Vietnam Journal of Mechanics, Vol. 45, No. 2 (2023), pp. 131 – 144
DOI: https://doi.org/10.15625/0866-7136/18148

ON THE TRADE-OFF PERFORMANCE CURVE OF VIBRATION ISOLATION CONTROLLED BY PASSIVE OR ADAPTIVE DAMPINGS

La Duc Viet[1,2,*]
[1]*Institute of Mechanics, Vietnam Academy of Science and Technology, 264 Doi Can, Hanoi, Vietnam*
[2]*Graduate University of Science and Technology, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam*
[*]E-mail: ldviet@imech.vast.vn

Received: 06 March 2023 / Published online: 30 June 2023

Abstract. This paper addresses an on-off adaptive damping isolation, in which the damping can be frequency dependent. The comparison with the passive isolation is illustrated by the trade off performance curve, which is shown the relation between the displacement and force transmissibility. The adaptive control is based on on-off dampings and a switching frequency. The analytical optimization of the adaptive damping law can be obtained. The optimized adaptive damping control showed a remarkable improvement comparing with the passive one. At last, a self-made magnetorheological fluid is introduced to show the capability to provide the variable damping.

Keywords: adaptive damping, vibration isolation, analytical optimization, on-off damping.

1. INTRODUCTION

Vibration isolation involves the insertion of the springs and dampers between the foundation and a vibrating machine to protect the foundation against large unbalanced forces. The conventional passive isolation suffers from the inherent trade-off between poor high frequency isolation and amplification of vibration at the fundamental mounted resonance frequency [1]. To overcome the trade-off, considerably improved performance can be provided by active, hybrid and semi-active approaches [2, 3]. In the most controlled system, the variable damping technologies are used. The main damping modulation technologies (electrohydraulic [4], magnetorheological [5], electrorheological [6], air-damping [7]) can be produced at low cost and with compact packaging. The on-off damping is the simplest variable damping technology, in which the bi-state damping device is quite cheap and easy to control [8]. Many on-off damping strategies have been developed extensively for the vehicle suspension problem [3] and tuned mass damper system [9–14]. However, most of them involve the feedback control using the displacement, velocity or acceleration measurements. In fact, in some applications such as the machine passing through resonance during start-up and stopping, the excitation frequency can be known. Therefore, we consider in this paper the feed forward adaptive damping law, in which the damping can be varied depending on the excitation frequency. The feed forward control is indeed much simpler and cheaper than the feedback one.

This paper proposes a simple adaptive damping law to improve the passive damping one. The analytical analysis and optimizing are presented. The performance improvement is shown through a so-called performance trade-off curve. The last section introduces a self-made magnetorheological fluid to show the capability to provide the variable damping.

2. PROBLEM STATEMENT

Let us consider the foundation or base of a vibrating machine is protected against large unbalanced forces as shown in Fig. 1.

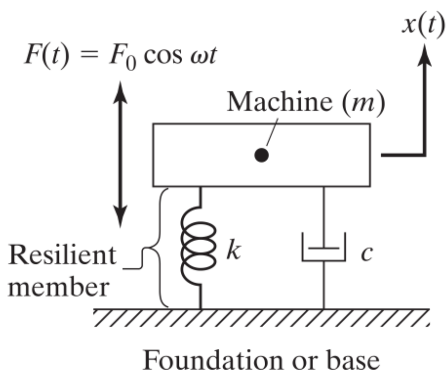

Fig. 1. Machine and resilient member on rigid foundation

The motion equation is given by [1]

$$m\ddot{x} + c\,(\omega)\,\dot{x} + kx = F_0 \cos \omega t, \tag{1}$$

in which m, k and c respectively as the mass, the stiffness and the damping of spring-mass-damper system, F_0 and ω respectively are the amplitude and frequency of the harmonically varying force. The damping $c(\omega)$ is assumed to be adaptive, which depends on the excitation frequency. The damping is passive when its value is a constant. From [1], the vibration amplitude of x in (1) is obtained by

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \omega^2 c^2}}, \tag{2}$$

while magnitude of the total transmitted force is given by

$$F_T = \frac{F_0\sqrt{k^2 + \omega^2 c^2}}{\sqrt{\left(k - m\omega^2\right)^2 + \omega^2 c^2}}. \tag{3}$$

Denote the following non-dimensional terms

$$r = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{k/m}}, \quad \zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}, \tag{4}$$

where r is the frequency ratio and ζ is the damping ratio. The transmissibility or transmission ratio of the isolator (T_f) is defined as the ratio of the magnitude of the force transmitted to that of the exciting force and can be expressed by

$$T_f = \frac{F_T}{F_0} = \sqrt{\frac{k^2 + \omega^2 c^2}{\left(k - m\omega^2\right)^2 + \omega^2 c^2}} = \sqrt{\frac{1 + (2\zeta r)^2}{\left(1 - r^2\right)^2 + (2\zeta r)^2}}. \tag{5}$$

The displacement transmissibility or amplitude ratio indicating the ratio of the amplitude of the mass, X, to the static deflection under the constant force F_0 can be expressed by

$$T_d = \frac{X}{\delta_{st}} = \frac{X}{F_0/k} = \sqrt{\frac{1}{\left(1 - r^2\right)^2 + (2\zeta r)^2}}. \tag{6}$$

Variations of T_f and T_d with r are shown in Fig. 2.

Fig. 2 shows that the damping reduces T_d for all frequencies, but reduces T_f only if $r < \sqrt{2}$. Above that value, the addition of damping increases T_f. To measure the metrics characterizing T_d and T_f, let us introduce following indexes

$$J_d = \int_0^\infty T_d^2 \mathrm{d}r, \tag{7}$$

$$J_f = \int_0^\infty T_f^2 \mathrm{d}r. \tag{8}$$

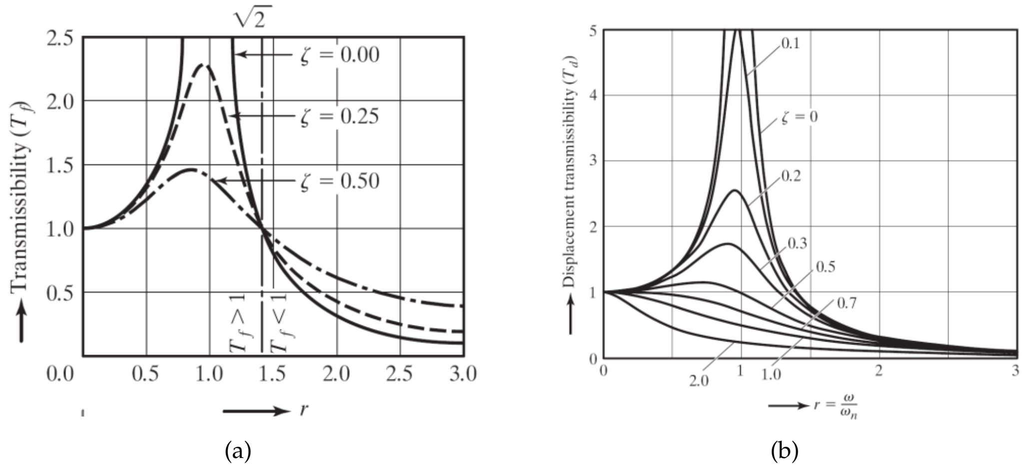

Fig. 2. Variations of (a) transmission ratio and (b) displacement transmissibility with r

By varying a certain parameter and plotting J_f versus J_d, we obtain a so-called performance trade-off curve of the system [3]. In the next section, we will discuss the trade-off curves of passive and adaptive system.

3. TRADE-OFF CURVES

3.1. Passive system

In the passive damping case (ζ is constant), we can calculate explicitly the indexes (7) and (8). Using the Appendix, we have

$$J_d = \int_0^\infty \frac{\mathrm{d}r}{(1-r^2)^2 + 4\zeta^2 r^2} = \frac{\pi}{4\zeta}, \tag{9}$$

$$J_f = \int_0^\infty \frac{(1+4\zeta^2 r^2)\,\mathrm{d}r}{(1-r^2)^2 + 4\zeta^2 r^2} = \pi \frac{1+4\zeta^2}{4\zeta}. \tag{10}$$

The performance trade-off curve of the passive system is shown in Fig. 3.

As seen in Fig. 3, although increasing damping reduces the performance index of displacement transmissibility (J_d), the performance index of force transmissibility (J_f) can only achieve the minimum when $\zeta = 0.5$. In the case $\zeta = 0.5$ we obtain $J_f = \pi \approx 3.14$ and $J_d = \dfrac{\pi}{2} \approx 1.57$. In the passive system, we must compromise in choosing the amount of damping to minimize J_f and to limit J_d.

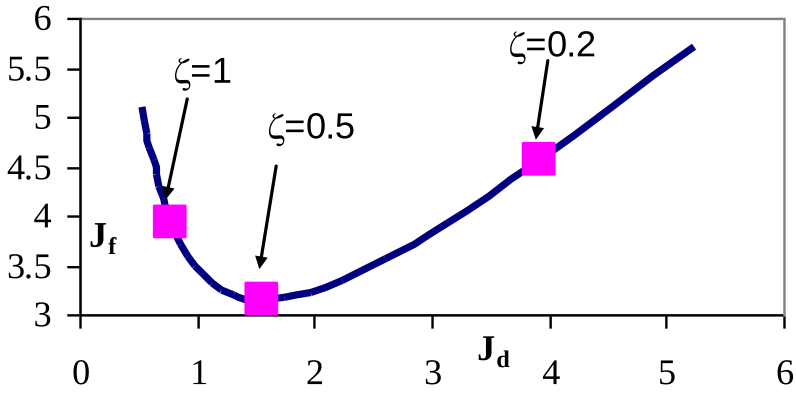

Fig. 3. Trade-off performance curve of passive system

3.2. Adaptive on-off damping system

Let us consider a class of adaptive on-off damping system, in which the damping is high at low frequency and conversely. The adaptive on-off strategy is as follow

$$\zeta = \begin{cases} \zeta_h, & r < r_s \\ \zeta_l, & r \geq r_s \end{cases} \tag{11}$$

In which ζ_h and ζ_l are high and low dampings respectively, r_s is the switching frequency ratio. The performance indexes are calculated as

$$J_d = \int_0^{r_s} \frac{\mathrm{d}r}{(1-r^2)^2 + 4\zeta_h r^2} + \int_{r_s}^{\infty} \frac{\mathrm{d}r}{(1-r^2)^2 + 4\zeta_l r^2}, \tag{12}$$

$$J_f = \int_0^{r_s} \frac{(1+4\zeta_h^2 r^2)\,\mathrm{d}r}{(1-r^2)^2 + 4\zeta_h r^2} + \int_{r_s}^{\infty} \frac{(1+4\zeta_l^2 r^2)\,\mathrm{d}r}{(1-r^2)^2 + 4\zeta_l r^2}. \tag{13}$$

By using the Appendix, we can determined analytically the indexes (12) and (13). Some results for various cases of on-off damping and switching frequency ratio are presented in Figs. 4, 5, and 6. As seen in the figures, the adaptive damping can improve remarkably the performance trade-off curve, by move it lower and further to the left. The performance is better if the off-damping ζ_l is lower (Fig. 4) and the on-damping ζ_h is higher (Fig. 5) and the switching frequency ratio r_s is chosen optimally (Fig. 6). We will show how good the performance in the limit case is, i.e ζ_l tends to zero, ζ_h tends to infinity and r_s is chosen optimally.

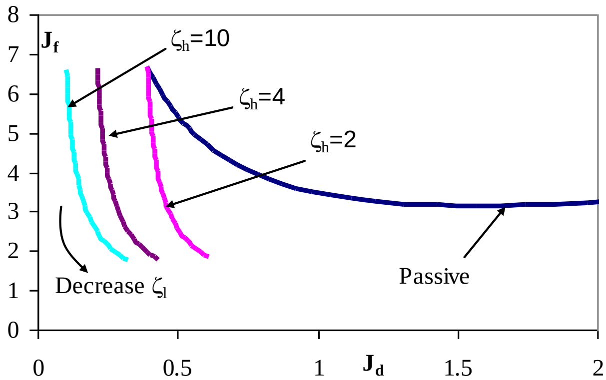

Fig. 4. Comparison between trade-off performance curves of passive and adaptive systems. For the adaptive system: $r_s = 2^{0.5}$, ζ_l varies from 0.45 to 2

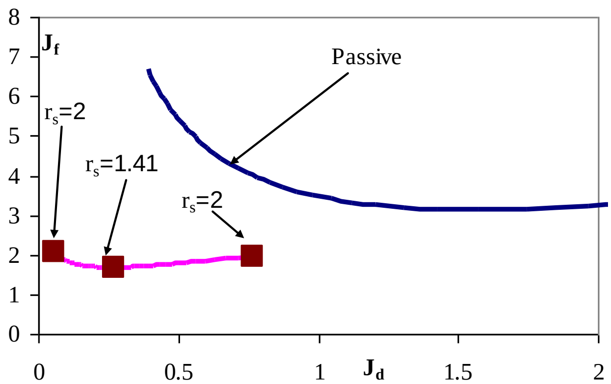

Fig. 5. Comparison between trade-off performance curves of passive and adaptive systems. For the adaptive system: $r_s = 2^{0.5}$, ζ_h varies from 1.5 to 2

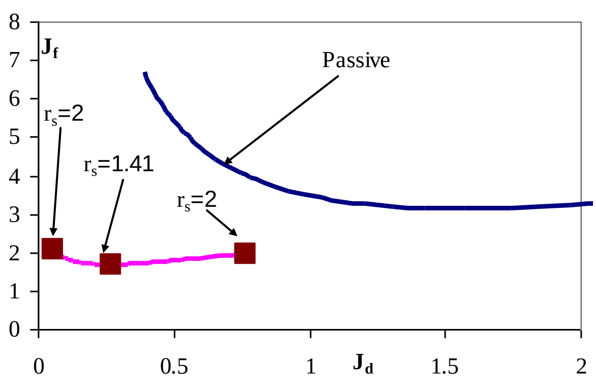

Fig. 6. Comparison between trade-off performance curves of passive and adaptive systems. For the adaptive system: $\zeta_l = 10^{-5}$, $\zeta_h = 10^5$

In this limit, $\zeta_l < 1$ and $\zeta_h > 1$, from the Appendix, we have

$$
\begin{aligned}
J_d = &\left\{ \frac{1}{4\zeta_h\sqrt{\zeta_h^2-1}} \left(\frac{1}{\sqrt{\zeta_h^2-1}-\zeta_h} \arctan \frac{r}{\sqrt{\zeta_h^2-1}-\zeta_h} - \frac{1}{\sqrt{\zeta_h^2-1}+\zeta_h} \arctan \frac{r}{\sqrt{\zeta_h^2-1}+\zeta_h} \right) \right\} \Bigg|_0^{r_s} \\
&+ \left\{ \frac{1}{8\sqrt{1-\zeta_l^2}} \ln \frac{\left(\sqrt{1-\zeta_l^2}+r\right)^2+\zeta_l^2}{\left(\sqrt{1-\zeta_l^2}-r\right)^2+\zeta_l^2} + \frac{1}{4\zeta_l} \left(\arctan \frac{r-\sqrt{1-\zeta_l^2}}{\zeta_l} + \arctan \frac{r+\sqrt{1-\zeta_l^2}}{\zeta_l} \right) \right\} \Bigg|_{r_s}^{\infty} \\
= &\frac{1}{4\zeta_h\sqrt{\zeta_h^2-1}} \left(\frac{1}{\sqrt{\zeta_h^2-1}-\zeta_h} \arctan \frac{r_s}{\sqrt{\zeta_h^2-1}-\zeta_h} - \frac{1}{\sqrt{\zeta_h^2-1}+\zeta_h} \arctan \frac{r_s}{\sqrt{\zeta_h^2-1}+\zeta_h} \right) \\
&- \frac{1}{8\sqrt{1-\zeta_l^2}} \ln \frac{\left(\sqrt{1-\zeta_l^2}+r_s\right)^2+\zeta_l^2}{\left(\sqrt{1-\zeta_l^2}-r_s\right)^2+\zeta_l^2} + \frac{1}{4\zeta_l} \left(\pi - \arctan \frac{r_s-\sqrt{1-\zeta_l^2}}{\zeta_l} - \arctan \frac{r_s+\sqrt{1-\zeta_l^2}}{\zeta_l} \right).
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
J_f = &\left\{ \begin{aligned} &\left(2\zeta_h^2 + \frac{1-4\zeta_h^2\left(2\zeta_h^2-1\right)}{4\zeta_h\sqrt{\zeta_h^2-1}}\right) \frac{1}{\sqrt{\zeta_h^2-1}-\zeta_h} \arctan \frac{r}{\sqrt{\zeta_h^2-1}-\zeta_h} \\ &+ \left(2\zeta_h^2 - \frac{1-4\zeta_h^2\left(2\zeta_h^2-1\right)}{4\zeta_h\sqrt{\zeta_h^2-1}}\right) \frac{1}{\sqrt{\zeta_h^2-1}+\zeta_h} \arctan \frac{r}{\sqrt{\zeta_h^2-1}+\zeta_h} \end{aligned} \right\} \Bigg|_0^{r_s} \\
&+ \left\{ \frac{1-4\zeta_l^2}{8\sqrt{1-\zeta_l^2}} \ln \frac{\left(\sqrt{1-\zeta_l^2}+r\right)^2+\zeta_l^2}{\left(\sqrt{1-\zeta_l^2}-r\right)^2+\zeta_l^2} + \left(\zeta_l + \frac{1}{4\zeta_l}\right) \left(\arctan \frac{r-\sqrt{1-\zeta_l^2}}{\zeta_l} + \arctan \frac{r+\sqrt{1-\zeta_l^2}}{\zeta_l} \right) \right\} \Bigg|_{r_s}^{\infty} \\
= &\left(2\zeta_h^2 + \frac{1-4\zeta_h^2\left(2\zeta_h^2-1\right)}{4\zeta_h\sqrt{\zeta_h^2-1}}\right) \frac{1}{\sqrt{\zeta_h^2-1}-\zeta_h} \arctan \frac{r_s}{\sqrt{\zeta_h^2-1}-\zeta_h} \\
&+ \left(2\zeta_h^2 - \frac{1-4\zeta_h^2\left(2\zeta_h^2-1\right)}{4\zeta_h\sqrt{\zeta_h^2-1}}\right) \frac{1}{\sqrt{\zeta_h^2-1}+\zeta_h} \arctan \frac{r_s}{\sqrt{\zeta_h^2-1}+\zeta_h} \\
&- \frac{1-4\zeta_l^2}{8\sqrt{1-\zeta_l^2}} \ln \frac{\left(\sqrt{1-\zeta_l^2}+r_s\right)^2+\zeta_l^2}{\left(\sqrt{1-\zeta_l^2}-r_s\right)^2+\zeta_l^2} + \left(\zeta_l + \frac{1}{4\zeta_l}\right) \left(\pi - \arctan \frac{r_s-\sqrt{1-\zeta_l^2}}{\zeta_l} - \arctan \frac{r_s+\sqrt{1-\zeta_l^2}}{\zeta_l} \right).
\end{aligned}
\tag{15}
$$

Take the limits of (14) and (15) we have

$$
\lim_{\substack{\zeta_l\to 0\\ \zeta_h\to\infty}} J_d = -\frac{1}{8} \ln \frac{(1+r_s)^2}{(1-r_s)^2} + \frac{1}{4}\left(\frac{1}{r_s-1} + \frac{1}{r_s+1}\right) + \lim_{\zeta_l\to 0} \frac{\pi}{8\zeta_l}\left(1-\operatorname{sgn}\left(r_s-1\right)\right), \tag{16}
$$

$$\lim_{\substack{\zeta_l \to 0 \\ \zeta_h \to \infty}} J_f = r_s - \frac{1}{8} \ln \frac{(1+r_s)^2}{(1-r_s)^2} + \frac{1}{4} \left(\frac{1}{r_s - 1} + \frac{1}{r_s + 1} \right) + \lim_{\zeta_l \to 0} \frac{\pi}{8\zeta_l} \left(1 - \operatorname{sgn}(r_s - 1)\right). \quad (17)$$

To limit the values of the performance indexes, the frequency ratio r_s must be larger than unity. Then we have

$$\frac{\partial \left(\lim\limits_{\substack{\zeta_l \to 0 \\ \zeta_h \to \infty}} J_d \right)}{\partial r_s} = \frac{-1}{(r_s^2 - 1)^2} < 0, \quad (18)$$

$$\frac{\partial \left(\lim\limits_{\substack{\zeta_l \to 0 \\ \zeta_h \to \infty}} J_f \right)}{\partial r_s} = \frac{r_s^2 \left(r_s^2 - 2\right)}{\left(r_s^2 - 1\right)^2}. \quad (19)$$

The formula (18) means J_d decreases as r_s increases. This is clear because increasing r_s in (11) means the on-damping is used more, which reduces the amplitude as shown in Fig. 2(b). The force transmissibility however is minimized when $r_s = \sqrt{2}$ as the result from (19). At this point we have

$$J_f\big|_{\zeta_l=0,\zeta_h=\infty,r_s=\sqrt{2}} = \sqrt{2} - \frac{1}{8} \ln \frac{\left(1+\sqrt{2}\right)^2}{\left(1-\sqrt{2}\right)^2} + \frac{1}{4} \left(\frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{2}+1} \right) \approx 1.68, \quad (20)$$

$$J_d\big|_{\zeta_l=0,\zeta_h=\infty,r_s=\sqrt{2}} = -\frac{1}{8} \ln \frac{\left(1+\sqrt{2}\right)^2}{\left(1-\sqrt{2}\right)^2} + \frac{1}{4} \left(\frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{2}+1} \right) \approx 0.27. \quad (21)$$

Compare with the passive case, it is seen that the indexes J_f and J_d can reduce 47% (from 3.14 to 1.68) and 83% (from 1.57 to 0.27), respectively.

4. ON THE SELF-MADE MAGNETORHEOLOGICAL FLUID

In the above sections, we have shown the remarkable benefit of using adaptive damping in the a vibration isolation. In this section, we present some preliminary results on the self-made magnetorheological fluid, which can be used in the future self-made variable fluid damper. Magnetorheological (MR) fluids are known for their ability to change their rheological behavior within milliseconds under the influence of magnetic field and therefore have been regarded as controllable fluids for fluid damper. Typically these fluids are non colloidal suspensions of micron sized magnetic particles (generally iron particles) in nonmagnetic carrier medium (mineral oil, synthetic oil or water). Under the influence of magnetic field, these particles polarize and forms columnar structure parallel to the applied field. Therefore MR fluid can be converted from a free flowing liquid to a plastic like solid by applying magnetic field and vice versa.

The magnetorheological fluids available in the market currently are expensive [15]. Therefore, for the future use, we developed a self-made margnetorheological fluid, which can be converted from the liquid state to the semi solid state under magnetic field and vice versa. The developed MR fluid consists of three parts: a continuous phase, magnetizable particles, and stabilizer additives. The iron powder reduced (Fig. 7(a)) was used as the dispersed phase. These magnetizable particles were dispersed in the silicone oil (Fig. 7(b)), which has often been used as lubricant in photocopying machines. To improve the fluid stability, the additive employed was stearic acid (Fig. 7(c)). The sample weight contains 62% iron particles, 33% silicone oil and 5% stearic acid. The mixture was stirred with the aid of an self-made overhead stirrer for half an hour until a homogeneous solution was obtained.

(a) Iron powder (b) Silicone oil (c) Stearic acid

Fig. 7. Three parts of MR fluid

The obtained MR fluid then is checked to see if it can be changed its state from liquid to semi-solid and vice versa. In Fig. 8(a), a magnet attached to the bottom of the cup can change the fluid to the semi-solid state, which can resist the gravity. Without the magnet, in Fig. 8(b), the fluid is changed to the liquid state and the free flow appears due to the gravity.

(a)

(b)

Fig. 8. (a) Semi solid state and (b) Liquid state of MR fluid in a cup

The same illustration of the MR fluid in the market can be seen in Fig. 9.

(a)

(b)

Fig. 9. (a) Semi solid state and (b) Liquid state of a commercial MR fluid (cut from a clip, source: Internet)

In brief, we have initially synthesized a low-cost MR fluid using some popular chemical compounds. Its change between the semi-solid and liquid states under magnetic field can be used in the future damper to make the variable damping devices.

5. CONCLUSIONS

This paper considers the problem of an on-off damping isolation regulated by a simple adaptive controller. The proposed controller changes the damping based on a switching frequency. The proposed controller has simple feed forward nature and can be analytical optimized. Comparing with the passive isolation, the reductions of the force and displacement transmissibility of the best adaptive isolation can achieve 47% and 83%, respectively. A self-made magnetorheological fluid is synthesized to show the capability of change between the semi-solid and liquid states under magnetic field. This state change can be used in the future damper to make the variable damping devices.

ACKNOWLEDGMENT

This paper is funded by Vietnam Academy of Science and Technology under grant number VAST01.04/21-22.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

REFERENCES

[1] S. S. Rao. *Mechanical vibrations*. Pearson Education, Harlow, UK, (2018).

[2] A. Preumont. *Vibration control of active structures*. Springer Netherlands, (2011). https://doi.org/10.1007/978-94-007-2033-6.

[3] S. M. Savaresi, C. Poussot-Vassal, C. Spelta, O. Sename, and L. Dugard. Semi-active suspension technologies and models. In *Semi-Active Suspension Control Design for Vehicles*, pp. 15–39. Elsevier, (2010). https://doi.org/10.1016/b978-0-08-096678-6.00002-x.

[4] S. K. Yalla and A. Kareem. Semiactive tuned liquid column dampers: experimental study. *Journal of Structural Engineering*, **129**, (2003), pp. 960–971. https://doi.org/10.1061/(asce)0733-9445(2003)129:7(960).

[5] Y. Liu, H. Matsuhisa, and H. Utsuno. Semi-active vibration isolation system with variable stiffness and damping control. *Journal of Sound and Vibration*, **313**, (2008), pp. 16–28. https://doi.org/10.1016/j.jsv.2007.11.045.

[6] J. Pons. *Emerging actuator technologies*. Wiley, (2005). https://doi.org/10.1002/0470091991.

[7] X. Sun and X. Jing. A nonlinear vibration isolator achieving high-static-low-dynamic stiffness and tunable anti-resonance frequency band. *Mechanical Systems and Signal Processing*, **80**, (2016), pp. 166–188. https://doi.org/10.1016/j.ymssp.2016.04.011.

[8] S. K. Yalla, A. Kareem, and J. C. Kantor. Semi-active tuned liquid column dampers for vibration control of structures. *Engineering Structures*, **23**, (2001), pp. 1469–1479. https://doi.org/10.1016/s0141-0296(01)00047-5.
[9] J.-H. Koo, M. Ahmadian, M. Setareh, and T. Murray. In search of suitable control methods for semi-active tuned vibration absorbers. *Journal of Vibration and Control*, **10**, (2004), pp. 163–174. https://doi.org/10.1177/1077546304032020.
[10] V. D. La. Semi-active on–off damping control of a dynamic vibration absorber using Coriolis force. *Journal of Sound and Vibration*, **331**, (2012), pp. 3429–3436. https://doi.org/10.1016/j.jsv.2012.03.026.
[11] L. D. Viet, N. B. Nghi, N. N. Hieu, D. T. Hung, N. N. Linh, and L. X. Hung. On a combination of ground-hook controllers for semi-active tuned mass dampers. *Journal of Mechanical Science and Technology*, **28**, (2014), pp. 2059–2064. https://doi.org/10.1007/s12206-014-0109-3.
[12] J. N. Potter, S. A. Neild, and D. J. Wagg. Generalisation and optimisation of semi-active, on–off switching controllers for single degree-of-freedom systems. *Journal of Sound and Vibration*, **329**, (2010), pp. 2450–2462. https://doi.org/10.1016/j.jsv.2009.12.011.
[13] Y.-J. Shen, L. Wang, S.-P. Yang, and G.-S. Gao. Nonlinear dynamical analysis and parameters optimization of four semi-active on-off dynamic vibration absorbers. *Journal of Vibration and Control*, **19**, (2012), pp. 143–160. https://doi.org/10.1177/1077546311433610.
[14] V. D. La and C. Adam. General on-off damping controller for semi-active tuned liquid column damper. *Journal of Vibration and Control*, **24**, (2016), pp. 5487–5501. https://doi.org/10.1177/1077546316648080.
[15] *https://www.shoplordmr.com/*.

APPENDIX

In this appendix, we calculate explicitly the following two integrals

$$I_1 = \int \frac{\mathrm{d}r}{\left(1-r^2\right)^2 + 4\zeta^2 r^2}, \tag{A.1}$$

$$I_2 = \int \frac{\left(1+4\zeta^2 r^2\right)\mathrm{d}r}{\left(1-r^2\right)^2 + 4\zeta^2 r^2}. \tag{A.2}$$

The integrals have different forms for three cases of ζ: the underdamped ($\zeta < 1$), the critically damped ($\zeta = 1$) and the overdamped ($\zeta > 1$).

If $\zeta < 1$, we can write (A.1) and (A.2) as

$$\begin{aligned} I_1 = &\int \frac{\sqrt{1-\zeta^2} - r}{4\sqrt{1-\zeta^2}\left(\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2\right)}\mathrm{d}r + \int \frac{\sqrt{1-\zeta^2} + r}{4\sqrt{1-\zeta^2}\left(\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2\right)}\mathrm{d}r \\ &+ \int \frac{\mathrm{d}r}{4\left(\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2\right)} + \int \frac{\mathrm{d}r}{4\left(\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2\right)}, \end{aligned} \tag{A.3}$$

$$I_2 = \int \frac{(1-4\zeta^2)\left(\sqrt{1-\zeta^2}-r\right)}{4\sqrt{1-\zeta^2}\left(\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2\right)}\mathrm{d}r + \int \frac{(1-4\zeta^2)\left(\sqrt{1-\zeta^2}+r\right)}{4\sqrt{1-\zeta^2}\left(\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2\right)}\mathrm{d}r + \int \frac{(\zeta^2+1/4)\,\mathrm{d}r}{\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2} + \int \frac{(\zeta^2+1/4)\,\mathrm{d}r}{\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2}. \tag{A.4}$$

The integrals (A.3) and (A.4) are determined explicitly by

$$I_1 = \frac{1}{8\sqrt{1-\zeta^2}} \ln \frac{\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2}{\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2} + \frac{1}{4\zeta}\left(\arctan\frac{r-\sqrt{1-\zeta^2}}{\zeta} + \arctan\frac{r+\sqrt{1-\zeta^2}}{\zeta}\right), \tag{A.5}$$

$$I_2 = \frac{1-4\zeta^2}{8\sqrt{1-\zeta^2}} \ln \frac{\left(\sqrt{1-\zeta^2}+r\right)^2+\zeta^2}{\left(\sqrt{1-\zeta^2}-r\right)^2+\zeta^2} + \left(\zeta+\frac{1}{4\zeta}\right)\left(\arctan\frac{r-\sqrt{1-\zeta^2}}{\zeta} + \arctan\frac{r+\sqrt{1-\zeta^2}}{\zeta}\right). \tag{A.6}$$

If $\zeta = 1$, the integrals (A.1) and (A.2) are calculated easily

$$I_1 = \int \frac{\mathrm{d}r}{(1-r^2)^2+4r^2} = \frac{r}{2\left(r^2+1\right)} + \frac{\arctan r}{2}, \tag{A.7}$$

$$I_2 = \int \frac{\left(1+4r^2\right)\mathrm{d}r}{(1-r^2)^2+4r^2} = \frac{-3r}{2\left(r^2+1\right)} + \frac{5\arctan r}{2}. \tag{A.8}$$

If $\zeta > 1$, we can write (A.1) and (A.2) as

$$I_1 = \frac{1}{4\zeta\sqrt{\zeta^2-1}} \int \frac{\mathrm{d}r}{r^2+\left(\sqrt{\zeta^2-1}-\zeta\right)^2} - \frac{1}{4\zeta\sqrt{\zeta^2-1}} \int \frac{\mathrm{d}r}{r^2+\left(\sqrt{\zeta^2-1}+\zeta\right)^2}, \tag{A.9}$$

$$I_2 = \left(2\zeta^2 + \frac{1-4\zeta^2\left(2\zeta^2-1\right)}{4\zeta\sqrt{\zeta^2-1}}\right) \int \frac{\mathrm{d}r}{r^2+\left(\sqrt{\zeta^2-1}-\zeta\right)^2} + \left(2\zeta^2 - \frac{1-4\zeta^2\left(2\zeta^2-1\right)}{4\zeta\sqrt{\zeta^2-1}}\right) \int \frac{\mathrm{d}r}{r^2+\left(\sqrt{\zeta^2-1}+\zeta\right)^2}. \tag{A.10}$$

The integrals (A.9) and (A.10) are determined explicitly by

$$I_1 = \frac{1}{4\zeta\sqrt{\zeta^2-1}}\left(\frac{1}{\sqrt{\zeta^2-1}-\zeta}\arctan\frac{r}{\sqrt{\zeta^2-1}-\zeta} - \frac{1}{\sqrt{\zeta^2-1}+\zeta}\arctan\frac{r}{\sqrt{\zeta^2-1}+\zeta}\right), \tag{A.11}$$

$$I_2 = \left(2\zeta^2 + \frac{1-4\zeta^2\left(2\zeta^2-1\right)}{4\zeta\sqrt{\zeta^2-1}}\right)\frac{1}{\sqrt{\zeta^2-1}-\zeta}\arctan\frac{r}{\sqrt{\zeta^2-1}-\zeta}$$
$$+\left(2\zeta^2 - \frac{1-4\zeta^2\left(2\zeta^2-1\right)}{4\zeta\sqrt{\zeta^2-1}}\right)\frac{1}{\sqrt{\zeta^2-1}+\zeta}\arctan\frac{r}{\sqrt{\zeta^2-1}+\zeta}. \tag{A.12}$$

Although the forms of the integrals are different depending on the comparison between ζ and unity, their values for $r = 0$ and $r = \infty$ do not depend on this classification. Indeed, for any amount of damping ζ, we have

$$I_1|_{r=0} = I_2|_{r=0} = 0, \quad I_1|_{r=\infty} = \frac{\pi}{4\zeta}, \quad I_2|_{r=\infty} = \pi\frac{1+4\zeta^2}{4\zeta}. \tag{A.13}$$