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A SINGLE DEGREE OF FREEDOM MODEL FOR CRACKED BEAM

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Abstract. This paper presents a simplified model of cracked beam by single-degree-of-freedom system. Equivalence between the beam and SDOF models means that they have the same fundamental natural frequency and similar frequency response functions (FRFs). Similarity of FRFs is checked by using the frequency-domain assurance criterion acknowledged herein as spectral similarity index (SSI). Finally, FRFs of both the cracked beam and its simplified SDOF model have been examined versus crack location and depth using the so-called spectral damage index (SDI). Numerical results show that SDI is significantly sensitive to crack and could be used as a novel indicator for crack detection in beam by measurements of frequency response functions.

Keywords: cracked beam, simplified model, frequency response function, frequency domain assurance criterion.

1. INTRODUCTION

Converting a continuous system to a discrete system with finite degrees of freedom is the simplest and most reasonable way to the dynamic analysis of elastic structures [1,2]. It is also the conventional and essential technique for solving partial differential equations using the well-developed theory of ordinary differential equations. However, the approach is feasible only in case a set of the orthogonal basic functions such as the eigenfunctions of a differential operator or the mode shapes of free vibration of the structure under consideration is available. This is incapable of applying for analysis of damaged structures because finding the basic functions of the structures is not a problem of less difficulty. In the latter case, the well-known finite element method [3] should be applied, but the finite element model exists only in an implicit numerical formulation, and it requires many degrees of freedom. Since most of the vibration energy is usually concentrated in the fundamental mode, it is sufficient to find a single degree of freedom (SDOF) model for dynamic analysis of the beam structures [4, 5]. Namely, it was proved both theoretically and experimentally in [6] that SDOF model is satisfactory for investigating free vibration of a cantilever beam with a breathing crack. Stochino and Carta [7] established a mass-spring oscillator model of a reinforced concrete beam under impulsive load that was validated by comparing its displacement maximum and midspan deflection calculated for the beam with the experimental result. Mousavi et al. [8] proposed an SDOF model of cracked beam subjected to a moving load and utilized it for detecting crack location and severity. They have determined equivalent stiffness and external force using midspan deflection of the beam under the moving load.

In the present paper, there is constructed an SDOF model of cracked simply supported beam based mainly on that natural frequency of SDOF oscillator is equal to an approximate fundamental eigenfrequency of cracked beam. The damping coefficient of equivalent oscillator is also selected from energy dissipation coeficients of the beam. The criterion accepted for validating the equivalent models is correlation between Frequency Response Functions (FRF) of beam and its SDOF model. Finally, sensitivity of the FRFs to crack is examined using so-called frequency domain assurance criterion [9, 10].

2. SINGLE DEGREE OF FREEDOM SYSTEMS

Let's consider one degree of freedom system given in Fig. 1, where m, c, k denote respectively mass, damping coefficient, stiffness of the system and x and $P(t)$ are displacements of the mass and external force respectively.

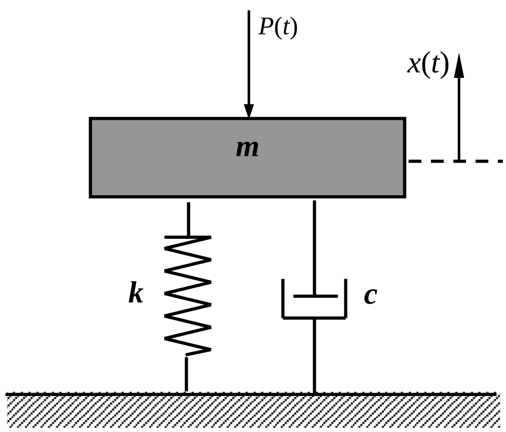

Fig. 1. Single degree of freedom system model

It is easily to write down equation of motion for the single degree of freedom (SDOF) system in the form

$$m\ddot{x}\left(t\right)+c\dot{x}\left(t\right)+kx\left(t\right)=P\left(t\right). \tag{1}$$

Introducing following notations

$$\omega_0^2=k/m, \quad 2\zeta=c/\sqrt{km}.$$

Eq. (1) can be rewritten as

$$\ddot{x}\left(t\right)+2\zeta\omega_0\dot{x}\left(t\right)+\omega_0^2x\left(t\right)=\left(1/m\right)P\left(t\right). \tag{2}$$

In the latter equation $\omega_0=\sqrt{k/m}$ is acknowledged as natural frequency and ζ is damping ratio of the system.

Supposing now that

$$P\left(t\right)=P_0e^{i\omega t}.$$

Eq. (2) becomes

$$\ddot{x}\left(t\right)+2\zeta\omega_0\dot{x}\left(t\right)+\omega_0^2x\left(t\right)=\hat{P}_0e^{i\omega t}, \quad \hat{P}_0=P_0/m. \tag{3}$$

Seeking solution of Eq. (3) in the form

$$x\left(t\right)=Xe^{i\omega t}, \tag{4}$$

one gets

$$X\left(\omega\right)=\hat{P}_0/\left[\omega_0^2-\omega^2+2i\zeta\omega_0\omega\right]. \tag{5}$$

Therefore,

$$A\left(\omega\right)=ReX\left(\omega\right)=\hat{P}_0/\left[\left(\omega_0^2-\omega^2\right)^2+4\zeta^2\omega_0^2\omega^2\right], \tag{6}$$

$$\varphi\left(\omega\right)=\text{atan}\left[2\zeta\omega_0\omega/\left(\omega_0^2-\omega^2\right)\right], \tag{7}$$

are the amplitude and dephase respectively of the forced vibration (4). It can be seen from the later equations that the forced vibration amplitude reaches its maximum for exciting frequency $\omega=\omega_r=\omega_0\sqrt{1-2\zeta^2}$ that is called resonant frequency and very close to the natural frequency in case of small damping ratio.

On the other hand, the function

$$FRF\left(\omega\right)=A_0/\left[\omega_0^2-\omega^2+2i\zeta\omega_0\omega\right], \tag{8}$$

is acknowledged as frequency response function of the SDOF system.

3. VIBRATORY MODEL OF CRACKED BEAMS

Now, let's consider a Euler–Bernoulli (EB) beam of meterial and geometry constants: E, G, ν, ρ are the elastic and shear modulus, Poisson coefficient and mass density; $\ell, A = b \times h, I = bh^3/12$ is the length, cross-section area and moment of inertia. Moreover, it is assumed that the beam is multiple cracked at positions $0 \leq e_1 < e_2 < \ldots < e_{n-1} < e_n \leq \ell$ and all the cracks are transverse and open with depths repectively $(a_1, \ldots, a_n)$ as shown in Fig. 2.

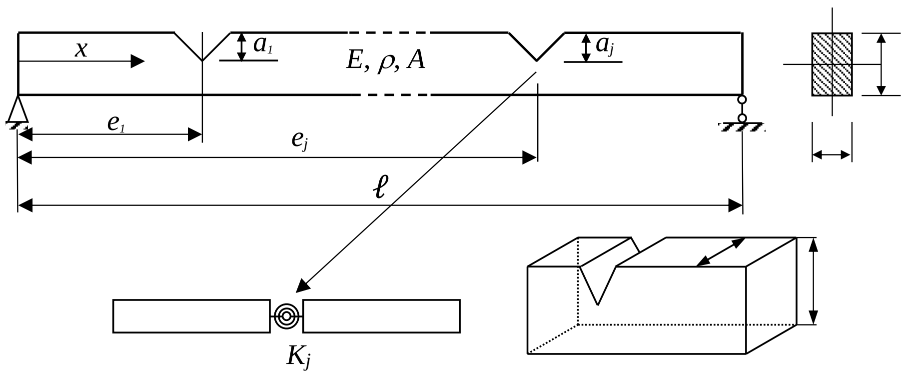

Fig. 2. Model of multiple cracked beam

3.1. Vibration mode and frequency equation

As well known, vibration shape of an EB-beam is difined as solution of equation

$$\frac{d^4\phi(x)}{dx^4} - \lambda^4\phi(x) = 0, \quad \lambda^4 = \rho A\omega^2/EI, \tag{9}$$

that is solved together with boundary conditions. Moreover, in the recently published review [3] general solution for vibration shape of multiple cracked EB beam has been obtained in the form

$$\phi(x, \omega) = C_1\Phi_1(\lambda x) + C_2\Phi_2(\lambda x) + C_3\Phi_3(\lambda x) + C_4\Phi_3(\lambda x), \tag{10}$$

where C_1, C_2, C_3, C_4 are constants, $\lambda = \left(\rho A\omega^2/EI\right)^{1/4}$ and $\Phi_k(\lambda x), k = 1, 2, 3, 4$ are

$$\Phi_1(\lambda x) = L_{01}(\lambda x) + \sum_{j=1}^{n} \mu_{j1} K\left(x - e_j\right), \quad \Phi_2(\lambda x) = L_{02}(\lambda x) + \sum_{j=1}^{n} \mu_{j2} K\left(x - e_j\right), \tag{11}$$

$$\Phi_3(\lambda x) = L_{03}(\lambda x) + \sum_{j=1}^{n} \mu_{j3} K\left(x - e_j\right), \quad \Phi_4(\lambda x) = L_{04}(\lambda x) + \sum_{j=1}^{n} \mu_{j4} K\left(x - e_j\right),$$

$$K(x) = \begin{cases} 0 & \text{for } x < 0 \\ S(x) & \text{for } x \geq 0 \end{cases}, \quad S(x) = (1/2\lambda)(\sinh \lambda x + \sin \lambda x), \tag{12}$$

$$\mu_{j1} = \gamma_j \left[L_{01}^{''} (\lambda e_j) + \sum_{k=1}^{j-1} \mu_{k1} S'' (e_j - e_k) \right],$$
$$\mu_{j2} = \gamma_j \left[L_{02}^{''} (\lambda e_j) + \sum_{k=1}^{j-1} \mu_{k2} S'' (e_j - e_k) \right], \tag{13}$$
$$\mu_{j3} = \gamma_j \left[L_{03}^{''} (\lambda e_j) + \sum_{k=1}^{j-1} \mu_{k3} S'' (e_j - e_k) \right], \quad \mu_{j4} = \gamma_j \left[L_{04}^{''} (\lambda e_j) + \sum_{k=1}^{j-1} \mu_{k4} S'' (e_j - e_k) \right].$$

The functions $L_{01}(\lambda x), L_{02}(\lambda x), L_{03}(\lambda x), L_{04}(\lambda x)$ are four independent solutions of free vibration problem for uncracked beam and $(n \times 4)$ - matrix of so-called damage indexes

$$[\boldsymbol{\mu}] = \{\mu_{jk}, j = 1, \ldots, n; k = 1, 2, 3, 4\},$$

given in Eq. (13) is calculated from crack parameters $(e_j, \gamma_j, j = 1, \ldots, n)$ by

$$\{\boldsymbol{\mu}\} = [\mathbf{G}]^{-1} [\mathbf{B}],$$

where $\mathbf{G} = [g_{ij}, i, j = 1, 2, 3, \ldots, n]$ is $n \times n$ - matrix with elements

$$g_{ij} = \{1 \text{ if } i = j; 0 \text{ for } i < j; -\gamma_i S'' (e_i - e_j) \text{ for } i > j\} \tag{14}$$

and matrix $[\mathbf{B}] = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4]$ of 4 vectors of dimension n

$$\{\mathbf{b}_1\} = \left\{\gamma_1 L_{01}^{''}(\lambda e_1), \ldots, \gamma_n L_{01}^{''}(\lambda e_n)\right\}^T, \quad \{\mathbf{b}_2\} = \left\{\gamma_1 L_{02}^{''}(\lambda e_1), \ldots, \gamma_n L_{02}^{''}(\lambda e_n)\right\}^T,$$
$$\{\mathbf{b}_3\} = \left\{\gamma_1 L_{03}^{''}(\lambda e_1), \ldots, \gamma_n L_{03}^{''}(\lambda e_n)\right\}^T, \quad \{\mathbf{b}_4\} = \left\{\gamma_1 L_{04}^{''}(\lambda e_1), \ldots, \gamma_n L_{04}^{''}(\lambda e_n)\right\}^T, \tag{15}$$

$$\gamma_j = 6\pi (1 - \nu^2) (h/L) f_b (a_j/h),$$
$$f_b(z) = z^2 \Big(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + 47.1063z^6 - 40.7556z^7 + 19.6z^8\Big). \tag{16}$$

Substituting vibration modes (10) into general boundary conditions given in the form

$$\phi^{(p_0)}(0, \omega) = 0, \quad \phi^{(q_0)}(0, \omega) = 0, \quad \phi^{(p_1)}(\ell, \omega) = 0, \quad \phi^{(q_1)}(\ell, \omega) = 0, \tag{17}$$

with derivative orders p_0, q, p_1, q_1 that could be equal to one of the values (0, 1, 2, 3), leads to the equations allowing one to find natural frequencies and mode shapes of multiple cracked EB-beams. Namely, suppose that vibration shape (10) satisfying boundary conditions (17) at the left end of beam can be represented as

$$\phi(x, \omega) = A L_1(\lambda x) + B L_2(\lambda x), \tag{18}$$

with

$$L_1(\lambda x) = L_{01}(\lambda x) + \sum_{j=1}^{n} \mu_{j1} K(x - e_j), \quad L_2(\lambda x) = L_{02}(\lambda x) + \sum_{j=1}^{n} \mu_{j2} K(x - e_j). \tag{19}$$

Then, remaining boundary conditions in (17) yield the equations

$$C_1 L_1^{(p_1)}(\lambda\ell) + C_2 L_2^{(p_1)}(\lambda\ell) = 0, \quad C_1 L_1^{(q_1)}(\lambda\ell) + C_2 L_2^{(q_1)}(\lambda\ell) = 0 \tag{20}$$

that result in the so-called frequency equation for multiple cracked beams in the form

$$L_1^{(p_1)}(\lambda\ell)\, L_2^{(q_1)}(\lambda\ell) - L_1^{(q_1)}(\lambda\ell)\, L_2^{(p_1)}(\lambda\ell) = 0. \tag{21}$$

In case of simply supported beams functions $L_{01}(\lambda x), L_{02}(\lambda x)$ can be chosen as

$$L_{01}(\lambda x) = \sinh \lambda x, L_{02}(\lambda x) = \sin \lambda x,$$

and, therefore, frequency equation (21) for simply supported beam with single crack is reduced to [11]

$$f_{ss}(\lambda) + \gamma_1 g_{ss}(\lambda, e_1) = 0, \tag{22}$$

where

$$f_{ss}(\lambda) = 2\lambda \sinh \lambda\ell \sin \lambda\ell,$$

$$g_{ss}(\lambda, e) = \sinh \lambda(\ell - e_1) \sinh \lambda e_1 \sin \lambda\ell - \sinh \lambda\ell \sin \lambda(\ell - e_1) \sin \lambda e_1.$$

3.2. Frequency response function

We consider now forced vibration in multiple cracked beams described by the equations [12]

$$EI\frac{\partial^4}{\partial x^4}\left(1 + i\mu_2\frac{\partial}{\partial t}\right) w(x,t) + \rho A\mu_1 \frac{\partial w(x,t)}{\partial t} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = P_0 e^{i\omega t}\delta(x - \ell/2). \tag{23}$$

Seeking solution of Eq. (23) in the form

$$w(x,t) = \phi(x)\, e^{i\omega t},$$

one gets following equation for vibration shape $\phi(x)$

$$\frac{d^4\phi(x)}{dx^4} - \hat{\lambda}^4\phi(x) = \hat{P}_0\delta(x - \ell/2), \tag{24}$$

where

$$\hat{\lambda}^4 = \frac{\rho A(\omega^2 - i\omega\mu_1)}{EI(1 + i\omega\mu_2)}, \quad \hat{P}_0 = \frac{P_0}{EI(1 + i\omega\mu_2)}.$$

First, let's consider complex frequency parameter

$$\hat{\lambda}^4 = \frac{\rho A(\omega^2 - i\omega\mu_1)}{EI(1 + i\omega\mu_2)} = \frac{\rho A\omega^2(\alpha - i\beta)}{EI} = \frac{\rho A\hat{\omega}^2}{EI}, \tag{25}$$

with

$$\hat{\omega}^2 = \omega^2(\alpha + i\beta), \tag{26}$$

$$\alpha = (1 + \mu_1\mu_2)/\left(1 + \omega^2\mu_2^2\right), \quad \beta = (\mu_1 + \omega^2\mu_2)/\omega\left(1 + \omega^2\mu_2^2\right).$$

It was well-known that general solution of Eq. (23) is

$$\phi\left(x\right)=\phi_0\left(x\right)+\hat{P}_0\int_0^x h\left(x-s\right)\delta\left(s-\ell/2\right)\mathrm{d}s=\phi_0\left(x\right)+\hat{P}_0 h\left(x-\ell/2\right),\tag{27}$$

where

$$h\left(x\right)=\left\{0\colon \text{ for } x<0;\left(\sinh\hat{\lambda}x-\sin\hat{\lambda}x\right)/2\hat{\lambda}^3\colon \text{ for} x\geq 0\right\},\tag{28}$$

and function $\phi_0\left(x\right)$ as mentioned above can be expressed in the form of Eqs. (18) and (19) satisfying given boundary conditions at the left end of the beam. Thus, now putting (27) with (18) into boundary conditions at the right end one gets the equations

$$AL_1^{(p_1)}\left(\hat{\lambda}\ell\right)+BL_2^{(p_1)}\left(\hat{\lambda}\ell\right)=-\hat{P}_0 h^{(p_1)}\left(\ell/2\right),\ AL_1^{(q_1)}\left(\hat{\lambda}\ell\right)+BL_2^{(q_1)}\left(\hat{\lambda}\ell\right)=-\hat{P}_0 h^{(q_1)}\left(\ell/2\right),$$

that give the constants A, B to be calculated as

$$A=\left(\hat{P}_0/\Delta\right)\hat{A},\quad B=\left(\hat{P}_0/\Delta\right)\hat{B},$$

with

$$\hat{A}=\left[h^{(q_1)}\left(\ell/2\right)L_2^{(p_1)}\left(\hat{\lambda}\ell\right)+h^{(p_1)}\left(\ell/2\right)L_2^{(q_1)}\left(\hat{\lambda}\ell\right)\right],\tag{29}$$

$$\hat{B}=\left[L_1^{(q_1)}\left(\hat{\lambda}\ell\right)h^{(p_1)}\left(\ell/2\right)-L_1^{(p_1)}\left(\hat{\lambda}\ell\right)h^{(q_1)}\left(\ell/2\right)\right],\tag{30}$$

$$\Delta=L_1^{(p_1)}\left(\hat{\lambda}\ell\right)L_2^{(q_1)}\left(\hat{\lambda}\ell\right)-L_1^{(q_1)}\left(\hat{\lambda}\ell\right)L_2^{(p_1)}\left(\hat{\lambda}\ell\right).\tag{31}$$

Obviously, the right-hand side of Eq. (31) becomes the left-hand side of Eq. (21) if the damping coefficients $\mu_1=\mu_2=0$. Finally, solution (27) gets the form

$$\phi\left(x,\omega\right)=\left(\hat{P}_0/\Delta\right)\left[h\left(x-\frac{\ell}{2}\right)\Delta+\hat{A}L_1\left(\hat{\lambda}x\right)+\hat{B}L_2\left(\hat{\lambda}x\right)\right].\tag{32}$$

Furthermore, solution (32) calculated at the beam middle would be

$$\phi\left(\ell/2,\omega\right)=\left(\hat{P}_0/\Delta\right)\left[\hat{A}L_1\left(\hat{\lambda}\ell/2\right)+\hat{B}L_2\left(\hat{\lambda}\ell/2\right)\right]$$

and therefore, the so-called frequency response function of midspan deflection of the beam is

$$FRF\left(\ell/2,\omega\right)=\phi\left(\ell/2,\omega\right)/\hat{P}_0=\frac{\hat{A}L_1\left(\hat{\lambda}\ell/2\right)+\hat{B}L_2\left(\hat{\lambda}\ell/2\right)}{L_1^{(p_1)}\left(\hat{\lambda}\ell\right)L_2^{(q_1)}\left(\hat{\lambda}\ell\right)-L_1^{(q_1)}\left(\hat{\lambda}\ell\right)L_2^{(p_1)}\left(\hat{\lambda}\ell\right)}.\tag{33}$$

4. A SIMPLIFIED MODEL OF CRACKED BEAM

This section is devoted to finding a single-degree-of-freedom system that could equivalently represent a simply supported Euler–Bernoulli beam with single crack. In principle, a SDOF system is determined by three parameters: mass m, stifness k and damping coefficient c. However, from the mechanics point of view, the system can be characterized by two features: natural frequency $\omega_0=\sqrt{k/m}$ and damping ratio $\zeta=c/2\sqrt{km}$ as shown in Eq. (2) or Eq. (8).

First, the natural frequency ω_E of the sought equivalent system can be accepted to be equal to the fundamental frequency of the cracked beam given by

$$\omega_E^2 = \omega_{01}^2 \left[1 - \sum_{j=1}^{n} \chi\left(a_j\right) \varphi\left(e_j\right) \right], \tag{34}$$

where $\omega_{01}^2 = \dfrac{EI}{\rho A}\left(\dfrac{\pi}{\ell}\right)^4$ is fundamental frequency of uncracked simply supported beam and the functions $\chi(a), \varphi(e)$ are determined from the condition

$$\left(\omega_E - \omega_1\right)^2 \to \min, \tag{35}$$

where $\omega_1 = \lambda_1^2\sqrt{EI/\rho A}$ with λ_1 being the first solution of Eq. (22). Solving the optimization problem (35) gives the solution

$$\chi(a) = f_b\left(a_1/h\right), \tag{36}$$

with

$$a_1 = g(a) = 0.04 + 1.4167a - 5.8333a^2 + 33.3333a^3 - 66.6667a^4, \tag{37}$$

and function $f_b(z)$ is defined in (16) and

$$\varphi(e) = 2.9e(1-e). \tag{38}$$

Thus, we found that

$$\omega_E^2 = \frac{k}{m} = \frac{EI}{\rho A}\left(\frac{\pi}{\ell}\right)^4 \vartheta = \omega_{01}^2\vartheta, \tag{39}$$

with

$$\vartheta = 1 - \sum_{j=1}^{n} \chi\left(a_j\right)\varphi\left(e_j\right). \tag{40}$$

Particularly, the mass and stiffness of the desired system can be accepted as [8]

$$m = \frac{48\rho A\ell}{\pi^4}, \quad k = \frac{48EI}{\ell^3}\vartheta. \tag{41}$$

On the other hand, complex eigenfrequency of damped SDOF system is a root of the equation

$$\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega = 0,$$

with respect to ω that gives rise

$$\hat{\omega}_0^2 = \omega_0^2\left[1 - 2\zeta^2 \pm 2i\zeta\sqrt{1-\zeta^2}\right]. \tag{42}$$

Comparing (42) with (26) allows one to obtain

$$\beta/\alpha = \frac{\mu_1 - \omega^2\mu_2}{\omega\left(1+\mu_1\mu_2\right)} = \frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2},$$

that yields

$$\zeta = \zeta_E = \left\{ \frac{1}{2} \left[1 - (1 - \mu_1 \mu_2) \,/\, \sqrt{1 + \mu_1^2 \mu_2^2 + \omega_0^2 \mu_2^2 + \mu_1^2 / \omega_0^2} \right] \right\}^{1/2}. \qquad (43)$$

Obviously, if $\mu_1 = \mu_2 = 0$, then $\zeta_E = 0$, otherwise,

$$\zeta_E = \left\{ \frac{1}{2} \left[1 - 1/\sqrt{1 + \mu_1^2/\omega_0^2} \right] \right\}^{1/2} \text{ when } \mu_2 = 0,$$

and

$$\zeta_E = \left\{ \frac{1}{2} \left[1 - 1/\sqrt{1 + \omega_0^2 \mu_2^2} \right] \right\}^{1/2} \text{ when } \mu_1 = 0. \qquad (44)$$

Substituting (42) and (43) into (8) leads the frequency response function of equivalent SDOF system to get the form

$$FRF\,(\omega) = P_E / \left[\omega_E^2 - \omega^2 + 2i\zeta_E \omega_E \omega \right], \qquad (45)$$

with constant P_E would be corrected by correlation between frequency response functions of the beam and its equivalent oscillator.

For checking similarity or equivalence of two frequency response functions $F_1\,(\omega)$, $F_2\,(\omega)$ the following coherence coefficient acknowledged as so-called assurance criterion can be utilized

$$COH\,(F_1, F_2) = \left[\left(\sum_{k=1}^{N} S_k Q_k \right)^2 / \left(\sum_{k=1}^{N} S_k^2 \times \sum_{k=1}^{N} Q_k^2 \right) \right]^{1/2}, \qquad (46)$$

where $S_k = F_1\,(\omega_k)$, $Q_k = F_2\,(\omega_k)\,, k = 1, \ldots, N$. Namely, two frequency response function are considered similar if coherence coefficient (46) is closed to unique and it is acknowledged as Spectral Similarity Index (SSI) of the frequency-dependent signals. Namely, two signals would be considered spectrally similar if the index is close to unique and even in case the coefficient (46) calculated with a frequency shift δ, i. e. $S_k = F_1\,(\omega_k)$, $Q_k = F_2\,(\omega_k + \delta)\,, k = 1, \ldots, N$ is about 1.

Moreover, the coherence coefficient (46), calculated for frequency response functions of undamaged $F_0\,(\omega)$ and damaged $F_c\,(\omega, e, a)$ circumstances of a given structure, is named by spectral damage index (SDI) of the structure

$$SDI\,(e, a) = \left[\left(\sum_{k=1}^{N} F_0\,(\omega_j)\, F_c\,(\omega_j, e, a) \right)^2 / \left(\sum_{k=1}^{N} F_0^2\,(\omega_j) \times \sum_{k=1}^{N} F_c^2\,(\omega_j, e, a) \right) \right]^{1/2} \qquad (47)$$

This is an indicator representing the damage-induced change of frequency response function of a system that will be examined below for both cracked beam and equivalent SDOF system.

5. NUMERICAL VALIDATION AND ANALYSIS

First, the frequency ratio $r = (\lambda_1/\lambda_{01})^4$ with λ_1 being first solution of Eq. (22) and $\lambda_{01} = (\pi/\ell)^4$ is computed versus crack location $e/\ell \in [0, 1]$ for different crack depth and compared with ratio

$$\omega_E^2/\omega_{01}^2 = [1 - \chi(a)\,\varphi(e)]\,,$$

given by Eq. (34). The comparison shown in Fig. 3 demonstrates that natural frequencies of both the beam and its reduced model are almost the same and they are similarly sensitive to crack.

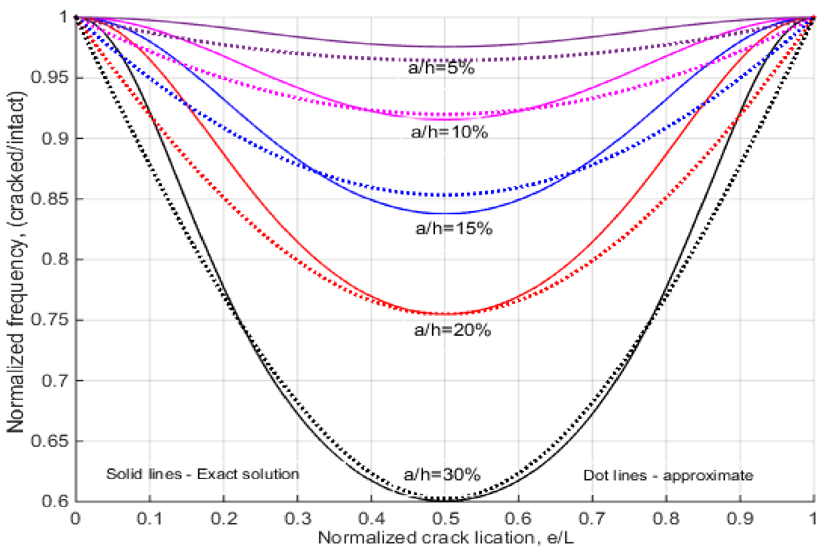

Fig. 3. Compared crack-induced change in exact and approximate fundamental frequencies for simply supported beam with single crack

Second, the frequency response functions of the systems are calculated for different crack locations and depths and presented in Figs. 4–6, corresponding respectively to cases of crack locations $e/\ell = 0.25; 0.5; 0.75$. Obviously, the Figues show that the difference between the FRFs is the shift of approximate FRFs to the right and it is endorsed by graphs given in the small boxes where there are given the frequency response function of the beam and shifted to the left ($\delta = |\omega_1 - \omega_E|$) frequency response function of the simplified SDOF model. It can be seen from the Figures that the frequency response functions are similar, and it is validated by their spectral similarity index calculated and depicted in Table 1. Note, the similarity of the FRFs is assured with the confidence of more than 0.99 for the crack of depth less than 30%. This means that the SDOF model is well accepted for the beam only in the case when the crack depth is within 20% of the beam thickness.

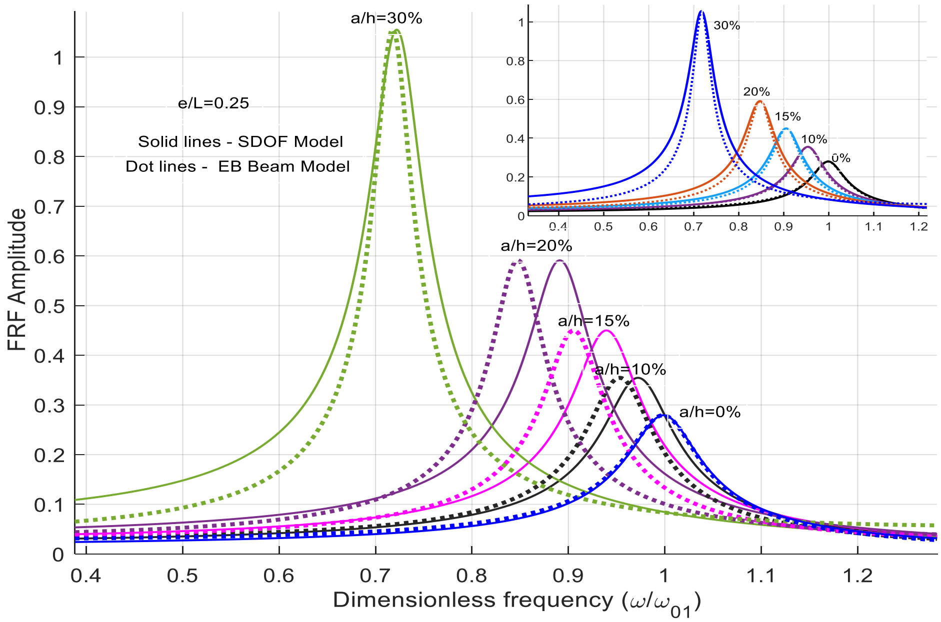

Fig. 4. Compared Frequency Response Functions of Beam and SDOF models with crack at position 0.25 of relative depth 0%–30%

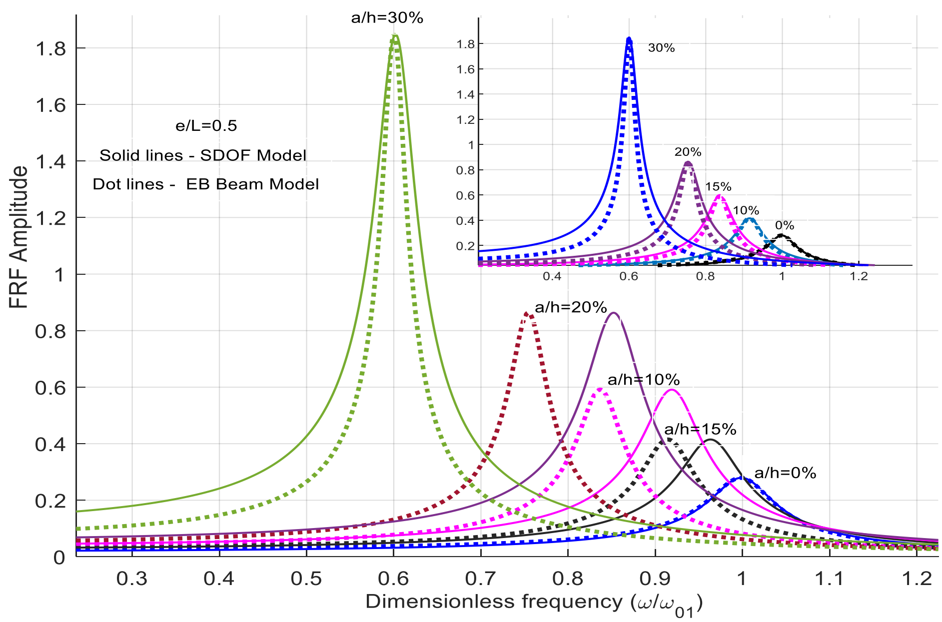

Fig. 5. Compared Frequency Response Functions of Beam and SDOF models with crack at position 0.5 of relative depth 0%–30%

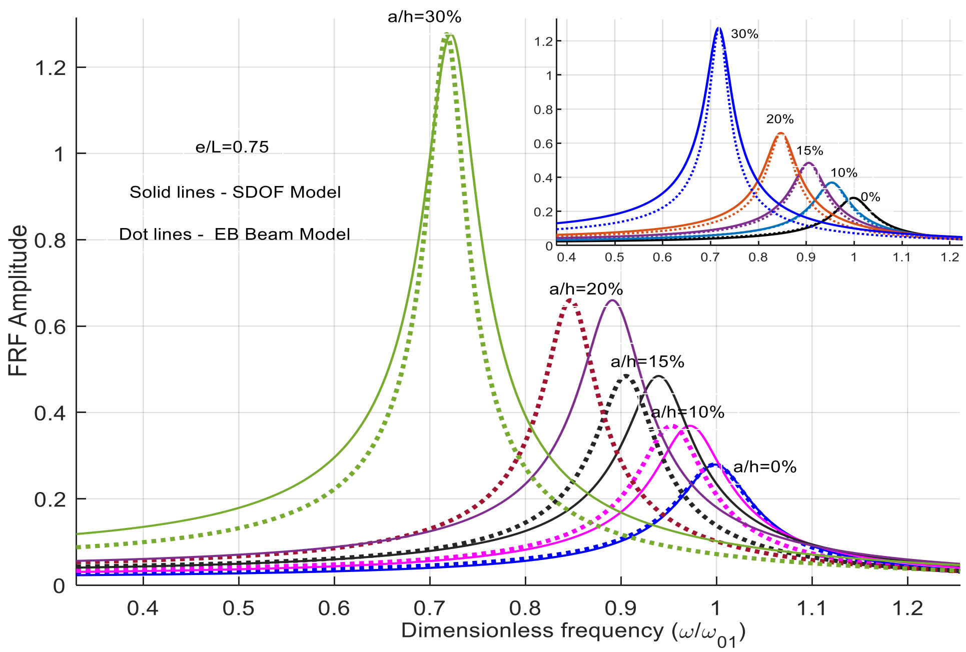

Fig. 6. Compared Frequency Response Functions of Beam and SDOF models with crack at position 0.75 of relative depth 0%–30%

Table 1. Spectral Similarity Index of frequency response functions for beam and SDOF models

Crack depth	Crack location								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0%	0.9971	0.9971	0.9971	0.9971	0.9971	0.9971	0.9971	0.9971	0.9971
10%	0.9990	0.9994	0.9991	0.9986	0.9982	0.9982	0.9985	0.9986	0.9982
15%	0.9996	0.9994	0.9983	0.9967	0.9958	0.9964	0.9979	0.9991	0.9990
20%	0.9984	0.9972	0.9953	0.9923	0.9908	0.9921	0.9955	0.9988	0.9996
30%	0.9882	0.9872	0.9874	0.9852	0.9842	0.9856	0.9900	0.9966	0.9999

Spectral damge indexes of beam and SDOF models calculated along crack location for different crack depth are exhibited in Fig. 7. Graphs given in the Figure show that sensitivity of FRFs of the models is comparable only in the case of crack depth being 30%. For crack depth less than 30% FRF of SDOF model is much less sensitive to crack than that of beam model. This may be explained by the fact that the reduced model has ignored the effect of crack on mode shape of the beam.

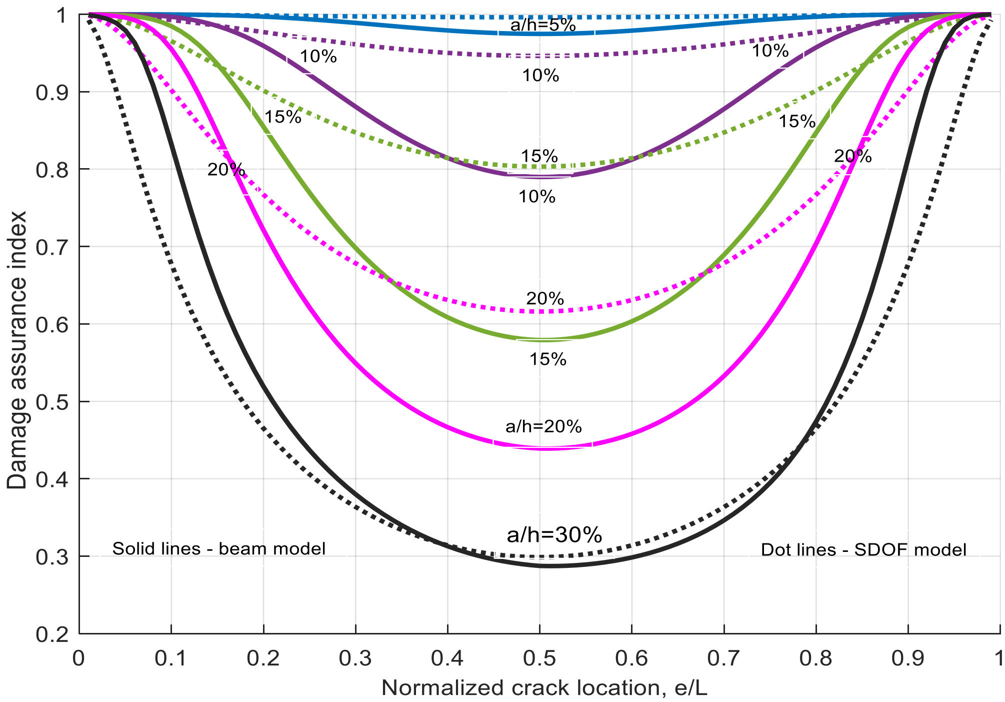

Fig. 7. Compared spectral damage indexes of beam and SDOF models along crack location and depth

6. CONCLUSION

In the present paper a single-degree-of-freedom system reprenting a simplified model of simply supported beam with multiple cracks has been constructed. The equivalence of the SDOF and beam models is defined by similarity of their frequency response functions and similarity of two frequency-dependent signals is checked by using an assurance criterion.

The sensitivity of frequency response functions of both models to crack is examined by the introduced herein so-called spectral damage index. Numerical results demonstrate that frequency response function of the reduced model is less sensitive to crack, but the spectral damage index shows to be useful indicator that could be for crack identification problem by measurements of frequency response functions.

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DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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