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INFLUENCE OF CHIRALITY ON BUCKLING OF INEXTENSIBLE RINGS SUBJECT TO DEAD LOADING

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Abstract. A variational approach is studied to understand buckling of inextensible rings made from chiral filaments and subject to dead loading. In opposite to previous literatures in which only in-plane bifurcation is allowed, i.e., the ring deforms only in its plane, in this work we consider both in-plane and out-of-plane bifurcation, i.e., the ring deforms both in its plane and out of its plane. For circular rings made from filaments without chirality, we find that they lose instability via out-of-plane bifurcation at critical values of loading smaller than those published. For circular rings made from filaments with chirality, they lose instability via coupling between in-plane and out-of-plane bifurcation at critical values of loading smaller than those at which bifurcation would happen without chirality. The destabilizing effect of chirality, however, can be reduced by increasing the twisting rigidity relative to the bending rigidity.

Keywords: elastic ring, special Cosserat rod, buckling, chirality, dead loading.

1. INTRODUCTION

Buckling of elastic rings is a traditional buckling problem starting with the pioneering work of Maurice Lévy [1] in buckling of an infinitely long cylindrical shell subject to pressure. The buckling of such rings has many applications ranging from the nano-scale up to the macro-scale, including designing nanorings in nanostructured devices [2], designing cylindrical shells in thin-walled structures such as boilers, pressure vessels [3], and various other settings. For a recent review of this problem, readers are advised to refer to Chapter 3 of [4] and references therein.

Buckling of elastic rings subject to three types of loadings including hydrostatic pressure, centrally directed loading, and dead loading is an important issue from the structure design point of view since complex loadings, such as air blast loadings in explosion, can be decomposed into these types of loadings for practical reason, before applying to structural analysis. Buckling of elastic rings subject to hydrostatic pressure, which remains normal to the tangent of the midline of the ring in its deformed configuration, is well studied [5–10] and has recently gained new interests in many physical and biological systems. A prominent example is the vertebrate gut tube anchored by the dorsal mesenteric membrane which creates a force responsible for chirality and looping of the gut tube [11]. Another example is the helical protein belts spanned by the lipid bilayers, which might have both flat and saddle shapes [12]. Inspired by experiments of rings made from fishing line and spanned by soapy water, Giomi & Mahadevan [13] numerically characterized the shapes of the rings and subsequently Chen & Fried [14] theoretically studied the stability and bifurcation of the circular configurations. Later, Fried et al. [15–18] showed that elastic rings made from filaments with a spontaneous twist, or a spontaneous curvature, or noncircular cross sections might buckle out of their plane at critical values of surface tension of the soapy water. Buckling of elastic rings subject to central loading, which remains directed toward the initial center of curvature of the rings, is also well considered [8–10, 19–21] and is experimentally realized in a recent work [22]. However, in-plane bifurcation is only considered in all previous literatures. Motivated by the work [23] on buckling of viscoelastic rings under central loadings which demonstrates that twist buckling dominates the instability, Hoang [24] considered both in-plane and out-of-plane bifurcation of rings subject to central loadings. He showed that elastic rings lose instability via out-of-plane bifurcation at critical values of loading smaller than those at which instability occurs via in-plane bifurcation and thus demonstrated the importance of out-of-plane bifurcation in bifurcation analysis of elastic rings. Buckling of elastic rings subject to dead loading, which remains normal to the tangent of the midline of the ring in its undeformed configuration, in contrast, is less explored [20, 21] and additionally only in-plane bifurcation are included, even in the recent work [22]. In this study, we allow for both in-plane and out-of-plane bifurcation and show that elastic rings subject to dead loadings buckle via out-of-plane bifurcation at critical values of loading lower than those at which buckling would occur if only in-plane bifurcation is included.

Chirality or "handedness", defined as the loss of mirror symmetry, was seen in various structures of different scales, from DNA to the horns of animals. Chirality of filaments produces many interesting phenomena including but not limited to the coupling between twist and stretch or between twist and bend of the filaments. While the twist-stretch coupling is well studied [25–29], the twist-bend coupling has only been explored in [24, 30] although the latter was introduced before the former [31–33]. Inspired by the work [34] in which the twist-bend coupling is used to understand kink instability in DNA rings, we explore the effect of this coupling on the buckling of elastic rings subject to dead loadings. From now on, the twist-bend coupling is implicitly understood when we refer to the chirality of filaments. We find that elastic rings made from filaments with chirality exhibit coupling between in-plane and out-of-plane bifurcation at critical values of magnitude of loading lower than those at which buckling would occur with no chirality.

In this work, we use a variational approach to study bifurcation of inextensible chiral elastic rings subject to dead loading. This approach is relatively straightforward in deriving the equilibrium and buckling conditions in closed forms and is suitable for buckling problems under static loading. The variational approach, however, is not applicable for buckling problems under dynamic loading; instead, the dynamic stability method or the numerical method should be applied for these problems.

This paper is structured as follows. Section 2 contains fundamental assumptions concerning kinematics and energetics of the inextensible rings. The general equilibrium conditions and their linearized equations for instability to occur appear in Section 3 and Section 4, respectively. Section 5 summaries our most important results.

2. BACKGROUND

We study an elastic ring compressed by a dead loading. The ring is made from an inextensible, unshearable, and chiral filament with centerline C. We assume that the curve C has no self-contact. Moreover, the ring is compressed by a dead loading of constant magnitude σ, depicted schematically in Fig. 1 and its direction remains parallel to its original direction and directs toward the initial radial direction of the ring.

2.1. Kinematics

We parameterize C as $\boldsymbol{r} = \boldsymbol{r}(s)$ where s is the arclength parametrization. To facilitate our analysis, we measure distance relative to the ring radius R. The positional vector of any material point on C of the undeformed ring is, thus

$$\boldsymbol{r} = \boldsymbol{e}_r(\theta), \qquad 0 \leq \theta \leq 2\pi,$$

where $\boldsymbol{e}_r$ is the unit vector in the radial direction and $\theta = R^{-1}s$ is the azimuthal angle of polar coordinates on the unit circle. Denoting u, v, w be the (dimensionless) displacements along tangent, radial, and transverse directions respectively, the positional vector of a material point on C of the deformed ring is

$$\boldsymbol{\xi} = (1+v)\boldsymbol{e}_r + u\boldsymbol{e}_\theta + w\boldsymbol{e}_3, \qquad \boldsymbol{e}_3 = \boldsymbol{e}_r \times \boldsymbol{e}_\theta, \tag{1}$$

where $\boldsymbol{e}_\theta$ is the unit tangent vector to C of the undeformed ring. Periodicity then requires u, v, w and their relevant derivatives are 2π–periodic functions of θ.

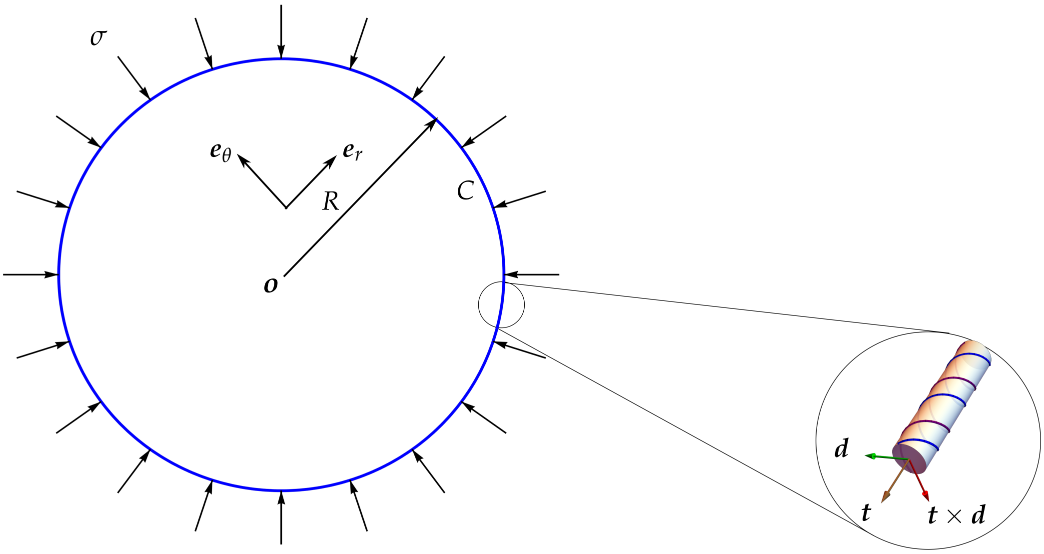


Fig. 1. A ring of radius R made from a chiral filament with centerline $C = \{\boldsymbol{r} : \boldsymbol{r} = R\boldsymbol{\xi}(R\theta), 0 \leq \theta \leq 2\pi\}$, endowed with a triad $\{\boldsymbol{t}, \boldsymbol{d}, \boldsymbol{t} \times \boldsymbol{d}\}$ of orthogonal directors, with $\boldsymbol{t}$ being tangent to C and $\boldsymbol{d}$ residing in the normal cross-section of the filament. The ring is compressed by a loading of constant magnitude σ and its direction always remains parallel to its original direction. A portion of the filament is zoomed in to show its chirality, illustrated here as "right-handed" chirality

We denote $\boldsymbol{t}$ a tangent vector to C, the inextensibility condition of C imposes

$$|\boldsymbol{t}| = |\boldsymbol{\xi}'| = 1, \tag{2}$$

with a prime being differentiation with respect to θ. Substitution of (1) into (2) yields the inextensibility in the form

$$v + u' + \frac{1}{2}\Big(v + u'\Big)^2 + \frac{1}{2}\Big(v' - u\Big)^2 + \frac{1}{2}w'^2 = 0. \tag{3}$$

The curvature and the torsion of C are, [35]

$$\kappa = |\boldsymbol{\xi}''|, \qquad \tau = \frac{\boldsymbol{\xi}' \times \boldsymbol{\xi}'' \cdot \boldsymbol{\xi}'''}{|\boldsymbol{\xi}''|^2}. \tag{4}$$

Substitution of (1) into (4) yields

$$\begin{cases} \kappa^2 = (1 + 2u' + v - v'')^2 + (u'' - u + 2v')^2 + w''^2, \\ \tau = w'(1 + u' - u''' - 2v'') + w''(v''' - 2v' - 3u'') + w'''(1 + v'' - u') + \text{h.o.t.} \end{cases} \tag{5}$$

We next introduce a unit director field $\boldsymbol{d}$ lying in the cross-section of the rod and admitting a representation

$$\boldsymbol{d} = \cos\psi\, \boldsymbol{p} + \sin\psi\, \boldsymbol{b}. \tag{6}$$

The twist density of C is, [35]

$$\omega = \boldsymbol{d} \times \boldsymbol{d}' \cdot \boldsymbol{t}. \tag{7}$$

From (6) and the Frenet-Serret relations,

$$\boldsymbol{t}' = \kappa \boldsymbol{p}, \qquad \boldsymbol{p}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \qquad \boldsymbol{b}' = -\tau \boldsymbol{p},$$

(7) yields

$$\omega = \tau + \psi' = w'(1 + u' - u''' - 2v'') + w''(v''' - 2v' - 3u'') + w'''(1 + v'' - u') + \psi'. \tag{8}$$

2.2. Energetics

For an inextensible, unshearable, and chiral rod, Healey [27] proved that the quadratic stored energy density function is

$$U = \frac{1}{2}\Big(|\boldsymbol{\kappa}|^2 + \alpha\,\omega^2 + 2\eta\,\omega(\boldsymbol{\kappa} \cdot \boldsymbol{d}) \Big), \tag{9}$$

where nondimensional parameters are [24]

$$\alpha = \frac{c}{a} > 0, \qquad \eta = \frac{e}{a}, \qquad \eta^2 \leq \alpha,$$

with a, c, e being, respectively, bending rigidity, twisting rigidity, and a bend-twist coupling coefficient due to the chirality of the ring and the vector curvature $\boldsymbol{\kappa}$ is given as

$$\boldsymbol{\kappa} = \boldsymbol{t}' = \kappa \boldsymbol{p}. \tag{10}$$

The total internal potential energy of the ring is, from (9) using (6) and (10),

$$V = \int_0^{2\pi} U = \int_0^{2\pi} \frac{1}{2}\Big(\kappa^2 + \alpha\,\omega^2 + 2\eta\,\omega\kappa \cos\psi \Big)\, \mathrm{d}\theta.$$

The (dimensionless) work done of the dead loading of constant magnitude σ is simply

$$W = -\nu \int_0^{2\pi} v \, \mathrm{d}\theta,$$

where (dimensionless) loading parameter is defined as

$$\nu = \frac{R^3 \sigma}{a} \geq 0.$$

The net potential energy of the chiral ring under compressive loadings is thus

$$E = V - W = \int_0^{2\pi} \frac{1}{2}\Big(\kappa^2 + \alpha\omega^2 + 2\eta\omega\kappa\cos\psi \Big)\, \mathrm{d}\theta + \nu \int_0^{2\pi} v \, \mathrm{d}\theta. \tag{11}$$

In the subsequent analysis, we find it convenient to treat the curvature κ and the twist density ω as independent variables constrained by $(5)_1$ and (8). So to incorporate these constraints as well as the constraint of inextensibility (3), we work with the augmented version of the dimensionless net potential energy (11),

$$\Phi = E + L, \tag{12}$$

where

$$
\begin{aligned}
L = & \int_0^{2\pi} \lambda \Big[v + u' + \frac{1}{2}\Big(v + u'\Big)^2 + \frac{1}{2}\Big(v' - u\Big)^2 + \frac{1}{2}w'^2 \Big] \, \mathrm{d}\theta \\
& + \int_0^{2\pi} \frac{1}{2}\rho \Big[\kappa^2 - (1 + 2u' + v - v'')^2 - (u'' - u + 2v')^2 - w''^2 \Big] \, \mathrm{d}\theta \\
& + \int_0^{2\pi} \vartheta \Big[\omega - w'(1+u'-u'''-2v'') - w''(v'''-2v'-3u'') - w'''(1+v''-u') - \psi' \Big] \mathrm{d}\theta,
\end{aligned} \tag{13}
$$

with λ, ρ, ϑ being Lagrange multipliers corresponding to the constraints.

3. EQUILIBRIUM CONDITIONS

Let $\boldsymbol{U} = \{\kappa, u, v, w, \omega, \psi, \lambda, \rho, \vartheta\}$ and $\delta\boldsymbol{U} = \{\delta\kappa, \delta u, \delta v, \delta w, \delta\omega, \delta\psi, \delta\lambda, \delta\rho, \delta\vartheta\}$ be state variables and their admissible variation. Taking the first variation of terms in (12), we obtain

$$
\delta V = \int_0^{2\pi} \left[(\kappa + \eta\, \omega \cos\psi)\delta\kappa + (\alpha\, \omega + \eta\, \kappa \cos\psi)\delta\omega - \eta\, \omega\kappa \sin\psi \delta\psi \right] \mathrm{d}\theta, \tag{14}
$$

and

$$
\delta W = -\nu \int_0^{2\pi} \big(\delta v + w \delta w \big) \, \mathrm{d}\theta. \tag{15}
$$

We similarly obtain the contributions of Lagrange multipliers in (13) to (12) which is used in conjunction with (14) and (15) to yield

$$
\begin{aligned}
\delta\Phi = & \int_0^{2\pi} \Big[\big((1+\rho)\kappa + \eta\omega\cos\psi\big)\delta\kappa + (\alpha\omega + \vartheta + \eta\kappa\cos\psi)\delta\omega + (\vartheta' - \eta\kappa\omega\sin\psi)\delta\psi \Big] \, \mathrm{d}\theta \\
& + \int_0^{2\pi} \big(I\,\delta u + J\,\delta v + K\,\delta w \big) \, \mathrm{d}\theta + \int_0^{2\pi} \Big[v + u' + \frac{1}{2}\Big(v+u'\Big)^2 + \frac{1}{2}\Big(v'-u\Big)^2 + \frac{1}{2}w'^2 \Big] \delta\lambda \, \mathrm{d}\theta \\
& + \int_0^{2\pi} \frac{1}{2} \Big[\kappa^2 - (1+2u'+v-v'')^2 - (u''-u+2v')^2 - w''^2 \Big] \delta\rho \, \mathrm{d}\theta \\
& + \int_0^{2\pi} \Big[\omega - w'(1+u'-u'''-2v'') - w''(v'''-2v'-3u'') - w'''(1+v''-u') - \psi' \Big] \delta\vartheta \, \mathrm{d}\theta,
\end{aligned} \tag{16}
$$

where

$$
\begin{cases}
I = \lambda(u - v') + \rho(u'' - u + 2v') + \big(\rho(u - u'' - 2v') + 3\vartheta w''\big)'' - \big(\vartheta w'\big)''' \\
\qquad - \big(\lambda(1 + v + u') - 2\rho(1 + v - v'' + 2u') - \vartheta(w' - w''')\big)', \\
J = \lambda(1+v+u') - \rho(1+2u'+v-v'') + \big(\rho(1 + 2u' + v - v'') + \vartheta(2w' - w''')\big)'' \\
\qquad + \big(\vartheta w''\big)''' + \big(\lambda(u - v') - 2\rho(u - u'' - 2v') - 2\vartheta w''\big)' + \nu, \\
K = \big(\vartheta(1 + v'' - u')\big)''' + \big(\vartheta(1 + u' - u''' - 2v'') - \lambda w'\big)' \\
\qquad + \big(\vartheta(3u'' + 2v' - v''') - \rho w''\big)''.
\end{cases}
$$

Setting to zero the first variation (16) then yields to the equilibrium conditions

$$\begin{cases} (1+\rho)\kappa + \eta\omega\cos\psi = 0, \\ \alpha\omega + \vartheta + \eta\kappa\cos\psi = 0, \\ (\alpha\omega + \eta\kappa\cos\psi)' + \eta\kappa\omega\sin\psi = 0, \\ I = J = K = 0, \end{cases} \tag{17}$$

and the constraints (3), $(5)_1$, (8).

4. LINEARIZED BIFURCATION ANALYSIS

Let us now study the buckling of an inextensible chiral elastic ring under dead loading. We are particularly interested in situations in which the filament has a (nondimensional) curvature $\kappa = 1$ and the cross section does not rotate about the centerline C. As shown later, we will see that the flat circular ring can buckle out of the plane of the ring even with zero twist density. For such situations, we have

$$\kappa = 1, \qquad u = v = w = 0, \qquad \omega = 0, \qquad \psi = 0. \tag{18}$$

With the help of (18), we see that the constraints (3), $(5)_1$, and (8) are satisfied trivially while the equilibrium conditions (17) are satisfied if Lagrange multipliers satisfy

$$\lambda = -(1+\nu), \qquad \rho = -1, \qquad \vartheta = -\eta. \tag{19}$$

We denote the state characterized by (18)–(19) the fundamental or trivial state $\boldsymbol{U}_0$. To study bifurcation, we need to seek the buckling solution $\boldsymbol{U}_1 = \{\kappa_1, u_1, v_1, w_1, \omega_1, \psi_1, \lambda_1, \rho_1, \vartheta_1\}$ satisfying the variational equation [36]

$$\Phi''\boldsymbol{U}_1\delta\boldsymbol{U} = 0.$$

Integration by parts while using the periodicity of $\boldsymbol{U}$, $\boldsymbol{U}_1$, $\delta\boldsymbol{U}$ and their higher derivatives, we arrive at

$$\Phi''\boldsymbol{U}_1\delta\boldsymbol{U} = \int_0^{2\pi} \Big[(\rho_1 + \eta\omega_1)\delta\kappa + (\alpha\omega_1 + \vartheta_1 + \eta\kappa_1)\delta\omega + \vartheta_1'\delta\psi + I_1\,\delta u + J_1\,\delta v + K_1\,\delta w$$
$$+ (v_1 + u_1')\delta\lambda + (\kappa_1 - 2u_1' - v_1 + v_1'')\delta\rho + (\omega_1 - w_1' - w_1''' - \psi_1')\delta\vartheta\Big]\,\mathrm{d}\theta = 0,$$

where

$$\begin{cases} I_1 = -\nu(u_1 - v_1') - 3(u_1'' + u_1'''') - \eta(w_1'' + w_1'''') - \lambda_1' + 2\rho_1', \\ J_1 = v_1'''' + (\nu-2)v_1 + (\nu-1)v_1'' + \lambda_1 - \rho_1 + \rho_1'', \\ K_1 = -\eta(u_1'' + u_1'''') + (1+\nu)w_1'' + w_1''''. \end{cases}$$

The above equation requires

$$\begin{cases} \vartheta_1' = 0, \\ v_1 = -u_1', \\ \kappa_1 = u_1' + u_1''', \\ \omega_1 = w_1' + w_1''' + \psi_1', \\ \rho_1 = -\eta(w_1' + w_1''' + \psi_1'), \\ \lambda_1 = \rho_1 - \rho_1'' - v_1'''' - (\nu-2)v_1 - (\nu-1)v_1'', \end{cases} \tag{20}$$

and

$$\begin{cases} \alpha(w_1' + w_1''' + \psi_1') + \eta(u_1' + u_1''') = 0, \\ -\eta(u_1'' + u_1'''') + (1+\nu)w_1'' + w_1'''' = 0, \\ -\nu u_1 + (\nu(m-2)-1)u_1'' - (\nu+2)u_1'''' - u_1^{(6)} - \eta(2w_1'' + 3w_1'''' + w_1^{(6)} + \psi_1'' + \psi_1'''') = 0. \end{cases} \tag{21}$$

Note that the governing equations (20) and (21) are linearizations of the general equilibrium equations (17) and constraints (3), $(5)_1$, (8) about the trivial solution. For bifurcation to occur, (20) and (21) should have nontrivial solutions of the following forms

$$\begin{cases} u_1 = a_1 \sin n\theta, \\ w_1 = b_1 \sin n\theta, \\ \psi_1 = c_1 \sin n\theta, \end{cases} \tag{22}$$

where n is the mode number and a_1, b_1, c_1 are constants. Since $n = 0$ and $n = 1$ represent respectively translational and rotational rigid bodies, we restrict to $n \geq 2$. Plugging (22) into (21), we arrive

$$c_1 = (n^2-1)(\eta\alpha^{-1}a_1 + b_1),$$

$$\begin{bmatrix} (n^2-1)\left(1 - \dfrac{\eta^2}{\alpha}\right) - \dfrac{(n^2-1)\nu}{n^2} & -\eta \\ -n^2\eta & n^2 - \dfrac{n^2\nu}{n^2-1} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}$$

Nontrivial solution exists if the determinant of coefficient matrix of the above equation is zero

$$f(\nu) \equiv \left[(n^2-1)\left(1 - \frac{\eta^2}{\alpha}\right) - \frac{(n^2-1)\nu}{n^2}\right]\left[n^2 - \frac{n^2\nu}{n^2-1}\right] - n^2\eta^2 = 0. \tag{24}$$

The corresponding buckling solutions are

$$\boldsymbol{\xi}_1(\theta) = a_1(-n\cos n\theta\, \boldsymbol{e}_r + \sin n\theta\, \boldsymbol{e}_\theta) + b_1 \sin n\theta\, \boldsymbol{e}_3, \qquad 0 \leq \theta \leq 2\pi. \tag{25}$$

We next discuss the buckling condition in two cases.

4.1. Bifurcation of inextensible achiral elastic rings

For this case, we set the chiral coefficient $e = 0$ or equivalently $\eta = 0$ and (24) reduces to

$$\nu = n^2 \qquad \text{or} \qquad \nu = n^2 - 1, \qquad n \geq 2. \tag{26}$$

The homogeneous system for a_1, b_1 in this case is uncoupling and hence, there is no couple between the in plane and out of plane bifurcations.

When $\nu = n^2$, the homogeneous system has solution $a_1 \neq 0, b_1 = 0$ and the circular ring bifurcates via in-plane perturbation to a flat but noncircular shape, given as

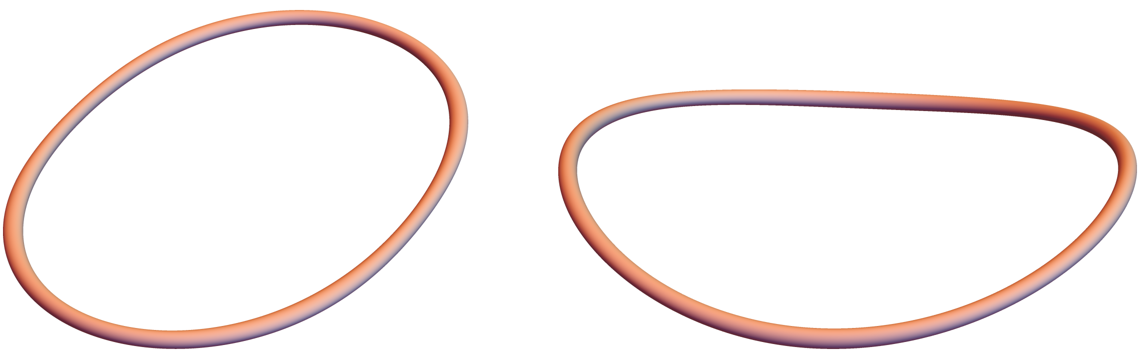
$$\boldsymbol{\xi}_1(\theta) = a_1(-n \cos n\theta \boldsymbol{e}_r + \sin n\theta \boldsymbol{e}_\theta), \qquad 0 \leq \theta \leq 2\pi.$$

For the lowest buckling mode $n = 2$, the buckling shape is in the form of an ovalization, as illustrated in Fig. 2(a).

When $\nu = n^2 - 1$, the homogeneous system has solution $a_1 = 0, b_1 \neq 0$ and the circular ring bifurcates via out-of-plane perturbation to a nonflat shape whose projection onto the initial ring plane is circular, given as

$$\boldsymbol{\xi}_1(\theta) = b_1 \sin n\theta \boldsymbol{e}_3, \qquad 0 \leq \theta \leq 2\pi.$$

For the lowest buckling mode $n = 2$, the buckling shape is a saddle one, as illustrated in Fig. 2(b).



(a) In-plane bifurcation (b) Out-of-plane bifurcation

Fig. 2. Graphical depictions of the buckling modes which achiral elastic rings bifurcate purely in its plane (a) or purely out of its plane (b), respectively, to an oval shape or a saddle shape for $n = 2$. The projection of the saddle shape onto the initial ring plane is circular

It is obvious from (26) that the critical loading is smallest when $n = 2$. For $n = 2$, the circular ring bifurcates to a saddle shape via out-of-plane bifurcation at the smallest critical value of dead loading $\nu_c = 3$. This value is lower than the smallest critical value of dead loading $\nu_c = 4$ reported in previous literatures [10,20,21,37] which restrict to in-plane bifurcation only. This new result demonstrates the role of out-of-plane bifurcation in bifurcation analysis of elastic rings in general and of those subject to dead loading in particular.

4.2. Bifurcation of inextensible chiral elastic rings

We first specialize to cases where $\sigma = 0$ or equivalently $\nu = 0$ and (24) reduces to

$$\eta^2 = \frac{(n^2-1)\alpha}{n^2-1+\alpha}, \qquad n \geq 2. \tag{27}$$

This implies that even when there are no loadings, the chiral ring still buckles if the chiral degree is large enough. Therefore, we assume the chiral rings are stable before the application of the loading so that

$$\eta^2 \leq \frac{\alpha(n^2-1)}{\alpha+(n^2-1)}, \qquad n \geq 2.$$

From (24) we see that,

$$f\left(\nu = n^2-1\right) = -n^2\eta^2 \leq 0,$$

so two solutions $\nu_{1,2}$ of (24) satisfy

$$\nu_1 \leq n^2 - 1 \leq \nu_2, \qquad n \geq 2.$$

Since the smallest critical loading at which elastic ring buckles is more important, we consider the lower critical loading $\nu = \nu_1$ among the two solutions of (24).

When $\nu = \nu_1$, the homogeneous system (23) is coupling for a_1, b_1 and has solution $a_1 \neq 0, b_1 \neq 0$ and hence the chiral ring bifurcates via a coupling between in-plane and out-of-plane perturbations to a nonflat shape whose projection onto the initial ring plane is noncircular, given as (25).

For $n \geq 2$, plots showing the effect of twisting-to-bending ratio α and chirality degree η on the critical loading $\nu_c = \nu_1$ are plotted in Fig. 3. We see from this figure that the smallest critical loading $\nu_c = \nu_1$ is attained when $n = 2$. This finding agrees with previous literatures on stability of circular rings subject to external pressure [1,5–10,19–21,24,38].

For the lowest buckling mode $n = 2$, the smallest critical loading is, from (24),

$$\nu_c = -\frac{2\eta^2}{\alpha} + \frac{1}{2}\left(7 - \sqrt{\frac{16\eta^4}{\alpha^2} + \left(16 - \frac{8}{\alpha}\right)\eta^2 + 1}\right), \tag{28}$$

and the buckling shape is a saddle one, as illustrated in Fig. 4 and given as

$$\boldsymbol{\xi}_1(\theta) = b_1\left[\left(\frac{3-\nu_c}{3\eta}\right)(-2\cos 2\theta\, \boldsymbol{e}_r + \sin 2\theta\, \boldsymbol{e}_\theta) + \sin 2\theta\, \boldsymbol{e}_3\right], \qquad 0 \leq \theta \leq 2\pi.$$

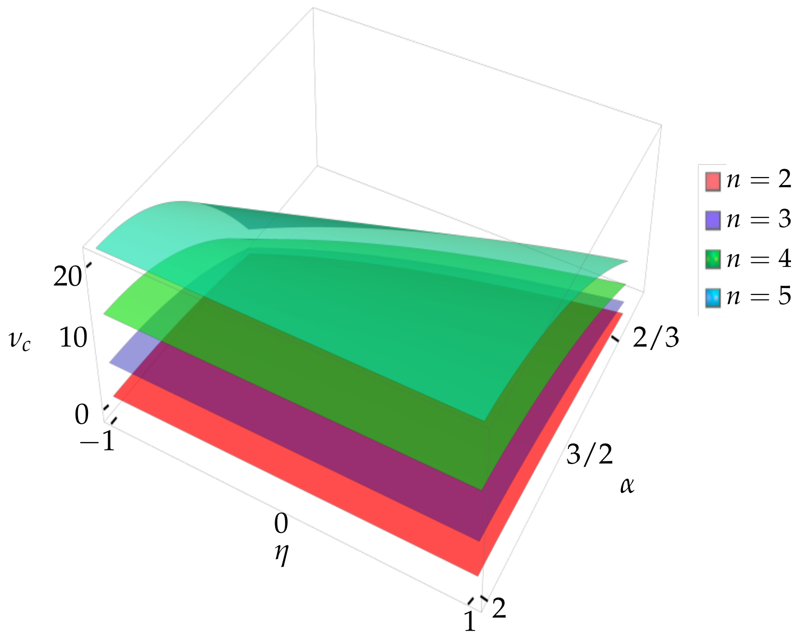


Fig. 3. Plots depicting the dependence of the critical values $\nu_c = \nu_1$ on the twist-to-bend ratio α and chirality η for mode numbers $n = 2, 3, 4$, and 5

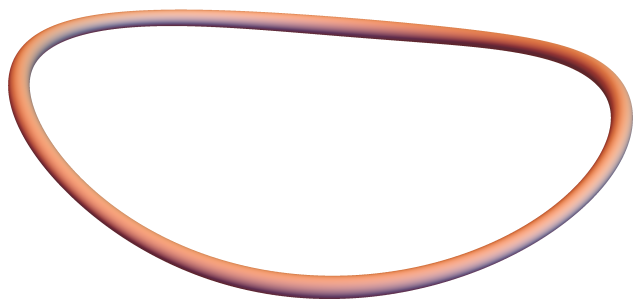


Fig. 4. Graphical depictions of the buckling modes which a chiral elastic ring bifurcates both in its plane (in-plane bifurcation) and out of its plane (out-of-plane bifurcation) to a saddle shape for $n = 2$. The projection of the saddle shape onto the initial ring plane is not circular

From (28), we see that the smallest critical value of dead loading is an even function of chirality η implying that the critical loading ν_c of ν for "left-handed" ($\eta < 0$) chirality is the same as that for "right-handed" ($\eta > 0$) chirality. Plots showing the effect of twisting-to-bending ratio α and chirality degree η on the critical loading $\nu_c = \nu_1$ when $n = 2$ are plotted in Fig. 5. From this figure, we re-obtain the two special cases already discussed. The first case is $\eta = 0$ and the smallest critical loading of $\nu = 3$, independent with α, in accordance with (26) for $n = 2$. The second case is $\nu = 0$ and the curves intersect the horizontal axis at the critical chirality degree of η, given as (27) for $n = 2$. When $\eta \neq 0$, the critical value of dead loading $\nu_c < 3$ suggests that circular rings made from filaments with chirality lose instability via coupling between in-plane and out-of-plane bifurcation at critical values of loading smaller than that at which bifurcation would happen without chirality via purely out-of-plane bifurcation. Additionally, the monotonic decrease of the curve $\nu_c = \nu_c(\eta)$ as the degree of chirality $|\eta|$ increases shows the destabilizing effect of chirality. Moreover, a curve corresponding to a smaller value of α is under that corresponding to higher value of α implying the stabilizing influence of the twisting-to-bending ratio. Thus, increasing the twisting rigidity over the bending rigidity will reduce the potentially destabilizing effect of chirality.

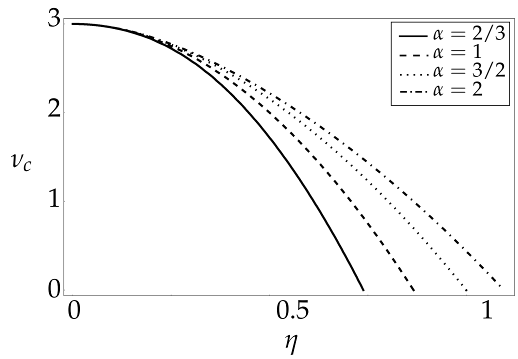


Fig. 5. Plots depicting the dependence of the lowest critical values ν_c of ν, given as (28), on the chirality coefficient η with various twist-to-bend ratio α

5. SUMMARY

A variational approach has been used to study bifurcation of inextensible chiral elastic rings subject to dead loading. Rings made from circular cross-section filaments with constant material properties are studied. In opposite to previous literatures which restrict to planar bifurcation, we allow for both in-plane and out-of-plane bifurcation. Moreover, we assume the filament is chiral which couples bending and twisting deformations. For filaments without chirality, circular rings subject to dead loading bifurcate to three dimensional shapes at the critical loading $\nu_c = 3$ less than that at which buckling would occur if only in plane bifurcation were allowed. For filaments with chirality, on the other hand, circular rings subject to dead loading bifurcate to three dimensional shapes at the critical loading ν_c of ν smaller than that at which buckling would occur without chirality. This results in a destabilizing effect of chirality on buckling of circular rings but its effect could be reduced by raising the twisting-to-bending ratio.

Based on the variational approach, our work is suitable to analyze bifurcation problems under static loading. When dynamic effects are included, i.e., dynamic loadings or vibration characteristics of elastic rings are considered, the variational approach is not applicable and our future work would be to use alternative methods such as dynamic stability analysis or numerical methods to provide more comprehensive analysis.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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