Vietnam Journal of Mechanics, Vol. 45, No. 2 (2023), pp. 145 – 163
DOI: https://doi.org/10.15625/0866-7136/18189

ABOUT THE NONLINEAR VIBRATION EQUATION OF THE BEAM

Nguyen Van Khang[1,*], **Nguyen Thi Van Huong**[1], **Nguyen Van Quyen**[1], **Pham Thanh Chung**[1]
[1]*Hanoi University of Science and Technology, Vietnam*
[*]E-mail: khang.nguyenvan2@hust.edu.vn

Received: 24 March 2023 / Published online: 30 June 2023

Abstract. In this paper, we present the establishment the vibration equation for the study of nonlinear beam effects. A illustrative example allows us to derive equations und use nonlinear vibration for calculating vibration of beams.

Keywords: beam, nonlinear vibration, vibration equation, resonance oscillation, asymptotic method.

1. INTRODUCTION

When studying the bending vibrations of beams, we assume that the beam cross-section is symmetrical about two axes. For example, the cross-section of a beam is circular, rectangular, or I-shaped. If the cross-section of the beam is not symmetrical about two axes, the beam will undergo bending, longitudinal, and torsional vibrations simultaneously. In this report, we do not consider the beam torsional vibration problem, only the beam bending and longitudinal vibration problem. Neglecting the rotational inertia and the sliding deformation of the beam axis, we have Euler–Bernoulli beams. If we are interested in rotational inertia and Gleitverformung deformation of beam axis, we have Timoshenko beam.

The problem of bending and longitudinal vibrations of linear beams has been studied very carefully. However, the bending and longitudinal vibrations of nonlinear beams are still a little explored area. In the book "Nonlinear Mechanics" [1], Kauderer presented the basics of the theory of nonlinear vibration of beams with relative care. In our country, the presentation of the mechanical principles of deformed solids was presented in [2] and some other documents. In [3, 4] the basic problems of setting up the equations of motion of non-linear beams were presented relatively carefully. In order to extend some assumptions about the relationship between stress and strain, the relationship between strain and displacement, in this paper we establish equations for bending and document vibrations of nonlinear beams of beams under the action of axial force. Studying the computation of the oscillation of this class of problems is of interest at Hanoi University of Science and Technology.

2. ESTABLISHING THE NONLINEAR BENDING VIBRATION EQUATION OF THE BEAM SUBJECTED TO THE LONGITUDINAL FORCE AT THE BEAM END

In this section, we establish the bending vibration equation of the beam taking into account the geometric nonlinearity, the physical nonlinearity and the longitudinal force acting at the beam end (Fig. 1).

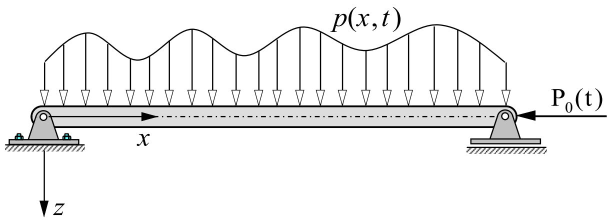

Fig. 1

Assume that in the un-deformed state, the geometric axis of the beam coincides with the axis x of the perpendicular coordinate system xyz. The ends of the beams have coordinates $x = 0$ and $x = l$. Assume that the principal axes of inertia at the intersection of the cross-section with the axis x are parallel to the axes y and z that the beam's axis is bent only in the plane of symmetry (x, z). The symbol $m(x)$ is the mass per unit length of the beam, r is the mass density, $A(r)$ is the cross-sectional area of the beam, the beam length is l, the beam is homogeneous. Using the Bernoulli hypothesis, consider the beam's cross-section to be consistently flat and perpendicular to the beam's deflection axis.

2.1. Dynamic balance equations of beams

To establish the beam bending vibration equation, imagine separating a small element of the beam at two cross-sections x and $x + \mathrm{d}x$. The symbol for the length of the element before deformation is $\mathrm{d}x$, after deformation is $\mathrm{d}s$. The symbol $w(x, t)$ is the deflection of the beam at the section x, $u(x, t)$ is the axial displacement of the beam at the section x, $j(x, t)$ is the angle of rotation of the beam at the section x. Neglect rotational inertia and shear deformation of beam shaft. Applying d'Alembert principle, set up the dynamic equations of the investigated beam element as shown in Fig. 2. In Fig. 2, we use the symbols: w is the deflection of the beam in the z direction, u is the displacement along the x axis.

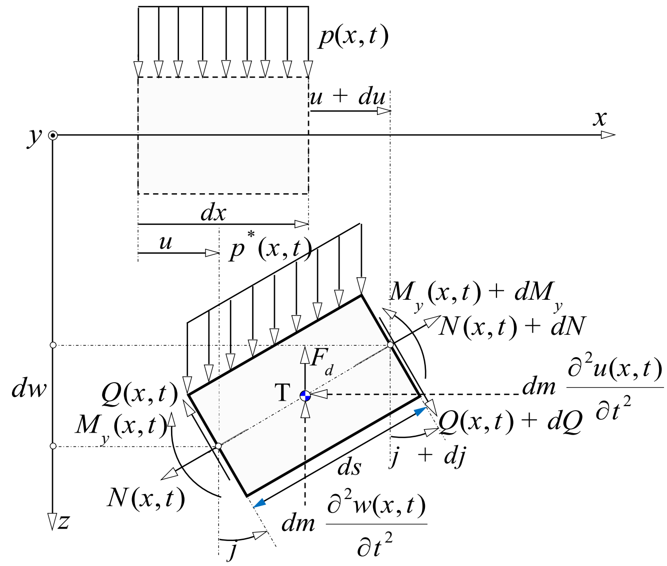

Fig. 2

From the condition of dynamic equilibrium in the direction z, we have

$$\begin{aligned}\sum F_{kz} = & -dm\frac{\partial^2 w}{\partial t^2} - F_d - Q\cos\varphi + (Q+dQ)\cos(\varphi+d\varphi) \\ & + N\sin\varphi - (N+dN)\sin(\varphi+d\varphi) + p^*(x,t)ds = 0.\end{aligned} \tag{1}$$

From the condition of dynamic equilibrium in the direction x, we have

$$\begin{aligned}\sum F_{kx} = & -dm\frac{\partial^2 u}{\partial t^2} - Q\sin\varphi + (Q+dQ)\sin(\varphi+d\varphi) \\ & - N\cos\varphi + (N+dN)\cos(\varphi+d\varphi) = 0.\end{aligned} \tag{2}$$

From the torque equilibrium condition, we get the equation

$$\sum \bar{m}_T\left(\vec{F}_k\right) = -M_y + M_y + dM_y - Q\frac{ds}{2} - (Q+dQ)\frac{ds}{2} = 0, \tag{3}$$

where N is the component of the normal force, Q is the shear force, M_y is the bending moment and F_d is the external resistance acting on the element of length dx. Suppose external resistance is proportional to velocity

$$F_d = c\frac{\partial w}{\partial t}\rho A(x)dx, \tag{4}$$

where c is a constant.

Notice that $\mu(x) = \rho A(x), dm = \mu(x)dx = \rho A(x)dx = \mu^*(x)ds, p^*(x,t)ds = p(x,t)dx$.

2.2. Some assumptions and approximate formulas describing the properties of the beam

2.2.1. *Stress–strain characteristic equation (constitutive equation)*

When considering the physical non-linearity of beams, the relationship between stress and strain is often expressed by a general formula in the form of a stress-strain curve, as follows [2]

$$f(\sigma_x, \dot{\sigma}_x, \varepsilon_x, \dot{\varepsilon}_x, t) = 0. \tag{5}$$

For the nonlinear elastic model, the stress-strain characteristic equations (5) have the following form:

- Stress-strain relationship of the nonlinear rheological model

$$\sigma_x = E(1 - a_3 E^2 \varepsilon_x^2)\varepsilon_x. \tag{6}$$

- Stress-strain relationship of the viscoelastic Kelvin–Voigt model

$$\dot{\sigma}_x + \gamma(1+K)\sigma_x = E(\dot{\varepsilon}_x + \gamma\varepsilon_x). \tag{7}$$

- Stress-strain relationship of the nonlinear rheological model

$$\sigma_x + b\dot{\sigma}_x = k_1\varepsilon_x + k_3\varepsilon_x^3 + h_1\dot{\varepsilon}_x + h_3\dot{\varepsilon}_3^3. \tag{8}$$

For the creeping beam model, the stress-strain characteristic equation (5) is commonly used in the following form [2]

$$\varepsilon = \frac{\sigma(t)}{E} + \int_0^t \Phi(t)\sigma(\tau)d\tau, \tag{9}$$

where the first term on the right-hand side represents the elastic strain that occurs instantaneously after the load is applied, the second term represents the linear time accumulation of strains from the differential variable. The function $\Phi(t)$ represents the creep deformation rate. We can also consider $\Phi(t)$ as the creep rate (the creep softness in a unit of time).

For creep beam model, in [3] stress-strain characteristic equation (5) is used in the form

$$\sigma_x = a_1\varepsilon_x + a_3\varepsilon_x^3 + \int_0^t K(t-\tau)\frac{d\varepsilon_x}{d\tau}d\tau, \tag{10}$$

where a_1, a_3 are the constants that characterize the physical properties of the material, the function $K(t-\tau)$ is the kernel of the genetic function determined by empirical formulas.

For the fractional viscoelastic nonlinear beam model, the stress-strain characteristic equation (5) has a very rich form [5, 6]. Below we introduce two types of stress-strain characteristic equations that are commonly used

$$\sigma_x = E\left(\varepsilon_x + \mu_\alpha \frac{d^\alpha \varepsilon_x}{dt^\alpha}\right), \tag{11}$$

where α is a real number ($0 < \alpha < 1$, or $1 < \alpha < 2$), and

$$\sigma(t) + a_1 D^\alpha \sigma(t) = b_0 \varepsilon(t) + b_1 D^\alpha \varepsilon(t), \tag{12}$$

where α is a real number.

2.2.2. *Approximate formula for determining the relative strain and displacement relationship*

Due to the assumption that the displacement in the direction y of the points on the beam is zero $(v = 0)$, from the displacement and deformation formulas in [1] we deduce

$$\varepsilon_{x0} = \frac{1}{2}\lambda_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{13}$$

Using Kirchhoff's hypothesis, considering partial derivatives of small u relative to partial derivatives of w, we have

$$\varepsilon_{x0} = \frac{1}{2}\lambda_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{14}$$

One can prove that ε_{x0} depends only on t, not on x. Therefore, from (14) we have

$$\varepsilon_{x0}(t) = \frac{1}{\ell}\int_0^\ell \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx = \frac{1}{\ell}[u(\ell,t) - u(0,t)] + \frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{15}$$

If $u(\ell,t) - u(0,t) = 0$ then from (15) deduce

$$\varepsilon_{x0}(t) = \frac{1}{\ell}\int_0^\ell \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{16}$$

According to [7], the relative strain of the layer a distance from the neutral axis (axis x) a segment z is of the form

$$\varepsilon_x(x,z,t) = \varepsilon_{x0}(x,t) - z\frac{\partial^2 w}{\partial x^2}. \tag{17}$$

From (16) and (17), we deduce the approximate formula determining the relationship between the relative length strain and displacement

$$\varepsilon_x(x,z,t) = \varepsilon_{x0}(t) - z\frac{\partial^2 w}{\partial x^2} = \frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - z\frac{\partial^2 w}{\partial x^2}. \tag{18}$$

2.2.3. *Approximate formula for determining axial forces*

Fig. 3 shows the deformation of a beam element dx at the survey location and at the initial position.

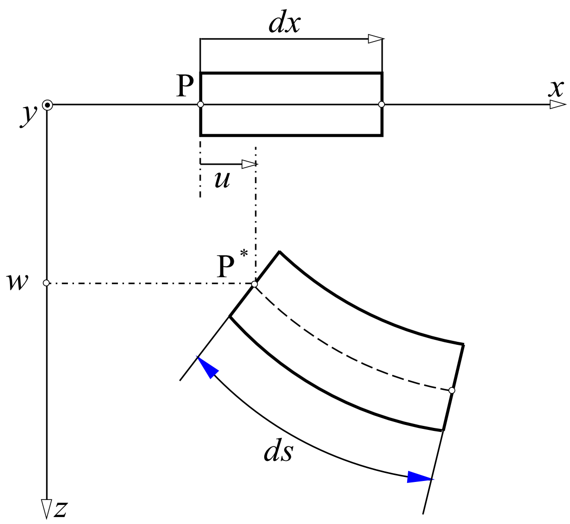

Fig. 3

According to Fig. 3, we have

$$H(x,t) = Q\sin\varphi + N\cos\varphi, \tag{19}$$

$$V(x,t) = Q\cos\varphi - N\sin\varphi, \tag{20}$$

where $H(x,t)$ is the force along the x axis, $V(x,t)$ is the force along the z axis.

First, we determine the elongation of the beam. Consider the beam elements before and after deformation as shown in Fig. 3. The coordinate of point $P(x,0)$ before deformation, after deformation $P \to P^*(x^*,z^*)$

$$x^* = x + u, \quad z^* = w. \tag{21}$$

From (21) we deduce

$$dx^* = \left(1 + \frac{\partial u}{\partial x}\right) dx, \quad dz^* = \frac{\partial w}{\partial x} dx. \tag{22}$$

The length of this element after deformation is

$$\begin{aligned} ds &= \sqrt{(dx^*)^2 + (dz^*)^2} = \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} dx, \\ &= \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} dx. \end{aligned} \tag{23}$$

Notice the formula $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ (when $|x| < 1$), we have

$$ds \approx \left\{1 + \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\right\} dx = 1 + \varepsilon_{x0} dx. \tag{24}$$

The elongation due to elastic deformation of the element is

$$d\Delta = ds - dx = \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx. \tag{25}$$

Due to assumption of Kirchhoff, $\left(\frac{\partial u}{\partial x}\right)^2$ is small compared to $\left(\frac{\partial w}{\partial x}\right)^2$, the expression (25) has the form

$$d\Delta = \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx. \tag{26}$$

From (26), the elongation due to total strain is

$$\Delta = u(\ell) - u(0) + \frac{1}{2}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{27}$$

If at the two ends of the beam $u(\ell) - u(0) = 0$, from (27) we deduce

$$\Delta = \frac{1}{2}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{28}$$

So the compressive force due to elastic deformation at the end of the beam has the following form

$$S = \frac{EA}{\ell}\Delta = \frac{EA}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{29}$$

Now we move on to determine the tensile force at the section x.

From Fig. 4, ignoring $\frac{\partial^2 u}{\partial t^2}$, we have

$$-H - P_0(t) + S = 0.$$

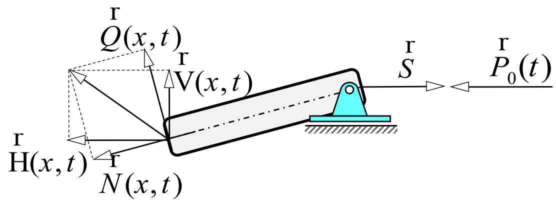

Fig. 4

Therefore $H = S - P_0(t)$

$$H(t) = \frac{EA}{2\ell} \int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - P_0(t). \tag{30}$$

Note: From formula (24), we have

$$ds = \left[1 + \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx. \tag{31}$$

2.3. General nonlinear bending vibration equation of elastic beam

Using approximate formulas

$$\sin x \approx x - \frac{x^3}{6}, \quad \cos x \approx 1 - \frac{x^2}{2}, \tag{32}$$

$$\begin{aligned}\sin(\varphi + d\varphi) &= \sin\varphi\cos(d\varphi) + \cos\varphi\sin(d\varphi) \\ &\approx \sin\varphi\left[1 - \frac{(d\varphi)^2}{2}\right] + \cos\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right],\end{aligned} \tag{33}$$

$$\begin{aligned}\cos(\varphi + d\varphi) &= \cos\varphi\cos(d\varphi) - \sin\varphi\sin(d\varphi) \\ &\approx \cos\varphi\left[1 - \frac{(d\varphi)^2}{2}\right] - \sin\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right].\end{aligned} \tag{34}$$

Substituting (33), (34) into Eqs. (1), (2) and paying attention to formula (31), we have

$$\begin{aligned}\rho A(x)dx\frac{\partial^2 w}{\partial t^2} &= -c\frac{\partial w}{\partial t}\rho A(x)dx - Q\cos\varphi \\ &\quad + (Q + dQ)\left\{\cos\varphi\left[1 - \frac{(d\varphi)^2}{2}\right] - \sin\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right]\right\} + N\sin\varphi \\ &\quad - (N + dN)\left\{\sin\varphi\left[1 - \frac{(d\varphi)^2}{2}\right] + \cos\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right]\right\} + p(x,t)dx \\ &\approx -c\frac{\partial w}{\partial t}\rho A(x)dx + dQ\cos\varphi - Q\sin\varphi d\varphi - dN\sin\varphi - N\cos\varphi d\varphi + p(x,t)dx \\ &\approx -c\frac{\partial w}{\partial t}\rho A(x)dx + d(Q\cos\varphi - N\sin\varphi) + p(x,t)dx.\end{aligned}$$

Simplifying the above equation, we have

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} = -\rho A(x)\left(c\frac{\partial w}{\partial t}\right) + \frac{\partial V}{\partial x} + p(x,t), \tag{35}$$

$$\begin{aligned}\rho A(x)dx\frac{\partial^2 u}{\partial t^2} &= Q\sin\varphi + (Q+dQ)\left\{\sin\varphi\left[1-\frac{(d\varphi)^2}{2}\right] + \cos\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right]\right\}\\ &\quad - N\cos\varphi + (N+dN)\left\{\cos\varphi\left[1-\frac{(d\varphi)^2}{2}\right] - \sin\varphi\left[d\varphi - \frac{(d\varphi)^3}{6}\right]\right\}\\ &\approx dQ\sin\varphi + Q\cos\varphi d\varphi + dN\cos\varphi - N\sin\varphi d\varphi\\ &\approx d(Q\sin\varphi + N\cos\varphi).\end{aligned}$$

So we have

$$\rho A(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial H}{\partial x}. \tag{36}$$

From Eq. (26) we have

$$dM_y = Qds + dQ\frac{ds}{2} \Rightarrow \frac{\partial M_y}{\partial s} = Q + \frac{dQ}{2} \approx Q \Rightarrow Q = \frac{\partial M_y}{\partial x}\frac{\partial x}{\partial s}. \tag{37}$$

In the sense of the derivation we have

$$\cos\varphi = \frac{\partial x}{\partial s}. \tag{38}$$

Substituting (38) into (37) we get

$$Q = \frac{\partial M_y}{\partial x}\cos\varphi. \tag{39}$$

From Fig. 4, we have $Q = H\sin\varphi + V\cos\varphi$. Therefore, expression (39) has the form

$$\frac{\partial M_y}{\partial x}\cos\varphi = H\sin\varphi + V\cos\varphi. \tag{40}$$

Divide both sides of (40) by $\cos\varphi$, we have

$$V = \frac{\partial M_y}{\partial x} - H\tan\varphi. \tag{41}$$

When $\frac{\partial^2 u}{\partial t^2}$ viewed as small, from (32) we deduce $\frac{\partial H}{\partial x} = 0 \Rightarrow H = H(t)$. Using the approximation

$$\tan\varphi \approx \varphi + \frac{\varphi^3}{3},$$

the formula (41) is rewritten as

$$V = \frac{\partial M_y}{\partial x} - H\left(\varphi + \frac{\varphi^3}{3}\right). \tag{42}$$

From (42) deduce the approximate formula

$$V(x,t) = \frac{\partial M_y}{\partial x} - H(t)\varphi. \tag{43}$$

Substituting the expression (43) into (35) we get

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} = -\rho A(x)\left(c\frac{\partial w}{\partial t}\right) + \frac{\partial^2 M_y}{\partial x^2} - H(t)\frac{\partial \varphi}{\partial x} + p(x,t). \tag{44}$$

Note that the relationship between deflection and rotation has the following form [7]

$$\tan\varphi = -\frac{\frac{\partial w}{\partial x}}{1+\frac{\partial u}{\partial x}}.$$

According to the assumption of Kichhhoff: $\dfrac{\partial u}{\partial x}$ is small compared to $\dfrac{\partial w}{\partial x}$ so tan $=$ $-\dfrac{\partial w}{\partial x}$. Replace $\tan\varphi = \varphi + \dfrac{\varphi^3}{3}$ we have $\varphi + \dfrac{\varphi^3}{3} = -\dfrac{\partial w}{\partial x}$.

Therefore

$$\varphi \approx -\frac{\partial w}{\partial x}. \tag{45}$$

Substituting expressions (26) and (41) into equation (40), we get a nonlinear vibrational equation of the beam

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} + \rho A(x)\left(c\frac{\partial w}{\partial t}\right) - \frac{\partial^2 M_y}{\partial x^2} + \left[P_0(t) - \frac{EA}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} = p(x,t), \tag{46}$$

within

$$M_y = \int_A z\sigma_x dA = \int_A z\sigma_x dydz = \int_A zf\left(\varepsilon_x, \dot{\varepsilon}_x\right) dydz. \tag{47}$$

Eq. (46) is the equation of flexural vibration of a nonlinear beam under the action of axial force $P_0(t)$ at the shaft end. Depending on the physical nonlinear stress-strain equation of beam $f(\sigma_x, \dot{\sigma}_x, \varepsilon_x, \dot{\varepsilon}_x, t) = 0$, we will calculate the derivative $\dfrac{\partial^2 M_y}{\partial x^2}$, then substitute into Eq. (46).

2.4. Nonlinear bending vibration equation of beam when choosing simple physical nonlinear law

We choose a fairly simple nonlinear physical law like formula (6)

$$\sigma_x = E(1 - a_3E^2\varepsilon_x^2)\varepsilon_x. \tag{48}$$

With a_3 is a constant. Which ε_x is calculated according to (18)

$$\varepsilon_x = \frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - z\frac{\partial^2 w}{\partial x^2}. \tag{49}$$

Here we will show how to calculate the derivative $\dfrac{\partial^2 M_y}{\partial x^2}$ to substitute Eq. (46). Substituting (49) into (48), we get

$$\begin{aligned}
\sigma_x &= f(\varepsilon_x) = E\left(1 - a_3 E^2 \varepsilon_x^2\right)\varepsilon_x \\
&= E\left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - z\frac{\partial^2 w}{\partial x^2}\right] - a_3E^3\left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - z\frac{\partial^2 w}{\partial x^2}\right]^3 \\
&= \left\{E\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx - a_3E^3\left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]^3\right\} \\
&\quad - zE\frac{\partial^2 w}{\partial x^2}\left[1 - 3a_3E^2\left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]^2\right] \\
&\quad - 3z^2a_3E^3\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + z^3a_3E^3\left(\frac{\partial^2 w}{\partial x^2}\right)^3.
\end{aligned} \tag{50}$$

Substituting (50) into (47), we get the expression for calculating bending moment

$$\begin{aligned}
M_y &= \int_A z\sigma_x dA \\
&= \int_A \left\{\begin{array}{c} z\left[E\varepsilon_{x0}(t) - a_3E^3\varepsilon_{x0}^3(t)\right] - z^2E\dfrac{\partial^2 w}{\partial x^2}\left[1 - 3a_3E^2\varepsilon_{x0}^2(t)\right] \\ -3z^3a_3E^3\varepsilon_{x0}(t)\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + z^4a_3E^3\left(\dfrac{\partial^2 w}{\partial x^2}\right)^3 \end{array}\right\} dA.
\end{aligned} \tag{51}$$

Because is a function of and is a function of and so we have

$$\begin{aligned}
\int_A z\left[E\varepsilon_{x0}(t) - a_3E^3\varepsilon_{x0}^3(t)\right] dA &= \left[E\varepsilon_{x0}(t) - a_3E^3\varepsilon_{x0}^3(t)\right]\int_A z dA = 0, \\
\iint_A \frac{\partial^2 w}{\partial x^2}\left[1 - 3a_3E^2\varepsilon_{x0}^2(t)\right] z^2 dA &= \frac{\partial^2 w}{\partial x^2}\left[1 - 3a_3E^2\varepsilon_{x0}^2(t)\right]\iint_A z^2 dA \\
&= \frac{\partial^2 w}{\partial x^2}\left[1 - 3a_3E^2\varepsilon_{x0}^2(t)\right] I_0, \\
\int_A z^3a_3E^3\varepsilon_{x0}(t)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dA &= a_3E^3\varepsilon_{x0}(t)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\int_A z^3 dA = 0, \\
\iint_A \left(\frac{\partial^2 w}{\partial x^2}\right)^3 z^4 dA &= \left(\frac{\partial^2 w}{\partial x^2}\right)^3\iint_A z^4 dA = \left(\frac{\partial^2 w}{\partial x^2}\right)^3 I_2.
\end{aligned}$$

In which we denote

$$I_0 = \iint_A z^2 dA, \quad I_2 = \iint_A z^4 dA. \tag{52}$$

So the expression for bending moment is

$$M_y = -EI_0 \frac{\partial^2 w}{\partial x^2}\left[1 - 3a_3 E^2 \varepsilon_{x0}^2(t)\right] + a_3 E^3 I_2 \left(\frac{\partial^2 w}{\partial x^2}\right)^3. \tag{53}$$

From (53) we have

$$\frac{\partial^2 M_y}{\partial x^2} = -EI_0 \frac{\partial^4 w}{\partial x^4}\left[1 - 3a_3 E^2 \varepsilon_{x0}^2(t)\right] + a_3 E^3 I_2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)^3. \tag{54}$$

Notice that with $g = g(x,t)$, we have

$$\frac{\partial}{\partial x} g^3 = 3g^2 \frac{\partial g}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} g^3\right] = \frac{\partial}{\partial x}\left[3g^2 \frac{\partial g}{\partial x}\right] = 6g \frac{\partial g}{\partial x}\frac{\partial g}{\partial x} + 3g^2 \frac{\partial^2 g}{\partial x^2}.$$

For $g(x,t) = \dfrac{\partial^2 w}{\partial x^2}$, we have

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)^3\right] = 6\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + 3\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \frac{\partial^4 w}{\partial x^4}. \tag{55}$$

Substituting the expression (55) into (54), we get

$$\begin{aligned}
\frac{\partial^2 M_y}{\partial x^2} &= -EI_0 \frac{\partial^4 w}{\partial x^4}\left[1 - 3a_3 E^2 \varepsilon_{x0}^2(t)\right] + 3a_3 E^3 I_2 \left[\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right]\frac{\partial^2 w}{\partial x^2} \\
&= -EI_0 \frac{\partial^4 w}{\partial x^4}\left\{1 - 3a_3 E^2 \left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]^2\right\} \\
&\quad + 3a_3 E^3 I_2 \left[\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right]\frac{\partial^2 w}{\partial x^2}.
\end{aligned} \tag{56}$$

Substituting (56) into the nonlinear oscillation equation (46), we get the bending oscillation equation of the geometric and nonlinear physical elastic beam according to formula (44) and subjected to axial force

$$\begin{aligned}
&\rho A(x)\frac{\partial^2 w}{\partial t^2} + \rho A(x)\left(c\frac{\partial w}{\partial t}\right) + EI_0 \frac{\partial^4 w}{\partial x^4}\left\{1 - 3a_3 E^3 I_0 \left[\frac{1}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]^2\right\} \\
&+ \left[P_0(t) - \frac{EA}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} = 3a_3 E^3 I_2 \left[\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + p(x,t).
\end{aligned} \tag{57}$$

2.5. Some special cases of physical nonlinear law

a) In case the calculation of the physical law follows the linear elastic law, only the geometric nonlinearity is taken into account and the beam is not subjected to the axial force acting at the top of the shaft.

Then $a_3 = 0, P_0(t) = 0$, from (53) we get Eq. (48)

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} + EI_0\frac{\partial^4 w}{\partial x^4} - \left[\frac{EA}{2\ell}\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} = p(x,t). \tag{58}$$

If we use the notation

$$N = \frac{EA}{2\ell}\int\limits_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx, \tag{59}$$

from Eq. (48) we get

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} + EI_0\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial w}{\partial t} + k_f w = p(x,t). \tag{60}$$

b) In case the physical nonlinearity obeys the rule (48), the geometric nonlinearity is ignored and the beam is not subjected to longitudinal forces acting at the shaft end

Then $P_0(t) = 0$ and if the term is omitted $\varepsilon_{x0}(t) = \dfrac{1}{2\ell}\displaystyle\int_0^\ell \left(\frac{\partial w}{\partial x}\right)^2 dx$, from (59) we get the equation

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} + EI_0\frac{\partial^4 w}{\partial x^4} = 3a_3E^3I_2\left[\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + p(x,t). \tag{61}$$

3. ILLUSTRATED EXAMPLE

In this section, we present an example illustrating the calculation of nonlinear oscillations according to the equations given in the previous paragraph. Considering the model of two hinged beams subjected to agitation of concentrated distributed forces as shown in Fig. 5. According to (60), the vibration equation of the beam paying attention to the geometric nonlinearity has the form

$$EI\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} + \beta\frac{\partial w}{\partial t} + k_f w = p(x,t). \tag{62}$$

Adding the fractional order term $\beta_\alpha\dfrac{\partial^\alpha w}{\partial t^\alpha}$ in the equation (62), we get the following partial derivative equation

$$EI\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} + \beta\frac{\partial w}{\partial t} + \beta_\alpha\frac{\partial^\alpha w}{\partial t^\alpha} + k_f w = p(x,t). \tag{63}$$

In (63) the axial force component Nhas the form

$$N = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{64}$$

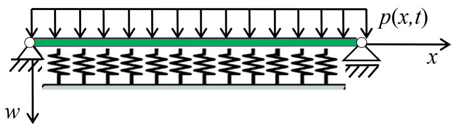

Fig. 5

3.1. Transforming a partial differential equation into a system of differential equations

Applying the Ritz–Galerkin method, we find the solution of the differential - integral equation (63) in the form

$$w(x,t) = \sum_{n=1}^{\infty} \Phi_n(x)q_n(t), \tag{65}$$

where $\Phi_n(x)$ is the form function of the beam with no longitudinal force and no elastic foundation. According to [8] the form function $\Phi_n(x)$ has the following property

$$\frac{d^4\Phi_n(x)}{dx^4} = \frac{\rho A}{EI}\omega_n^2\Phi_n(x). \tag{66}$$

In which we denote

$$\omega_n^2 = \frac{n^4\pi^4 EI}{\rho A L^4}. \tag{67}$$

Using the formula to find the root (67), we can calculate the axial force easily

$$N = \frac{EA}{2L}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\int_0^L \frac{d\Phi_i(x)}{dx}\frac{d\Phi_j(x)}{dx}dx\right] q_i(t)q_j(t). \tag{68}$$

To simplify the expression we put $\xi = \dfrac{x}{L}$. Therefore

$$\int_0^L \frac{d\Phi_i(x)}{dx}\frac{d\Phi_j(x)}{dx}dx = \frac{1}{L}\int_0^L \frac{d\Phi_i(\xi)}{d\xi}\frac{d\Phi_j(\xi)}{d\xi}d\xi.$$

If we enter the symbol

$$K_{ij} = \int_0^L \frac{d\Phi_i(\xi)}{d\xi}\frac{d\Phi_j(\xi)}{d\xi}d\xi, \tag{69}$$

then

$$\int_0^L \frac{d\Phi_i(x)}{dx}\frac{d\Phi_j(x)}{dx}dx = \frac{1}{L}K_{ij}. \tag{70}$$

Substituting (70) into (68) we get the expression to determine the axial force

$$N = \frac{EA}{2L^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}q_i(t)q_j(t). \tag{71}$$

Using formula (65), Eq. (63) is transformed to the form

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\rho A\ddot{q}_n(t) + \beta\dot{q}_n(t) + k_f q_n(t) + \rho A\omega_n^2 q_n(t) + \beta_\alpha\frac{\partial^\alpha q_n(t)}{\partial t^\alpha}\right]\Phi_n(x) \\ -\frac{EA}{2L^2}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}q_i(t)q_j(t)q_n(t)\frac{d^2\Phi_n(x)}{dx^2} = p(x,t).\end{aligned} \tag{72}$$

Notice that we have the expression

$$\frac{d^2\Phi_n(x)}{dx^2} = \frac{1}{L^2}\frac{d^2\Phi_n(\xi)}{d\xi^2}.$$

So Eq. (72) can be rewritten as

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\rho A\ddot{q}_n(t) + \beta\dot{q}_n(t) + k_f q_n(t) + \rho A\omega_n^2 q_n(t) + \beta_\alpha\frac{\partial^\alpha q_n(t)}{\partial t^\alpha}\right]\Phi_n(\xi) \\ &-\frac{EA}{2L^4}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}q_i(t)q_j(t)q_n(t)\frac{d^2\Phi_n(\xi)}{d\xi^2} = p(\xi,t).\end{aligned} \tag{73}$$

Multiplying equation (73) by the form function $\Phi_m(\xi)$ and integrating over the length of the beam from 0 to L, using the orthogonality of the form function, we get the formula

$$\begin{aligned}&\left[\rho A\ddot{q}_m(t) + \beta\dot{q}_n(t) + k_f q_m(t) + \rho A\omega_m^2 q_m(t) + \beta_\alpha\frac{\partial^\alpha q_n(t)}{\partial t^\alpha}\right]\int_0^1 \Phi_m^2(\xi)d\xi \\ &-\frac{EA}{2L^4}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}q_i(t)q_j(t)q_n(t)\int_0^1 \Phi_m(\xi)\frac{d^2\Phi_n(\xi)}{d\xi^2}d\xi = \frac{1}{L}\int_0^1 \Phi_m(\xi)p(\xi,t)d\xi.\end{aligned} \tag{74}$$

In some recent documents, we often choose the normalization function according to the following conditions

$$\int_0^1 \Phi_m^2(\xi)d\xi = 1. \tag{75}$$

If we use the notation

$$R_{mn} = \int_0^1 \Phi_m(\xi)\frac{d^2\Phi_n(\xi)}{d\xi^2}d\xi, \tag{76}$$

then Eq. (74) has the form

$$\begin{aligned}&\ddot{q}_m(t) + \frac{\beta}{\rho A}\dot{q}_m(t) + \omega_m^2 q_m(t) + \frac{k_f}{\rho A}q_m(t) + \frac{\beta_\alpha}{\rho A}\frac{\partial^\alpha q_m(t)}{\partial t^\alpha}\\ &-\frac{E}{2L^4\rho}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}R_{mn}q_i(t)q_j(t)q_n(t) = \frac{1}{\rho AL}\int_0^1 \Phi_m(\xi)p(\xi,t)d\xi.\end{aligned} \tag{77}$$

Notice that

$$\begin{aligned}\int_0^1 \Phi_m(\xi)\frac{d^2\Phi_n(\xi)}{d\xi^2}d\xi &= \int_0^1 \Phi_m\frac{d}{d\xi}\left(\frac{d\Phi_n}{d\xi}\right)d\xi\\ &= \underbrace{\Phi_m\frac{d\Phi_n}{d\xi}\bigg|_{\xi=0}^{\xi=1}}_{=0} - \int_0^1 \frac{d\Phi_n}{d\xi}\frac{d\Phi_m}{d\xi}d\xi = -K_{nm}.\end{aligned} \tag{78}$$

Paying attention to the boundary conditions of the hinged beam at both ends. So Eq. (77) now has the form

$$\begin{aligned}&\ddot{q}_m(t) + \frac{\beta}{\rho A}\dot{q}_m(t) + \left(\omega_m^2 + \frac{k_f}{\rho A}\right)q_m(t) + \frac{\beta_\alpha}{\rho A}\frac{\partial^\alpha q_m(t)}{\partial t^\alpha}\\ &+\frac{E}{2\rho L^4}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}K_{mn}q_iq_jq_n = \frac{1}{\rho AL}\int_0^1 \Phi_m(\xi)p(\xi,t)d\xi = h_m(t),\end{aligned} \tag{79}$$

where the index m takes the values from 1 to m_N. The right-hand side function in (79) has the form

$$h_m(t) = \frac{1}{\rho AL}\int_0^1 \Phi_m(\xi)p(\xi,t)d\xi. \tag{80}$$

If we choose the normalized form functions according to the condition

$$\Phi_m(\xi) = \sqrt{2}\sin(m\pi\xi), \tag{81}$$

then from (69) we have

$$K_{nm} = K_{mn} = \int_0^1 \frac{d\Phi_n}{d\xi}\frac{d\Phi_m}{d\xi}d\xi = \begin{cases}\pi^2m^2 & \text{when} \quad m=n\\ 0 & \text{when} \quad m\neq n\end{cases} \tag{82}$$

Eq. (79) now takes the form

$$\ddot{q}_m(t) + \frac{\beta}{\rho A}\dot{q}_m(t) + \omega_m^2\left(1 + \frac{\alpha}{m^4}\right) q_m(t) + \frac{\beta_\alpha}{\rho A}\frac{\partial^\alpha q_m(t)}{\partial t^\alpha} + \frac{\omega_m^2}{2R^2m^2}\sum_{n=1}^{M} n^2 q_n^2 q_m = h_m(t). \quad (83)$$

Inside

$$\omega_m^2 = \omega_0^2 m^4, \quad \omega_0^2 = \frac{\pi^4 EI}{\rho A L^4}, \quad k = \frac{k_f}{\rho A \omega_0^2}, \quad R = \sqrt{\frac{I}{A}}, \quad (84)$$

where ω_0 is the fundamental frequency.

To simplify the calculation, we choose $m_N = 1$, from (83) we have an equation as follows

$$\ddot{q}_1(t) + \frac{\beta}{\rho A}\dot{q}_1(t) + \omega_0^2(1+k)\, q_1(t) + \frac{\omega_0^2}{2R^2}q_1^3(t) + \frac{\beta_p}{\rho A}\frac{\partial^p q_1(t)}{\partial t^p} = h_1(t). \quad (85)$$

Eq. (85) is a Duffing equation with the addition of a fractional derivative resistance term. Shifting a few terms of Eq. (85) to the right hand side we get an equation of the form

$$\ddot{q}_1(t) + \omega_0^2 q_1(t) = -k\omega_0^2 q_1(t) - \frac{\omega_0^2}{2R^2}q_1^3(t) - \frac{\beta}{\rho A}\dot{q}_1(t) - \frac{\beta_p}{\rho A}\frac{\partial^p q_1(t)}{\partial t^p} + h_1(t). \quad (86)$$

The function $h_1(t)$ on the right hand side now has the form

$$h_1(t) = \frac{1}{\rho A L}\int_0^1 \Phi_1(\xi) p(\xi,t)\, d\xi = \frac{1}{\rho A L}\int_0^1 \sqrt{2}\sin(\pi\xi) p(\xi,t)\, d\xi. \quad (87)$$

Consider the case of beams subjected to uniformly distributed external loads

$$p(x,t) = P_0\cos\Omega t \quad \Rightarrow \quad p(\xi,t) = P_0\cos\Omega t. \quad (88)$$

Then $h_1(t)$ has the form

$$h_1(t) = \frac{1}{\rho A L}\int_0^1 \sqrt{2}\sin(\pi\xi) P_0\cos\Omega t\, d\xi = \frac{2P_0\sqrt{2}}{\pi\rho A L}\cos\Omega t. \quad (89)$$

3.2. Calculation of nonlinear oscillation

To study the main resonance oscillation of the system (86), when $\Omega \approx \omega_0$, we set

$$\Omega^2 = \omega_0^2 + \varepsilon\sigma, \quad (90)$$

where ε is a small parameter, σ representing the difference between Ω with ω_0.

With some assumptions about the parameters, Eq. (86) can now be reduced to the form

$$\begin{aligned}\ddot{q}_1(t) + \Omega^2 q_1(t) = & -\varepsilon\left[k\left(\Omega^2 - \varepsilon\sigma\right) - \sigma\right] q_1(t) - \varepsilon\frac{\left(\Omega^2 - \varepsilon\sigma\right)}{2R^2} q_1^3(t) \\ & - \frac{\varepsilon\beta}{\rho A}\dot{q}_1(t) - \frac{\varepsilon\beta_p}{\rho A}\frac{\partial^p q_1(t)}{\partial t^p} + \varepsilon\frac{2P_0\sqrt{2}}{\pi\rho A L}\cos\Omega t,\end{aligned} \tag{91}$$

where ε is the small parameter. To simplify mathematical expressions we denote $q = q_1(t)$. Ignoring the effect of higher order infinity ε^2, Eq. (91) is rewritten as

$$\ddot{q}(t) + \Omega^2 q(t) = \varepsilon f\left(t, q, D^p q, \dot{q}\right). \tag{92}$$

Inside

$$\begin{aligned}& f(t, q, D^p q, \dot{q}) = -\left(k\Omega^2 - \sigma\right) q(t) - \alpha q^3(t) - \delta\dot{q}(t) - \delta_p\frac{\partial^p q(t)}{\partial t^p} + E\cos\Omega t, \\ & \alpha = \frac{\Omega^2}{2R^2}, \quad \delta = \frac{\beta}{\rho A}, \quad \delta_p = \frac{\beta_p}{\rho A}, \quad E = \frac{2P_0\sqrt{2}}{\pi\rho A L}.\end{aligned} \tag{93}$$

Using the asymptotic method, we can solve the oscillation equation (92). The amplitude-frequency curve is plotted in Fig. 6. In it, we choose the following parameters

$$\delta_p = 0.1, \quad p = 0.5, \quad E = 1, \quad \alpha = 1, \quad \delta = 0.2, \quad k = 0.1, \quad \omega_0 = 1, \quad \Omega \approx \omega_0.$$

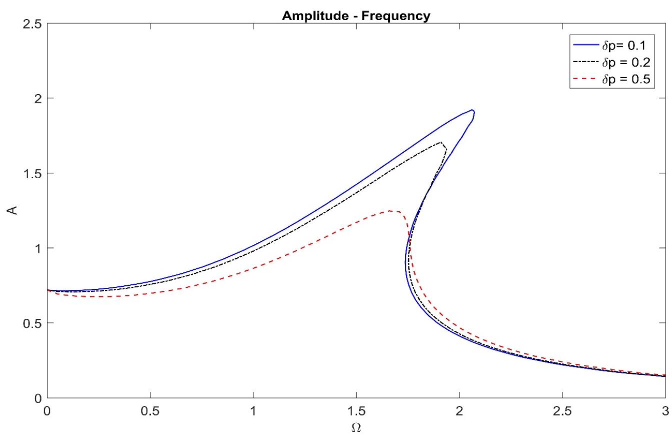

Fig. 6. The amplitude-frequency curve (dotted lines represent stable conditions)

4. CONCLUSION

In this paper, we present the establishment of a relatively general nonlinear beam vibration equation for the study of nonlinear beam effects. Eq. (46) allows us to derive equations or use nonlinear vibration calculations of beams.

A relatively brief illustrative example is presented at the end of the article. Readers who are interested in calculating the nonlinear vibrations of elastic beams can find them in the literature [9, 10].

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

FUNDING

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

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