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THREE VERSIONS OF GALERKIN'S METHOD APPLIED TO THE STATIC DEFLECTION OF A STEPPED BEAM

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Abstract. This work deals with the application of Galerkin's method for stepped structures to evaluate the static deflection under distributed loading. In this study, we compare two different implementations of the well-known method to the exact analytical result in order to prove that only the second method is able to give a good approximation to the solution of the problem.

Keywords: stepped beams, Galerkin's method, generalized functions.

1. INTRODUCTION

The Galerkin [1] method is a century-old celebrated numerical method to solve differential equations and boundary value problems. The theoretical aspects of this method were treated by Mikhlin [2] and Leipholz [3–6]. Singer [7] established its equivalence to the Rayleigh–Ritz method for conservative problems (see also the paper by Bailey [8]). Gander and Wanner [9] provided detailed discussion of the method whereas the centennial of the method along with the importance of this algorithm was highlighted by Repin [10]. Bastatsky and Khvoles [11] pioneered another avenue in dealing with implementation of Galerkin's method to stepped structures. The interested reader can consult also with papers by Vainberg, D.V., Roitfarb [12] and Maurini et al. [13]. Elishakoff et al. [14–16] and Amato et al. [17] pursued the investigation along the ideas of Ref [11]. They applied the method in order to find the eigenvalues of stepped elastic structure, i.e., the natural frequencies. The authors show that the method fails when applied in its original manner to stepped structures: Generalized functions are required.

The study of stepped structure is a topic of interest for lightweight vehicles and for airplane in general. Application of Galerkin's method to aerospace problems was studied by Avalos et al. [18], Toulorge and Desmet [19], Raju and Phillips [20], Leissa et al. [21], Helenbrook and Atkins [22], Blonigan et al. [23] in various contexts.

By analyzing the static deflection of a stepped beam under uniform loading, we show how a simple implementation of Galerkin's method yields poor convergence to the exact solution whereas a modified version exhibits excellent convergence.

2. PROBLEM STATEMENT

We consider the static deflection of the uniformly loaded Bernoulli–Euler beam shown in Fig. 1 which is simply supported at both end points. The beam is composed of three segments occupying the domains (1) $0 < x < 9a/2$, (2) $9a/2 < x < 11a/2$ and (3) $11a/2 < x < 10a$. Segments 1 and 3 are identical with bending stiffness D_1 while in segment 2 the bending stiffness is D_2. All three segments are of circular cross section where segments 1 and 3 have radius r and segment 2 has radius $r/2$. Assuming that all segments have the same Young's modulus, it follows that $D_1 = 16D_2$.

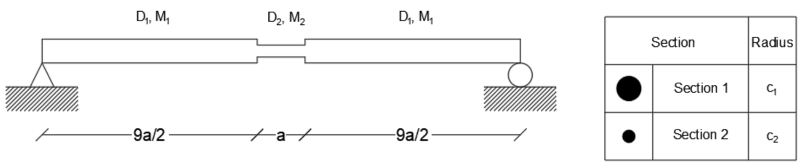

Fig. 1. Three-segment stepped beam

3. EXACT SOLUTION

Assuming a uniform loading per unit length q_0, the governing equations for the static deflections w_i in the ith segment are given by

$$16D_2 \frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} = q_0, \quad D_2 \frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} = q_0, \quad 16D_2 \frac{\mathrm{d}^4 w_3}{\mathrm{d}x^4} = q_0. \tag{1}$$

Straightforward integration yields

$$\begin{aligned}
w_1 &= c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{q_0}{384 D_2} x^4, \\
w_2 &= c_5 + c_6 x + c_7 x^2 + c_8 x^3 + \frac{q_0}{24 D_2} x^4, \\
w_3 &= c_9 + c_{10} x + c_{11} x^2 + c_{12} x^3 + \frac{q_0}{384 D_2} x^4,
\end{aligned} \tag{2}$$

where $c_1, c_2, \ldots, c_{12}$ are constants of integration which are determined from the boundary conditions and continuity requirements at the segment joints.

The simply supported end conditions require

$$w_1(0) = w_1{}''(0) = 0, \quad w_3(10a) = w_3{}''(10a) = 0. \tag{3}$$

Continuity of displacement at joints

$$w_1\left(9a/2\right) = w_2\left(9a/2\right) \quad \text{and} \quad w_2\left(11a/2\right) = w_3\left(11a/2\right). \tag{4}$$

Continuity of slope at joints

$$w_1'(9a/2) = w_2'(9a/2) \quad \text{and} \quad w_2'(11a/2) = w_3'(11a/2). \tag{5}$$

Continuity of bending moment at joints

$$16D_2 w_1''(9a/2) = D_2 w_2''(9a/2) \quad \text{and} \quad D_2 w_2''(11a/2) = 16D_2 w_3''(11a/2). \tag{6}$$

Continuity of shear force at joints

$$16D_2 w_1'''(9a/2) = D_2 w_2'''(9a/2) \quad \text{and} \quad D_2 w_2'''(11a/2) = 16D_2 w_3'''(11a/2). \tag{7}$$

Applying the constraints (3)–(7) to the deflection (2) we arrive at the set of linear equations

$$\begin{gathered}
c_1 = c_3 = 0, \\
c_9 + 10ac_{10} + 100a^2c_{11} + 1000a^3c_{12} + \frac{625a^4q_0}{24D_2} = 0, \\
c_{11} + 30ac_{12} + \frac{25a^2q_0}{16D_2} = 0, \\
c_1 + \frac{9a}{2}c_2 + \frac{81a^2}{4}c_3 + \frac{729a^3}{8}c_4 - c_5 - \frac{9a}{2}c_6 - \frac{81a^2}{4}c_7 - \frac{729a^3}{8}c_8 - \frac{32805a^4q_0}{2048D_2} = 0, \\
c_5 + \frac{11a}{2}c_6 + \frac{121a^2}{4}c_7 + \frac{1331a^3}{8}c_8 - c_9 - \frac{11a}{2}c_{10} - \frac{121a^2}{4}c_{11} - \frac{1331a^3}{8}c_{12} + \frac{73205a^4}{2048D_2} = 0, \\
c_2 + 9ac_3 + \frac{243a^2}{4}c_4 - c_6 - 9ac_7 - \frac{243a^2}{4}c_8 - \frac{3645a^3q_0}{256D_2} = 0, \\
c_6 + 11ac_7 + \frac{363a^2}{4}c_8 - c_{10} - 11ac_{11} - \frac{363a^2}{4}c_{12} + \frac{6655a^3q_0}{256D_2} = 0, \\
32c_3 + 432ac_4 - 2c_7 - 27ac_8 = 0, \quad 2c_7 + 33ac_8 - 32c_{11} - 528ac_{12} = 0, \\
c_8 - 16c_4 = 0, \quad c_8 - 16c_{12} = 0.
\end{gathered} \tag{8}$$

Solving the above system of equations and inserting the results into (2), we obtain

$$\begin{aligned}
w_1(x) &= \frac{q_0}{768D_2}\left(6485a^3x - 40ax^3 + 2x^4\right) \quad (0 < x < 9a/2),\\
w_2(x) &= -\frac{q_0}{6144D_2}\left(579555a^4 - 256000a^3x + 5120ax^3 - 256x^4\right) \quad (9a/2 < x < 11a/2),\\
w_3(x) &= \frac{q_0}{768D_2}\left(44850a^4 - 2485a^3x - 40ax^3 + 2x^4\right) \quad (11a/2 < x < 10a).
\end{aligned} \tag{9}$$

For purposes of comparison later in this article, the deflection at the beam midpoint is given by

$$w_2(5a) = \frac{220445a^4q_0}{6144D_2} \cong 35.8797\frac{a^4q_0}{D_2}. \tag{10}$$

4. GALERKIN SOLUTION – NONSYMMETRIC STIFFNESS MATRIX

For a beam of variable stiffness $D(x)$ subject to a uniform loading q_0, the governing equation for the static deflection assumes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[D(x)\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right] = q_0. \tag{11}$$

We approximate the deflection as a series of comparison functions

$$w(x) \cong \sum_{i=1}^{n} c_i\psi_i(x) = \sum_{i=1}^{n} c_i \sin\frac{i\pi x}{10a}. \tag{12}$$

Rendering the error residual orthogonal to each comparison functions, we arrive at the system of equations

$$\mathbf{Kc} = \mathbf{q}, \tag{13}$$

where

$$\begin{aligned}
k_{ij} &= \int_0^{10a} \psi_i(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[D(x)\psi_j''(x)\right]\mathrm{d}x,\\
q_i &= q_0\int_0^{10a} \psi_i(x)\mathrm{d}x.
\end{aligned} \tag{14}$$

Employing the comparison functions (12) we find

$$q_i = \frac{10aq_0}{\pi}\frac{\left[1-(-1)^i\right]}{i}. \tag{15}$$

Recall that

$$D(x) = \begin{cases} 16D_2, & 0 < x < 9a/2 \\ D_2, & 9a/2 < x < 11a/2 \\ 16D_2, & 11a/2 < x < 10a \end{cases} \tag{16}$$

It follows from (14) that

$$k_{ij} = 16D_2 \int_0^{9a/2} \psi_i(x)\psi_j^{(4)}(x)\mathrm{d}x + D_2 \int_{9a/2}^{11a/2} \psi_i(x)\psi_j^{(4)}(x)\mathrm{d}x + 16D_2 \int_{11a/2}^{10a} \psi_i(x)\psi_j^{(4)}(x)\mathrm{d}x. \tag{17}$$

Performing the integrations, we obtain

$$k_{ij} = \begin{cases} \dfrac{3\pi^3 D_2 j^4}{400a^3(j^2-i^2)} \left[(i+j)\left(\sin\dfrac{11\pi}{20}(i-j) - \sin\dfrac{9\pi}{20}(i-j)\right)\right. \\ \qquad \left. +(i-j)\left(\sin\dfrac{9\pi}{20}(i+j) - \sin\dfrac{11\pi}{20}(i+j)\right)\right], & (i \neq j) \\ \dfrac{\pi^3 D_2 i^3}{4000a^3}\left[29\pi i - 15\sin\left(\dfrac{9\pi i}{10}\right) + 15\sin\left(\dfrac{11\pi i}{10}\right)\right], & (i = j) \end{cases} \tag{18}$$

5. GALERKIN SOLUTION – SYMMETRIC STIFFNESS MATRIX

Integrating by parts twice, and invoking the boundary conditions, it follows from Eq. (14) that

$$k_{ij} = \int_0^{10a} D(x)\psi''_i(x)\psi''_j(x)\mathrm{d}x. \tag{19}$$

Note that this results in a symmetric stiffness matrix. Performing the integration and simplifying, we obtain

$$k_{ij} = \begin{cases} \dfrac{3\pi^3 D_2 i^2 j^2}{400a^3(i^2-j^2)} \left[(i+j)\left(\sin\left(\dfrac{9\pi(i-j)}{20}\right) - \sin\left(\dfrac{11\pi(i-j)}{20}\right)\right)\right. \\ \qquad \left. +(i-j)\left(\sin\left(\dfrac{11\pi(i+j)}{20}\right) - \sin\left(\dfrac{9\pi(i+j)}{20}\right)\right)\right], & (i \neq j) \\ \dfrac{\pi^3 D_2 i^3}{4000a^3}\left[29\pi i - 15\sin\left(\dfrac{9\pi i}{10}\right) + 15\sin\left(\dfrac{11\pi i}{10}\right)\right], & (j = i) \end{cases} \tag{20}$$

6. GALERKIN SOLUTION EMPLOYING GENERALIZED FUNCTIONS

In this section we represent the variable bending stiffness in terms of the genialized Heaviside function and perform the necessary differentiations without invoking integration by parts. Expanding the derivatives in Eq. (11) we obtain

$$D(x)\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+2D'(x)\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}+D''(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=q_0. \tag{21}$$

For the problem under consideration,

$$\begin{aligned} D(x) &= 16D_2-15D_2H(x-9a/2)+15D_2H(x-11a/2),\\ D'(x) &= -15D_2\delta(x-9a/2)+15D_2\delta(x-11a/2),\\ D''(x) &= -15D_2\delta'(x-9a/2)+15D_2\delta'(x-11a/2), \end{aligned} \tag{22}$$

where $H(x), \delta(x)$ denotes the Heaviside step function and Dirac Delta functions respectively. Employing the approximation (12), and rendering the error residual in (21) orthogonal to each comparison function, we arrive at the equations

$$\sum_{j=1}^{n}\left(\alpha_{ij}+2\beta_{ij}+\gamma_{ij}\right)c_j=\mu_i \quad (i=1,2,\ldots,n), \tag{23}$$

where

$$\begin{aligned} \alpha_{ij} &= \int_0^{10a}\psi_i(x)D(x)\psi_j^{(4)}\mathrm{d}x,\\ \beta_{ij} &= \int_0^{10a}\psi_i(x)D'(x)\psi_j^{(3)}\mathrm{d}x,\\ \gamma_{ij} &= \int_0^{10a}\psi_i(x)D''(x)\psi_j^{(2)}\mathrm{d}x,\\ \mu_i &= q_0\int_0^{10a}\psi_i(x)\mathrm{d}x. \end{aligned} \tag{24}$$

Performing the integrations, we obtain

$$\alpha_{ij}=\begin{cases} \dfrac{3\pi^3D_2j^4}{400a^3(i^2-j^2)}\left[(i+j)\sin\left(\dfrac{9\pi(i-j)}{20}\right)-(i+j)\sin\left(\dfrac{11\pi(i-j)}{20}\right)\right.\\ \qquad\left.+(i-j)\sin\left(\dfrac{11\pi(i+j)}{20}\right)-(i-j)\sin\left(\dfrac{9\pi(i+j)}{20}\right)\right], & (i\neq j)\\ \dfrac{\pi^3D_2i^3}{400a^3}\left(29\pi i-15\sin\left(\dfrac{9\pi i}{10}\right)+15\sin\left(\dfrac{11\pi i}{10}\right)\right), & (j=i) \end{cases} \tag{25}$$

$$\beta_{ij}=\frac{3\pi^3D_2j^3}{200a^3}\left[\sin\left(\frac{9\pi i}{20}\right)\cos\left(\frac{9\pi j}{20}\right)-\sin\left(\frac{11\pi i}{20}\right)\cos\left(\frac{11\pi j}{20}\right)\right], \tag{26}$$

$$\begin{aligned}\gamma_{ij}=\frac{3\pi^3D_2j^2}{200a^3}&\left[j\sin\left(\frac{11\pi i}{20}\right)\cos\left(\frac{11\pi j}{20}\right)-j\sin\left(\frac{9\pi i}{20}\right)\cos\left(\frac{9\pi j}{20}\right)\right.\\ &\left.+i\cos\left(\frac{11\pi i}{20}\right)\sin\left(\frac{11\pi j}{20}\right)-i\cos\left(\frac{9\pi i}{20}\right)\sin\left(\frac{9\pi j}{20}\right)\right], \end{aligned} \tag{27}$$

$$\mu_i = \frac{10aq_0}{\pi} \frac{1-(-1)^i}{i}. \tag{28}$$

It can be shown that $\alpha_{ij} + 2\beta_{ij} + \gamma_{ij} = k_{ij}$ and $\mu_i = q_i$ (see Eqs. (20) and (15)). Hence for this problem, the classical Galerkin method employing a symmetric stiffness matrix is equivalent to Galerkin's method in which the beam bending stiffness is represented in terms of generalized functions.

7. NUMERICAL RESULTS

For both Galerkin solutions, it turns out that $c_{2n} = 0\,(n = 1, 2, \ldots)$. Therefore, when we speak about an approximation based on n terms, it is understood that each of these terms are not identically zero. Fig. 2 shows plots of the non-dimensional deflection based on the first Galerkin solution (nonsymmetric stiffness matrix) for 5, 10 and 40 trial functions along with the exact solution. Similarly, Fig. 3 corresponds to the second Galerkin solution (symmetric stiffness matrix). Clearly, superior convergence is obtained with the second solution. This is not surprising since the first method invokes the 4$^{\text{th}}$ derivative of the comparison functions while the second method only requires 2$^{\text{nd}}$ derivatives.

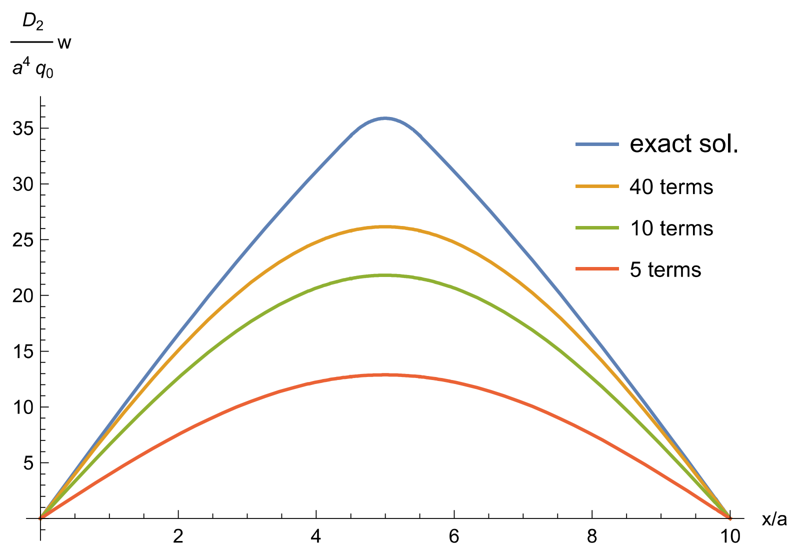

Fig. 2. Deflection under uniform loading - Galerkin method

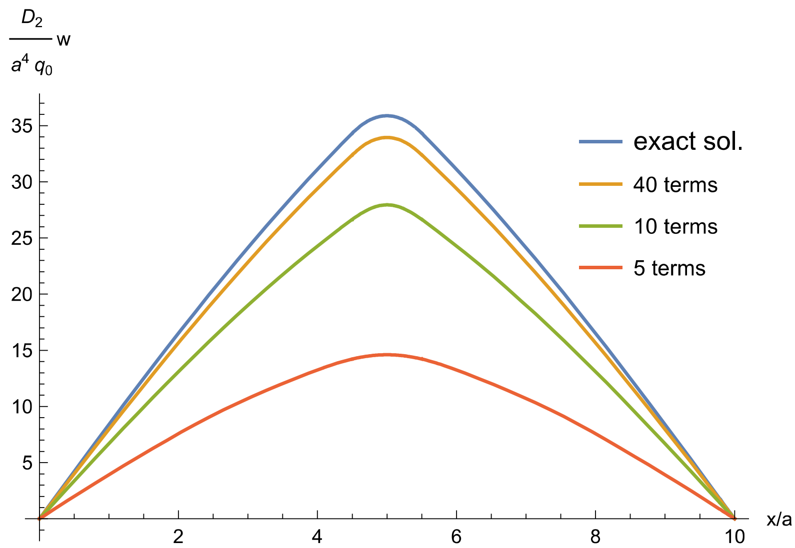

Fig. 3. Deflection under uniform loading - modified Galerkin method

Table 1 shows the nondimensional midpoint deflection of the beam for the two Galerkin solutions along with the percentage error as a function of the number of terms (n) retained in the expansion. Here we see quantitatively the advantage of the second approach based on the symmetric stiffness matrix.

Table 1. Nondimensional Deflection at Midpoint

n	Galerkin	%Error	Modified Galerkin	%Error
5	12.88	64.1	14.60	59.3
10	21.81	39.2	27.95	22.1
20	24.71	31.1	31.94	10.9
40	26.16	27.1	33.95	5.4
80	26.88	25.1	34.93	2.6
100	27.02	24.7	35.13	2.1
150	27.21	24.2	35.38	1.4
200	27.30	23.9	35.51	1.0

In conclusion, Fig. 4 depicts the relative error of the two considered method with respect to the exact solution when increasing the number of terms in Galerkin expansion.

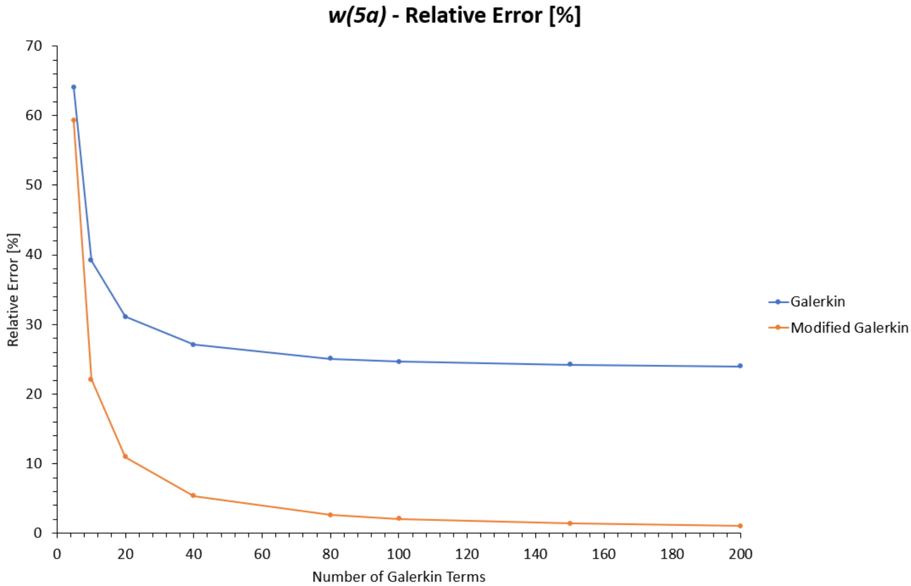

Fig. 4. Percent relative error at beam midpoint

8. CONCLUSION

This work demonstrates that a naive application of Galerkin's method to the problem of the static deflection of a uniformly loaded stepped beam can produce erroneous results. Integration by parts, resulting in a symmetric stiffness matrix, yields a sequence of approximations which converges to the exact solution. It is also shown that this modified method is equivalent to a straightforward application of Galerkin's method which employs symbolic functions to represent the discontinuous bending stiffness.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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