Design*, **30**, (2009), pp. 741–747. https://doi.org/10.1016/j.matdes.2008.05.015.

[24] L. O. Larbi, A. Kaci, M. S. A. Houari, and A. Tounsi. An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams. *Mechanics Based Design of Structures and Machines*, **41**, (2013), pp. 421–433. https://doi.org/10.1080/15397734.2013.763713.

[25] H. Su and J. R. Banerjee. Development of dynamic stiffness method for free vibration of functionally graded Timoshenko beams. *Computers & Structures*, **147**, (2015), pp. 107–116. https://doi.org/10.1016/j.compstruc.2014.10.001.

[26] X. Wang, X. Liang, and C. Jin. Accurate dynamic analysis of functionally graded beams under a moving point load. *Mechanics Based Design of Structures and Machines*, **45**, (2016), pp. 76–91. https://doi.org/10.1080/15397734.2016.1145060.

[27] J. Yang and Y. Chen. Free vibration and buckling analyses of functionally graded beams with edge cracks. *Composite Structures*, **83**, (2008), pp. 48–60. https://doi.org/10.1016/j.compstruct.2007.03.006.

[28] Ş. D. Akbaş. Free vibration characteristics of edge cracked functionally graded beams by using finite element method. *International Journal of Engineering Trends and Technology*, **4**, (10), (2013), pp. 4590–4597.

[29] K. Aydin. Free vibration of functionally graded beams with arbitrary number of surface cracks. *European Journal of Mechanics - A/Solids*, **42**, (2013), pp. 112–124. https://doi.org/10.1016/j.euromechsol.2013.05.002.

[30] N. T. Khiem, H. T. Tran, and D. Nam. Modal analysis of cracked continuous Timoshenko beam made of functionally graded material. *Mechanics Based Design of Structures and Machines*, **48**, (2019), pp. 459–479. https://doi.org/10.1080/15397734.2019.1639518.

[31] Z. Yu and F. Chu. Identification of crack in functionally graded material beams using the p-version of finite element method. *Journal of Sound and Vibration*, **325**, (2009), pp. 69–84. https://doi.org/10.1016/j.jsv.2009.03.010.

[32] A. Banerjee, B. Panigrahi, and G. Pohit. Crack modelling and detection in Timoshenko FGM beam under transverse vibration using frequency contour and response surface model with GA. *Nondestructive Testing and Evaluation*, **31**, (2015), pp. 142–164. https://doi.org/10.1080/10589759.2015.1071812.

[33] N. T. Khiem and N. N. Huyen. A method for crack identification in functionally graded Timoshenko beam. *Nondestructive Testing and Evaluation*, **32**, (2016), pp. 319–341. https://doi.org/10.1080/10589759.2016.1226304.

[34] M. K. Khorramabadi and A. R. Nezamabadi. Stability of functionally graded beams with piezoelectric layers based on the first order shear deformation theory. *International Journal of Electrical and Computer Engineering*, **4**, (11), (2010), pp. 1641–1644.

[35] Y. S. Li, W. J. Feng, and Z. Y. Cai. Bending and free vibration of functionally graded piezoelectric beam based on modified strain gradient theory. *Composite Structures*, **115**, (2014), pp. 41–50. https://doi.org/10.1016/j.compstruct.2014.04.005.

[36] K. Bendine, F. B. Boukhoulda, M. Nouari, and Z. Satla. Active vibration control of functionally graded beams with piezoelectric layers based on higher order shear deformation theory. *Earthquake Engineering and Engineering Vibration*, **15**, (2016), pp. 611–620. https://doi.org/10.1007/s11803-016-0352-y.

[37] N. T. Khiem, T. T. Hai, and L. Q. Huong. Effect of piezoelectric patch on natural frequencies of Timoshenko beam made of functionally graded material. *Materials Research Express*, **7**, (2020). https://doi.org/10.1088/2053-1591/ab8df5.

[38] N. T. Khiem, D. T. Huan, and T. T. Hieu. Vibration of cracked FGM beam with piezoelectric layer under moving load. *Journal of Vibration Engineering & Technologies*, **11**, (2022), pp. 755–769. https://doi.org/10.1007/s42417-022-00607-8.

[39] N. T. Khiem, N. N. Huyen, and N. T. Long. Vibration of cracked Timoshenko beam made of functionally graded material. In *Conference Proceedings of the Society for Experimental Mechanics Series*, Springer International Publishing, (2017), pp. 133–143. https://doi.org/10.1007/978-3-319-54735-0_15.

[40] T. T. Hai, P. T. Hang, and N. T. Khiem. A novel criterion for crack identification in beam-like structures using distributed piezoelectric sensor and controlled moving load. *Journal of Sound and Vibration*, **572**, (2024). https://doi.org/10.1016/j.jsv.2023.118155.