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AN APPLICATION OF RAYLEIGH QUOTIENT FOR CRACK DETECTION IN SIMPLY SUPPORTED BEAM

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Abstract. An explicit expression of natural frequencies through crack parameters is derived for multiple cracked beams with simply supported boundaries using the Rayleigh quotient. The obtained expression provides not only a simple tool for calculating natural frequencies of multiple cracked beams, but also allows employing the so-called crack scanning method for detecting multiple cracks in simply supported beams from measured natural frequencies. A numerical example demonstrates that the crack scanning method, in combination with the Rayleigh quotient, enables consistent identification of cracks with 1% relative depth.

Keywords: multiple cracked beam, Rayleigh quotient, crack identification, frequency-based method.

1. INTRODUCTION

The detection of cracks in structures and machinery is a vital concern because cracks may lead to catastrophic accidents if they are not recognized early. While conventional nondestructive techniques (such as ultrasonic or radiography, etc.) are limited in application for local damage detection, methods based on measurements of vibratory signals are global and more efficient, especially, in use for large and complicated engineering structures. Among the methods proposed to detect damage in structures using changes in their dynamic characteristics [1–3], many techniques are based on measurements of modal parameters such as natural frequencies and mode shapes. The mode shape, as a spatial characteristic of a structure, has proven to be a useful indicator for damage localization in structures [4–8], but it is more difficult to measure. Sometimes, measurement errors may hide changes in mode shape induced by damages. Natural frequencies have been used since early on [9–12] for damage assessment of structures and are still being used today [13, 14]. The practice of modal testing has confirmed that natural measured in comparison with mode shape. The most significant drawback of the frequency-based approach is that natural frequencies are only slightly sensitive to small damages, and the same change in frequencies might be caused by different damages. This leads to difficulty in detecting damage at its early stage (small extent) and non-uniqueness of the damage identification problem when using only natural frequencies. Therefore, seeking ways to overcome the shortcomings of using natural frequencies in damage detection problems, from one point of view, is more promising than solving problems related to the difficulty and erroneousness in mode shape measurements.

The theoretical basis of the frequency-based method for damage detection is the so-called characteristic equation that relates natural frequencies to damage parameters. Different forms of the characteristic equation were conducted in [15–18] for a beam-like structure with a single crack. Then, the equation was established for beams with multiple cracks [19–22]. Although the characteristic equation has been obtained explicitly, natural frequencies could only be computed numerically as implicit functions of damage parameters. This implicit interpretation of natural frequencies determined numerically from the characteristic equation makes it difficult to solve the problem of damage detection from natural frequencies. An explicit expression of natural frequencies in terms of crack magnitude was derived approximately in [23] for the case of small cracks using the perturbation method, and this expression has been used for crack detection in [24, 25]. A system of linear equations relating the shift of natural frequencies with variations of both crack magnitudes and positions was conducted in [26], but again numerically using the finite element model. By introducing the so-called element damage index, Liang et al. [27] and Patil and Maiti [28] were able to express natural frequency shifts in terms of damage indices in an explicit form of linear equations that provide a useful tool for damage localization by measured natural frequencies. Although the conventional Rayleigh method was employed early on for determining the fundamental frequency of a cracked beam by Shen and Pierre [29], an explicit expression of the fundamental frequency of a beam with a single crack in terms of both position and size of the crack was obtained much later by Fernandez-Saez et al. [30]. Recently, an expansion of the Rayleigh quotient for calculating natural frequencies of cracked beams has been developed in [31], but like the former results, it has not been straightforward to use for the crack detection problem.

In the present paper, the Rayleigh quotient for multiple cracked simply supported beams is conducted and employed to obtain an explicit expression of an arbitrary natural frequency in terms of crack position and size. The obtained expression is then used for crack identification in simply supported beams by natural frequencies.

2. RAYLEIGH QUOTIENT FOR MULTIPLE CRACKED BEAM

Let's consider a uniform Euler-Bernoulli beam with elasticity modulus E, mass density ρ, length L, cross-section area $F = b \times h$, moment of inertia I, and arbitrarily given boundary conditions at $x = 0$ and $x = 1$. For the beam, the k-th natural frequency and mode shape, denoted by $\omega_k, \phi_k(x)$, satisfy equations

$$\phi_k^{(IV)}(x) - \lambda_k^4 \phi_k(x) = 0, \quad x \in (0,1), \quad \lambda^4 = L^4 \rho F \omega_k^2 / EI. \tag{1}$$

Suppose, moreover, that the beam has been damaged to crack at positions $0 < e_1 < \ldots < e_n < 1$ with the unknown depth $(a_1, \ldots, a_n)$. If the spring model of the cracks is adopted, the so-called crack magnitude $(\gamma_1, \ldots, \gamma_n)$ can be used instead of the crack depth as follows

$$\gamma_j = EI/LK_j = (6\pi h/L) I_c(a_j/h), \tag{2}$$

where function [27,29]

$$I_c(z) = 1.8624z^2 - 3.95z^3 + 16.375z^4 - 37.226z^5 + 76.81z^6 \\ -126.9z^7 + 172z^8 - 143.97z^9 + 66.56z^{10}. \tag{3}$$

In this case, the mode shape $\phi_k(x)$ should satisfy the following conditions at the cracks

$$\phi_k(e_j^-) = \phi_k(e_j^+), \quad \phi_k''(e_j^-) = \phi_k''(e_j^+), \quad \phi_k'''(e_j^-) = \phi_k'''(e_j^+), \\ [\phi_k'(e_j^+) - \phi_k(e_j^-)] = \gamma_j \phi_k''(e_j), \quad j = 1, \ldots, n. \tag{4}$$

Now, multiply both sides of Eq. (1) by $\phi_k(x)$ and integrate along the beam length to obtain

$$\int_0^1 \phi_k^{(IV)}(x)\phi_k(x)\mathrm{d}x - \lambda_k^4 \int_0^1 \phi_k^2(x)\mathrm{d}x = 0. \tag{5}$$

Note that for the functions $\phi(x), \phi'(x), \phi''(x), \phi'''(x)$ that are all continuous in the segment (a, b) it can be easily obtained

$$\int_a^b \phi^{(IV)}(x)\phi(x)\mathrm{d}x = \int_a^b \phi''^2(x)\mathrm{d}x + [B(b) - B(a)], \tag{6}$$

where $B(x) = \phi'''(x)\phi(x) - \phi''(x)\phi'(x)$.

Applying the equality (6) for the integral $\int_{e_{j-1}}^{e_j} \phi_k^{(IV)}(x)\phi_k(x)\mathrm{d}x$ and then summing for $j = 1, \ldots, n+1$ with notice that $e_0 = 0, e_{n+1} = 1$ give

$$\int_0^1 \phi_k^{(IV)}(x)\phi_k(x)\mathrm{d}x = \sum_{j=1}^{n+1}\int_{e_{j-1}}^{e_j} \phi_k^{(IV)}(x)\phi_k(x)\mathrm{d}x$$
$$= \int_0^1 \phi_k''^2(x)\mathrm{d}x + [B_k(1) - B_k(0)] + \sum_{j=1}^{n}[B_k(e_j^+) - B_k(e_j^-)].$$

Using the conditions (4) the latter equation can be rewritten as

$$\int_0^1 \phi_j^{(IV)}(x)\phi_k(x)\mathrm{d}x = \int_0^1 \phi_k''^2(x)\mathrm{d}x + \sum_{j=1}^{n}\gamma\phi_k^{2\prime\prime}(e_j) + [B_k(1) - B_k(0)]. \quad (7)$$

Therefore, Eq. (5) becomes

$$\lambda_k^4 = \left[\int_0^1 \phi_k^{2\prime\prime}(x)\mathrm{d}x + \sum_{j=1}^{n}\gamma_j\phi_k^{2\prime\prime}(e_j) + [B_k(1) - B_k(0)]\right] / \int_0^1 \phi_k^2(x)\mathrm{d}x, \quad (8)$$

which is the Rayleigh quotient extended for multiple cracked uniform beams. The quotient (8) is unable yet to use for either calculating natural frequencies or crack detection from given natural frequencies because the mode shape of the beam is unknown. For the case of undamaged beams, the well-known Rayleigh quotient was taken in use for calculating natural frequency of beam by a proper choosing trial shape function, and this is called the Rayleigh method. In this paper, the Rayleigh method is extended not only for calculating the natural frequencies of multiple cracked beams but also for solving the problem of crack detection from measured natural frequencies.

For this purpose, the beam length, $L = 1$, is divided into n segments $(x_{j-1}, x_j), j = 1, \ldots, n$ with $x_{j-1} < e_j < x_j, x_0 = 0, x_n = 1$ and mode shape functions are selected in the form

$$\phi_k(x) = \phi_{0k}(x) + \hat{\phi}_k(x), \quad x \in (x_{j-1}, x_j), \quad j = 1, \ldots, n, \quad (9)$$
$$\hat{\phi}_k(x) = A_{kj}x^3 + Bx^2 + C_{kj}x + D_{kj} + \gamma_j\phi_k''(e_j)\begin{cases}(e_j - x), & x_{j-1} \le x \le e_j\\ 0, & e_j \le x \le x_j\end{cases}$$

where $\phi_{0k}(x)$ is k-th mode shapes of intact beam satisfying given boundary conditions. Such chosen shape functions (9) satisfy the conditions (4) at crack positions and the continuity conditions at x_j would be satisfied under conditions

$$\begin{aligned}&A_{kj} = A_{k,j+1}, \quad B_{kj} = B_{k,j+1}, \quad C_{kj} = C_{k,j+1} - \gamma_{j+1}\phi_k''(e_{j+1}),\\ &D_{kj} = D_{k,j+1} + \gamma_{j+1}\phi_k''(e_{j+1})e_{j+1}, \quad j = 1, \ldots, n-1.\end{aligned} \quad (10)$$

Assuming that

$$A_{k0} = A_{k1}, \quad B_{k0} = B_{k1}, \quad C_{k0} = C_{k1} - \gamma_1\phi_k''(e_1), \quad D_{k0} = D_{k1} + e_1\gamma_1\phi_k''(e_1), \quad (11)$$

one obtains

$$A_{kj} = A_{k0},\ B_{kj} = B_{k0},\ C_{kj} = C_{k0} + \sum_{i=1}^{j} \gamma_i \phi_k''(e_i),\ D_{kj} = D_{k0} - \sum_{i=1}^{j} e_i \gamma_i \phi_k''(e_i),\ j = 1, \dots, n. \tag{12}$$

So, it remains unknown four constants $A_{k0}, B_{k0}, C_{k0}, D_{k0}$ that can be determined from boundary conditions as follows. For simply supported beam, the boundary conditions

$$\phi_k(0) = \phi_k''(0) = \phi_k(1) = \phi_k''(1) = 0, \tag{13}$$

give rise

$$A_{k0} = B_{k0} = D_{k0} = 0, \quad C_{k0} = -\sum_{j=1}^{n} \gamma_j (1 - e_j) \phi_k''(e_j). \tag{14}$$

So that one gets finally

$$A_{kj} = B_{kj} = 0, \quad C_{kj} = \sum_{i=1}^{n} (e_i - 1)\gamma_i \phi_k''(e_i) + \sum_{i=1}^{j} \gamma_i \phi_k''(e_i), \quad D_{kj} = -\sum_{i=1}^{j} \gamma_i e_i \phi_k''(e_i). \tag{15}$$

The coefficients $A_{k0}, B_{k0}, C_{k0}, D_{k0}$ found above for different cases of boundary conditions allow obtaining a more explicit expression of the right-hand side in Eq. (8).

First, it is easy to verify that for any homogeneous (traditional) boundary conditions one has

$$B_k(0) = \phi'''(0)\phi(0) - \phi''(0)\phi'(0) = 0, \quad B_k(1) = \phi'''(1)\phi(1) - \phi''(1)\phi'(1) = 0. \tag{16}$$

So that the numerator and denominator of the quotient (8) can be calculated as

$$\text{Numerator} = \int_0^1 \phi_{0k}''^2(x)\mathrm{d}x + \sum_{j=1}^{n} \gamma_j \phi_{0k}''^2(e_j), \tag{17}$$

$$\text{Denominator} = \int_0^1 \phi_{0k}^2(x)\mathrm{d}x + 2\lambda_{0k}^{-4} \sum_{j=1}^{n} \gamma_j \phi_{0k}''^2(e_j) + \sum_{i,j=1}^{n} \gamma_i \gamma_j q_{ij} \phi_{0k}''(e_i)\phi_{0k}''(e_j), \tag{18}$$

with functions $q_{ij}(e_1, \dots, e_n), i, j = 1, \dots, n$ calculated for the simple supports (SS) boundary conditions respectively as

$$q_{ji} = \frac{1}{3} \begin{cases} e_j^2(1 - e_j)^2 & : \ j = i \\ e_j^3(e_i - 1) + e_i e_j & : \ i \succ j \\ e_i e_j (1 - 3e_j + e_j^2) & : \ i \prec j \end{cases} \tag{19}$$

Hence, the quotient (8) can be written in the form

$$\lambda_k^4 = \frac{\int\limits_0^1 \phi_{0k}''^2(x)\mathrm{d}x + \sum\limits_{j=1}^n \gamma_j \phi_{0k}''^2(e_j)}{\int\limits_0^1 \phi_{0k}^2(x)\mathrm{d}x + 2\lambda_{0k}^{-4}\sum\limits_{j=1}^n \gamma_j\phi_{0k}''^2(e_j) + \sum\limits_{i,j=1}^n \gamma_i\gamma_j q_{ij}\phi_{0k}''(e_i)\phi_{0k}''(e_j)}. \tag{20}$$

This is an expression of eigenvalues of multiple cracked beams through the crack positions and magnitudes and modal parameters of undamaged beam.

In the case of undamaged beam, when $\gamma_j = 0, j = 1, \ldots, n$, the latter equation is reduced to the well-known classical Rayleigh quotient

$$\lambda_k^4 = \lambda_{0k}^4 = \int\limits_0^1 \phi_{0k}''^2(x)\mathrm{d}x \Big/ \int\limits_0^1 \phi_{0k}^2(x)\mathrm{d}x\,. \tag{21}$$

Furthermore, since the mode shape contains an arbitrary constant one can without loss of generality assume that $\phi_{0k}(x) = N_k\bar{\phi}_{0k}(x)$ with $\int\limits_0^1 \bar{\phi}_{0k}^2(x)\mathrm{d}x = 1$. Therefore, $\lambda_{0k}^4 = \int\limits_0^1 \bar{\phi}_{0k}''^2(x)\mathrm{d}x$ and Eq. (21) becomes

$$\frac{\lambda_k^4}{\lambda_{0k}^4} = \frac{1 + \lambda_{0k}^{-4}\sum\limits_{j=1}^n \gamma_j\bar{\phi}_{0k}''^2(e_j)}{1 + 2\lambda_{0k}^{-4}\sum\limits_{j=1}^n \gamma_j\bar{\phi}_{0k}''^2(e_j) + \sum\limits_{i,j=1}^n \gamma_i\gamma_j q_{ij}\bar{\phi}_{0k}''(e_i)\bar{\phi}_{0k}''(e_j)}. \tag{22}$$

Furthermore, expanding the right-hand side of Eq. (22) in the Taylor series and using the notations $\hat{\phi}_{0k}''(x) = \lambda_{0k}^{-4}\bar{\phi}_{0k}''(x)$ for modal curvature of the normalized mode shape $\bar{\phi}_{0k}(x)$ one obtains

$$\frac{\lambda_k^{*4}}{\lambda_{0k}^4} = 1 - \lambda_{0k}^{-4}\sum_{j=1}^n \gamma_j\hat{\phi}_{0k}''^2 - \vartheta_k \sum_{j,i=1}^n q_{ji}\gamma_j\gamma_i\hat{\phi}_{0k}''(e_j)\hat{\phi}_{0k}''(e_i), \tag{23}$$

where ϑ_k stands for a factor introduced to minimize the truncated error in the Taylor series. The obtained expression (23) is a novel explicit representation of eigenvalues in terms of crack parameters for cracked beam that provides a simple tool not only for calculating natural frequencies beam with given crack parameters but also is straightforward to develop a procedure for crack evaluation from given natural frequencies.

3. THE CRACK SCANNING TECHNIQUE

Supposing that m natural frequencies $\omega_1^*, \ldots, \omega_m^*$ of a uniform beam with given end conditions are known, the problem is to evaluate the number, location, and depth of cracks possibly occurred in the beam. This problem is distinguished from the conventional model-based crack detection problem by the fact that the number of potential cracks in beam is unknown. This problem is solved by using a method that could be so-called crack scanning method. The essential content of the crack scanning method consists of following tasks:

(1) Introducing a mesh of positions $(e_1, \ldots, e_n)$ of cracks with unknown magnitude $(\gamma_1, \ldots, \gamma_n)$.

(2) Constructing a model of beam with the assumed crack grid that enables to relate the unknown crack magnitudes to natural frequencies of the beam.

(3) Estimating the unknown crack magnitudes based on the constructed model and given measured frequencies.

(4) Using the points $e_1, \ldots, e_{n_c}$ from the mesh where the crack magnitudes have been positive-definitely estimated as a new mesh and returning to step 2 until the mesh is unchanged.

(5) The desired crack magnitudes and positions would be detected as result of step 4 and depth of the detected cracks is calculated from the estimated magnitudes.

Obviously, the crucial point of the crack scanning method is deriving the diagnostic equations, that is accomplished as follows:

For the measured natural frequencies $\omega_1^*, \ldots, \omega_m^*$, associated eigenvalues can be easily calculated as

$$\lambda_k^* = (L^4 \rho F \omega_k^{*2} / EI)^{1/4}, \quad k = 1, \ldots, m. \tag{24}$$

The modal parameters of intact beam $\{\lambda_{0k}, \bar{\phi}_{0k}(x), k = 1, \ldots, m\}$ are given in any textbook on the Dynamics of Structures, for instance, the reference [32]. Now we can rewrite Eq. (23) as

$$[\mathbf{A} + \mathbf{B}(\gamma)]\gamma = \mathbf{b}, \tag{25}$$

$$\mathbf{A} = [a_{kj} = \bar{\phi}_{k0}''^2(e_j)], \quad \mathbf{B}(\gamma) = [b_{kj} = \alpha_k \hat{\phi}_{0k}''(e_j) \sum_{i=1}^{n} q_{ji} \gamma_i \hat{\phi}_{0k}''(e_i)],$$

$$\mathbf{b} = \{b_1, \ldots, b_m\}^T, \quad b_k = (1 - \delta_k), \tag{26}$$

$$\alpha_k = \vartheta_k \lambda_{0k}^4, \quad \delta_k = \lambda_{0k}^4 - \lambda_k^{*4}, \quad k = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{27}$$

This provides a nonlinear equation for estimating crack magnitude vector $\gamma = (\gamma_1, \ldots, \gamma_n)$ that can be solved by using the iteration method

$$[\mathbf{A}_{i-1}]\gamma^{(i)} = \mathbf{b}, \tag{28}$$

$$\mathbf{A}_{i-1} = \mathbf{A} + \mathbf{B}(\gamma^{(i-1)}), \quad \gamma^{(0)} = 0, \quad i = 1, 2, 3, \ldots \tag{29}$$

The iteration process is stopped when $\left\| \gamma^{(N)} - \gamma^{(N-1)} \right\| \leq tolerance$. The obtained solution $\hat{\gamma}$ gives the first estimation $(\hat{\gamma}_1, \ldots, \hat{\gamma}_n)$ of the crack magnitudes that are employed for performing steps 4. As result of step 4, crack positions $(e_1, \ldots, e_{\bar{n}_c})$ and magnitude $(\bar{\gamma}_1, \ldots, \bar{\gamma}_{\bar{n}_c})$ are finally determined together with desired number of cracks $\bar{n}_c$. The crack depths are estimated as required in step 5 by solving Eq. (2) with respect to a_j for the given crack magnitude $\bar{\gamma}_j$.

It could be noted that the length of the crack scanning mesh (n) must be very large compared to the limited number (m) of measured natural frequencies. Therefore, the system of equations (30) is in fact under-determinate so the problem is ill-conditioned. To avoid this trouble it can be used the well-developed regularization method that suggests replacing the equation system (30) by

$$[\mathbf{A}_{i-1}^T \mathbf{A}_{i-1} + \beta \mathbf{I}]\gamma^{(i)} = \mathbf{b}, \tag{30}$$

with β is a positive parameter termed by the regularization factor and $\mathbf{I}$ is the unique $n \times n$-matrix. Hence, during solution of the problem one has two parameters ϑ and β that can be selected to improve the results. The first one can be chosen according to sensitivity, and the other - to measurement noise level of the measured natural frequency.

4. NUMERICAL EXAMPLES

For comparison with the earlier results, firstly, the natural frequencies of simply supported beam with two cracks given in [28] are taken in use for crack detection by using the scanning method. Results of the initial evaluation are presented in Fig. 1(a) which shows four peaks 0.25; 0.45; 0.6 and 0.8. Using the later positions as a new mesh for the crack scanning procedure yields the desired crack positions and magnitudes, as shown in Fig. 1(b).

Obviously, the actual cracks are exactly localized by using only four frequencies. Graphics in the lower boxes of Fig. 1 illustrate the similar results of crack detection by using natural frequencies calculated from Eq. (25), the Rayleigh quotient for multiple cracked beams. Furthermore, results of crack detection for the beam with two and three cracks of the identical depth 1% by the natural frequencies computed from the Rayleigh quotient are shown in Figs. 2 and 3. The depth of the detected cracks is calculated and given in Table 1 in comparison with the actual one. Apparently, the scanning method

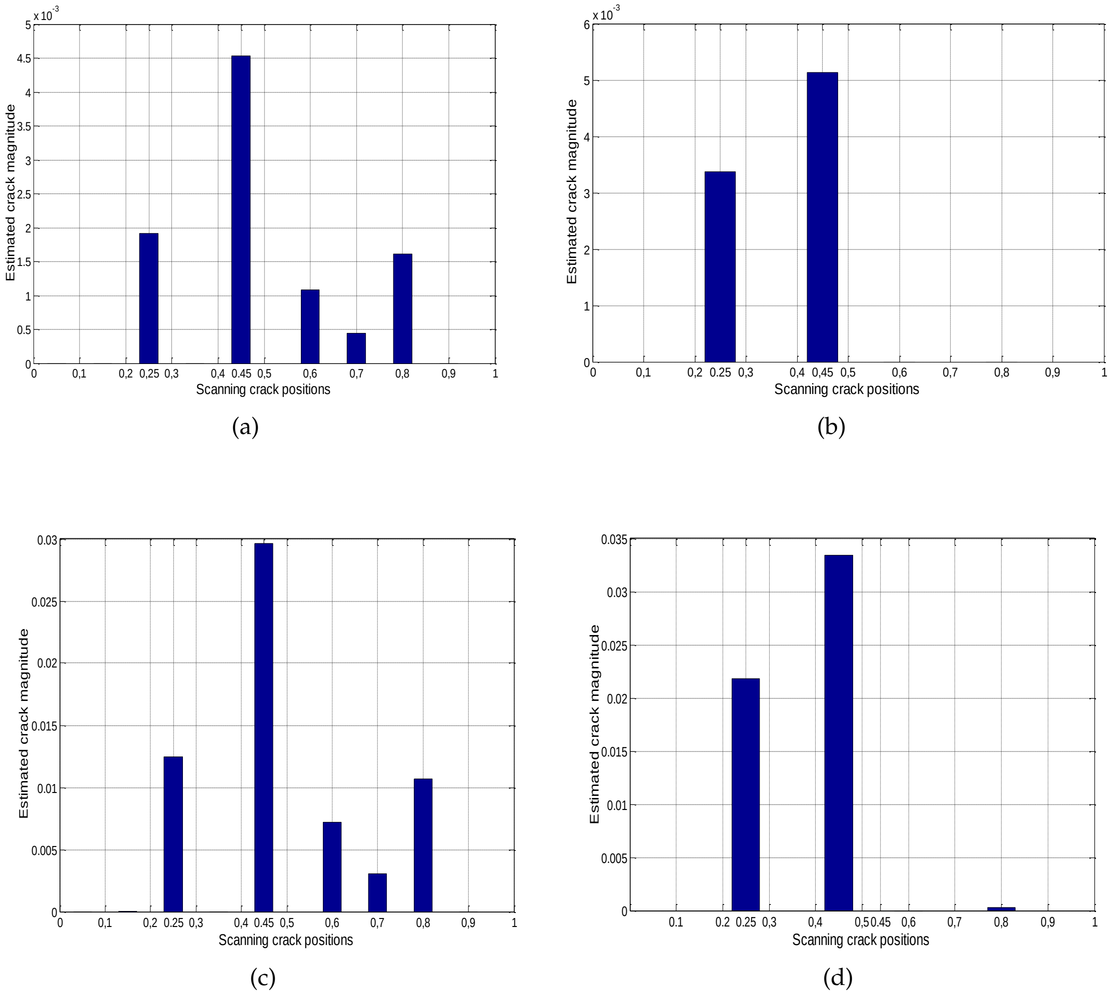

Fig. 1. Results of crack detection for simply supported beam with two cracks at positions (0.25; 0.45) of the depth (0.0797; 0.0986) by using the natural frequencies given in [28] (a, b) and Rayleigh quotient (c, d)

gives more accurate results in crack detection in comparison with the damage index method developed by Patil and Maiti [28] which could complete the crack localization in a beam segment by using the graphical method. Moreover, if natural frequencies calculated by the Rayleigh quotient are used for the crack scanning procedure, both the crack position and depth were exactly predicted even for the crack depth of 1%. These results demonstrate not only the acceptability of the Rayleigh quotient established herein for calculating natural frequencies but also the usefulness of the crack scanning method for multi-crack evaluation in simply supported beam.

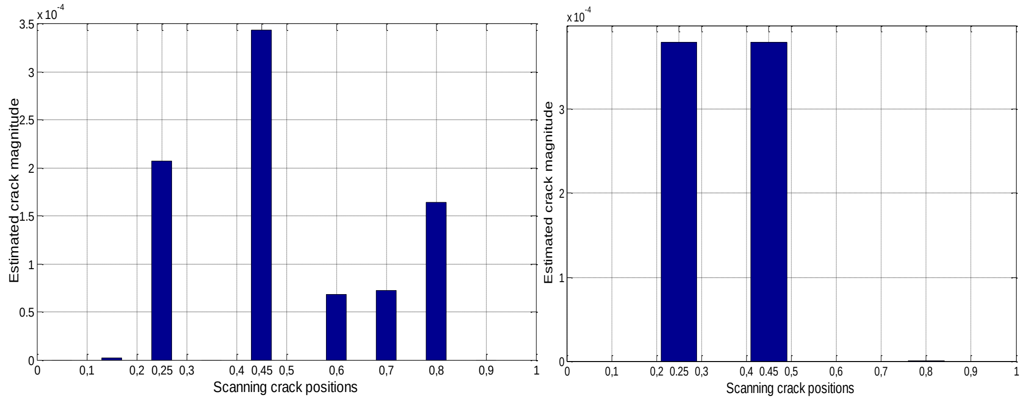

Fig. 2. Results of crack detection for simply supported beam with two cracks at positions (0.25; 0.45) of the same depth 1% by the natural frequencies computed from the Rayleigh quotient

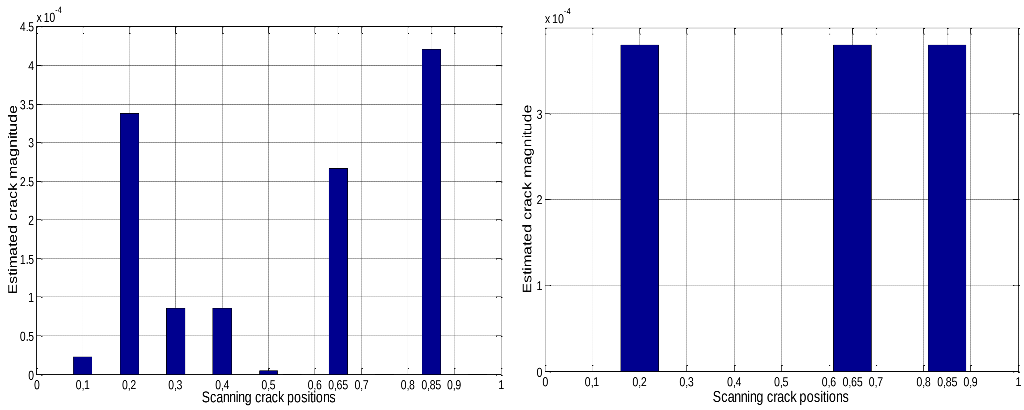

Fig. 3. Results of crack detection for simply supported beam with three cracks at positions (0.2; 0.65; 0.85) of the same depth 1% by the natural frequencies computed from the Rayleigh quotient

Table 1. Results of multiple crack evaluation in a simply supported beam

Crack scenarios		Crack position (e/L)			Crack depth (a/h)	
2 cracks	Actual	0.25	0.45		0.0797	0.0986
	Detected ([28])	**0.25**	**0.45**		**0.0716**	**0.0890**
	Detected (Present)	**0.25**	**0.45**		**0.0795**	**0.0986**
	Actual	0.25	0.45		0.01 (1%)	
	Detected (Present)	**0.25**	**0.45**		**0.01**	
3 cracks	Actual	0.2	0.65	0.85	0.01 (1%)	
	Detected (Present)	**0.25**	**0.65**	**0.85**	**0.01**	

5. CONCLUSION

In this paper, the Rayleigh quotient for a simply supported beam with arbitrary number of cracks has been derived and used for obtaining an explicit expression of natural frequencies through the crack parameter. The obtained interpretation of natural frequencies provides not only a simple tool for calculating natural frequencies with given crack parameters but also is straightforward to apply the so-called crack scanning technique for crack identification from measured natural frequencies. The numerical results obtained in this paper demonstrated that the crack scanning method can consistently evaluate multiple cracks of depth within 1% beam thickness by using a limited number of measured natural frequencies.

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DECLARATION OF COMPETING INTEREST

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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