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FEM SHAKEDOWN ANALYSIS OF STRUCTURES UNDER RANDOM STRENGTH WITH CHANCE CONSTRAINED PROGRAMMING

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Abstract. Direct methods, comprising limit and shakedown analysis, are a branch of computational mechanics. They play a significant role in mechanical and civil engineering design. The concept of direct methods aims to determine the ultimate load carrying capacity of structures beyond the elastic range. In practical problems, the direct methods lead to nonlinear convex optimization problems with a large number of variables and constraints. If strength and loading are random quantities, the shakedown analysis can be formulated as stochastic programming problem. In this paper, a method called chance constrained programming is presented, which is an effective method of stochastic programming to solve shakedown analysis problems under random conditions of strength. In this study, the loading is deterministic, and the strength is a normally or lognormally distributed variable.

Keywords: limit analysis, shakedown analysis, chance constrained programming, stochastic programming, reliability of structures.

1. INTRODUCTION

The plastic collapse limit and the shakedown limit, which define the load-carrying capacity of structures, are important in assessing the structural integrity. Due to the high expenses of experimental setups and the time-consuming full elastic-plastic cyclic loading analysis, the determination of these limits by numerically direct plasticity methods is of great interest. Lower bound limit analysis determines directly the largest load, which is safe against plastic collapse as a statically formulated maximum problem. Alternatively, lower bound limit analysis determines the least collapse load as a kinematically formulated minimum problem. Both optimization problems are convex, so that they have the same solution by duality, which is therefore an exact solution of classical plasticity. Shakedown analysis extends the optimization approach to time-variant loading and is used for limit state design to check against failure by alternating plasticity and incremental plastic collapse (ratcheting). A structure is safe against plastic failure if initial plastic deformations cease because the structure "shakes down" to elastic behavior. The theory and several numerical methods can be found in [1–6]. Under uncertainty, it is important that uncertain quantities like Young's modulus and details of the load history do not affect the limit and shakedown load. For a prescribed structural reliability (or prescribed failure probability) the limit and shakedown load can be obtained by stochastic optimization.

2. DETERMINISTIC PROGRAMMING FOR LIMIT AND SHAKEDOWN OF STRUCTURES

2.1. Limit analysis

Consider a structure made of elastic-perfectly plastic or rigid-perfectly plastic material and there is a set of forces **F** acting on it. A common assumption is that all the components of the set of forces change proportionally to a certain load factor α. This case is referred to as proportional loading. In matrix notation we can write

$$\mathbf{F} = \alpha \mathbf{F}_0, \tag{1}$$

where $\mathbf{F}_0$ is some fixed reference load vector. In Fig. 1 it is $\mathbf{F}_0 = (F, 2F)$.

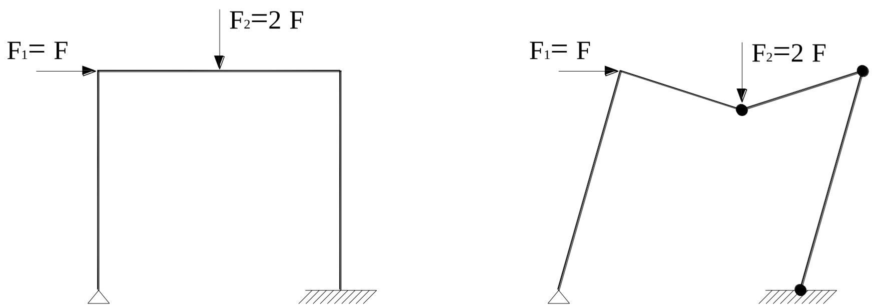

Fig. 1. Portal frame with the "combined" failure mechanism under the load $\mathbf{F}_0 = (F, 2F)$

If the value of α remains sufficiently low, response of the structure is elastic. As α increases and reaches a special value, the first point in the body reaches the plastic state. This state of stress is called elastic limit. Further increase of α will lead to the expansion of plastic region in the structure. In the frame in Fig. 1 localized plastic hinges gradually form and the structure becomes statically determinate before the next occurring plastic hinge transform the structure into a collapse mechanism. At this limit state, the structure is collapsed by applied forces. The value $\alpha = \alpha_{\text{lim}}$ corresponding to the plastic collapse state is called the safety factor of the structure or the limit load factor It is unique and independent of Young's modulus and residual stresses, since the structure was statically determinate before collapse. This property is useful because the influence of some uncertain quantities is eliminated from the limit analysis problem.

The numerical limit analysis can be based on two theorems.

- Lower bound theorem:

If a stress field $\boldsymbol{\sigma}$ can be found, which satisfies the statically admissible state, then the corresponding multiplier α cannot exceed the limit multiplier α_{lim}.

- Upper bound theorem:

Any multiplier α corresponding to a kinematically admissible state is not less than the limit multiplier α_{lim}.

There are two basic approaches to limit analysis corresponding to the two above theorems. *The static approach* is based on the lower bound theorem, which calculates the safety factor for the maximum statically admissible load multiplier α by solving a maximum optimization problem

$$\alpha_{\text{lim}} = \max \alpha \\ \text{s.t.:} \begin{cases} -\nabla \cdot \sigma = \alpha \bar{\mathbf{b}} & \text{in} \quad V \\ \sigma \cdot \mathbf{n} = \alpha \bar{\mathbf{t}} & \text{on} \quad S_t \\ f(\sigma) \leq 0 & \text{in} \quad V \end{cases} \tag{2}$$

The constraints in (2) are the Cauchy equations of equilibrium, static boundary conditions and conditions of plastic admissibility, respectively. With $\mathbf{F}_0 = \left(\bar{\mathbf{b}}, \bar{\mathbf{t}}\right)$ in (1) the body force $\bar{\mathbf{b}}$ and the traction $\bar{\mathbf{t}}$ in normal direction $\mathbf{n}$ to the static boundary S_t increase proportionally with α.

The second *kinematic approach* is based on the upper bound theorem that calculates the safety factor by searching for the minimum kinematically admissible load multiplier by the minimum problem

$$\alpha_{\text{lim}} = \min \alpha \\ \alpha = \int_V D(\dot{\varepsilon}) \mathrm{d}V \\ \text{s.t.:} \begin{cases} \dot{\varepsilon} = (\nabla \dot{\mathbf{u}})_{sym} & \text{in} \quad V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on} \quad S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \mathrm{d}V + \int_{S_u} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \mathrm{d}S - 1 = 0 & \end{cases} \tag{3}$$

where $\int_V D(\dot{\varepsilon}) \mathrm{d}V$ is the *plastic dissipation power.* The constraints in (3) are the strain-displacement relations, the kinematic boundary conditions on the boundary S_u, and the normalized positive external power.

2.2. Shakedown analysis of structures

For limit analysis, all the components of the set of forces acting upon the structures change monotonically. In practice, however, certain types of loads on structures are far from monotonic. Moreover, the load acting on the structures may be repeated (cyclic) many times or varying arbitrarily in a certain convex load domain $\mathcal{L}$.

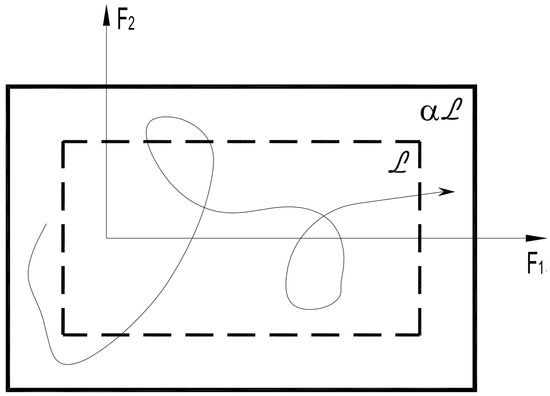

Fig. 2. Convex load domain $\mathcal{L}$ for two forces acting on structure

As the load intensities become higher than the elastic limit, plastic deformation occurs. It may happen that, after some plastic deformation in the initial load cycles, the structural behaviour becomes eventually elastic. Such stabilization of plastic deformations is called (*elastic*) *shakedown* or *adaptation*. If we consider the structure at elastic shakedown regime, we can extend the load capacity by increasing the load domain by a shakedown load factor α to $\alpha\mathcal{L}$ as shown in Fig. 2. Low cycle fatigue, ratchetting and collapse are failure modes which are not allowed to happen in the safe structures. There are two fundamental theorems for shakedown analysis. Melan's static theorem provides a lower bound to the shakedown load factor so that all load histories in $\alpha\mathcal{L}$ shakedown by convexity of $\mathcal{L}$ [7]. From Melan's shakedown theorem, smallest load factor α^- can be found by solving the mathematical programming

$$\alpha^- = \max \alpha$$
$$\text{s.t.} \left\{ \begin{array}{llll} \nabla \cdot \boldsymbol{\sigma}^E = -\mathbf{f}_v\,, & \nabla \cdot \bar{\boldsymbol{\rho}} = 0 & \text{in} & V \\ \nabla \cdot \boldsymbol{\sigma}^E = -\mathbf{f}_t\,, & n^T \bar{\boldsymbol{\rho}} = 0 & \text{on} & \partial V \\ f\left[\alpha \boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\rho}}\right] - \sigma_y \leq 0 & & \text{in} & V, \quad \forall t \in [0, \infty) \end{array} \right. \tag{4}$$

Considering that the inequality condition needs to be checked only in the m load vertices of a convex load domain, problem (4) can be written in finite element formulation

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \left\{ \begin{array}{l} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{B}^T \bar{\boldsymbol{\rho}} = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i\right] \leq r_i, \quad \forall k = \overline{1, m}, \ \ \forall i = \overline{1, NG} \end{array} \right. \tag{5}$$

where:

+ $\mathbf{B} = [w_1 \mathbf{B}_1, w_2 \mathbf{B}_2, \ldots, w_i \mathbf{B}_i, \ldots, w_{NG} \mathbf{B}_{NG}]$;

+ $\bar{\boldsymbol{\rho}}^T = \left[\bar{\boldsymbol{\rho}}_1^T, \bar{\boldsymbol{\rho}}_2^T, \ldots, \bar{\boldsymbol{\rho}}_i^T, \ldots, \bar{\boldsymbol{\rho}}_{NG}^T\right]$;

+ $\bar{\boldsymbol{\rho}}$ is the discretized residual stress field with its components computed at Gauss points;
+ $\boldsymbol{\sigma}_{ik}^{E}$ is the fictitious elastic stress vector at Gauss point i corresponding to the vertex $\hat{P}_k$ of the load domain $\mathcal{L}$;
+ $\mathbf{B}_i$ is the deformation matrix $\mathbf{B}(\mathbf{x})$ at Gauss point i;
+ w_i is the weight factor of the Gauss point i;
+ NG denotes the total number of Gauss points of the discretized structure;
+ r_i is the yield stress of the material at Gauss point i.

The second approach is based on Koiter's kinematic theorem. This is an upper bound nonlinear programming problem. If the von Mises yield condition is used, then the discretized formulation of the upper bound problem is [8]

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^{T} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, NG} \\ \mathbf{D}_v \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{0} & \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\varepsilon}}_{ik}^{T} \boldsymbol{\sigma}_{ik}^{e} - 1 = 0 \end{cases} \tag{6}$$

where:

+ $\dot{\boldsymbol{\varepsilon}}_{ik}$ is strain rate vector at Gauss point i, corresponding to load vertex $\hat{P}_k$

$$\dot{\boldsymbol{\varepsilon}}_{ik} = \left[\dot{\varepsilon}_{11}^{i}, \dot{\varepsilon}_{22}^{i}, \dot{\varepsilon}_{33}^{i}, \gamma_{12}^{i}, \gamma_{23}^{i}, \dot{\varepsilon}_{31}^{i} \right]_k^T ;$$

+ $\boldsymbol{\sigma}_{ik}^{e}$ is the fictitious elastic stress vector at Gauss point i corresponding to the vertex $\hat{F}_k$;
+ $\dot{\mathbf{u}}$ is the displacement rate vector;
+ ε_0^2 is the small positive number to avoid the singularity of the dissipation function;
+ $\mathbf{D}$, $\mathbf{D}_v$ are square matrices, in a 3D model they have the form

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \mathbf{D}_v = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. STOCHASTIC PROGRAMMING FOR LIMIT AND SHAKEDOWN ANALYSIS UNDER RANDOM STRENGTH

3.1. Lower bound approach

For a real structure the strength r_i of the material is uncertain it can be modelled through random variables $r_i = r_i(\omega)$ on a certain probability space. Under uncertainty, the inequalities of (5) are not always satisfied so that the structure may fail with the probability P_f because the stress is not plastically admissible. The structure must be safe with a required reliability $\psi = 1 - P_f$ that is the probability of the i^{th} yield condition being satisfied is greater than some reliability level ψ. Problem (5) becomes a chance constrained stochastic program

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{0} \\ \text{Prob}\left[f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i \right) - r_i(\omega) \le 0 \right] \ge \psi, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,NG} \end{cases} \tag{7}$$

If the strength $r_i(\omega)$ of the material follows a Gaussian distribution $r_i \sim N(\mu_i, \sigma_i)$ with mean value μ_i and standard deviation σ_i. Let Φ denote the cumulative distribution function (CDF) of the standard normal distribution. Introducing the new variable $\kappa = \Phi^{-1}(\psi)$ so that $\psi = \Phi(\kappa)$, the chance constrained program (7) can be convert into equivalent deterministic programming after some transformations

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i \right] \le \mu_i - \kappa \sigma_i, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,NG} \end{cases} \tag{8}$$

If the strength of the material $r_i(\omega)$ is distributed lognormally $\ln r_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ with parameters μ_i and σ_i, we get the equivalent deterministic formulation of the problem

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i \right] \le e^{\mu_i - \kappa \sigma_i}, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,NG} \end{cases} \tag{9}$$

3.2. Upper bound approach

If the strength r_i is an uncertain quantity, the objective function of the kinematic problem is a stochastic variable and the upper bound problem (6) also becomes a stochastic programming problem. We can state the problem in such a way that one looks for a minimum lower bound η of the objective function under the constraint that the probability ψ of violation of that bound is prescribed [9,10]

$$\alpha^{+} = \min \eta$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^T \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2} \geq \psi \right) = \eta \\ \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} = \mathbf{0} \quad \forall i = \overline{1, NG} \\ \mathbf{D}_v \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{0} \quad \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\varepsilon}}_{ik}^T \boldsymbol{\sigma}_{ik}^e - 1 = 0 \end{cases} \tag{10}$$

If yield stress $r_i(\omega)$ is distributed normally, program (10) can be converted into an equivalent deterministic program [11]

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} \sqrt{\frac{2}{3}} (\mu_i - \kappa \sigma_i) \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^T \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} = \mathbf{0} \quad \forall i = \overline{1, NG} \\ \mathbf{D}_v \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{0} \quad \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\varepsilon}}_{ik}^T \boldsymbol{\sigma}_{ik}^e - 1 = 0 \end{cases} \tag{11}$$

In the case of a lognormal distribution of strength, problem (10) can be convert into the equivalent deterministic program by duality [11]

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} e^{(\mu_i - \kappa \sigma_i)} \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^T \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} = \mathbf{0} \quad \forall i = \overline{1, NG} \\ \mathbf{D}_v \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{0} \quad \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\varepsilon}}_{ik}^T \boldsymbol{\sigma}_{ik}^e - 1 = 0 \end{cases} \tag{12}$$

3.3. Duality between lower bound and upper bound

One can show that problem (11) is dual to problem (8) for normal distributions, and problem (12) is dual to problem (9) for lognormal distributions. The solution of the lower bound and upper bound converge to the same load factor $\alpha = \alpha^{+} = \alpha^{-}$, which therefore is the exact solution of the FEM discretization. This fact is used to create a dual chance constrained programming algorithm, which calculates upper bound and lower bound shakedown load factors at the same time. More detail is given in [7,8].

4. NUMERICAL EXAMPLE

4.1. Limit analysis of a two span beam

In the first example, we consider a two span continuous beam with rectangular cross-section on a pin joint and two roller joints, (Sikorski and Borkowski 1990 [12]).

The beam is subjected to two concentrated forces shown in Fig. 3. Each span has random yield moments characterized by the mean values $\bar{M}_{0,1} = 2.0\,\text{kNm}$, $\bar{M}_{0,2} = 3.0\,\text{kNm}$ and the standard deviations $\sigma_i = 0.1\bar{M}_{0,i}$ $(i = 1, 2)$. For the numerical analysis the corresponding yield stress is obtained as $\sigma_y = 2000\,\text{kN/m}^2$ from the plastic section modulus $Z_p = bh^2/4$ with height $h = 0.244\,\text{m}$ and width $b = 0.1\,\text{m}$ of the section. Let us determine the limit load factor for the failure probability $P_f = 1.0 \cdot 10^{-4}$ or the reliability level $\psi = 1 - P_f = 0.9999$ so that $\kappa = \Phi^{-1}(\psi) = \Phi^{-1}(0.9999) = 3.719$.

The numerical solution converges as shown in Fig. 3 to the deterministic limit load factor $\alpha = 2.19$ and to the probabilistic limit load factors $\alpha = 1.38$ and $\alpha = 1.51$ for normally and lognormally distributed strength, respectively. The limit loads in [12] and the analytical limit loads are based on beam theory and are therefore different from the numerical limit loads. All limit loads are summarized in Table 1. Fig. 4 and Fig. 5 show the dependency of the stochastic limit load on the coefficient of variation and the failure probability, respectively.

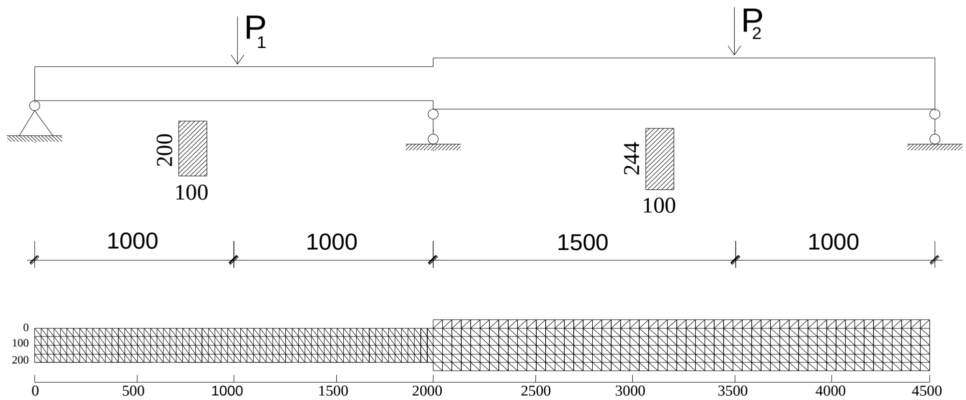

Fig. 3. Two span beam and the mesh using three-node triangular elements; all dimensions in mm

Table 1. Limit load factor of the two span beam

Lower bound	Upper bound		Method, reference
2.19	2.19	deterministic	numerical, Tran et al. [13]
2	2		analytical, Sikorski, Borkowski [12]
1.15	1.36	normal	Sikorski, Borkowski [12]
1.38	1.38		numerical, Tran et al. [13]
1.256	1.256		analytical, Tran et al. [13]
1.509	1.509	lognormal	numerical
1.373	1.373		analytical

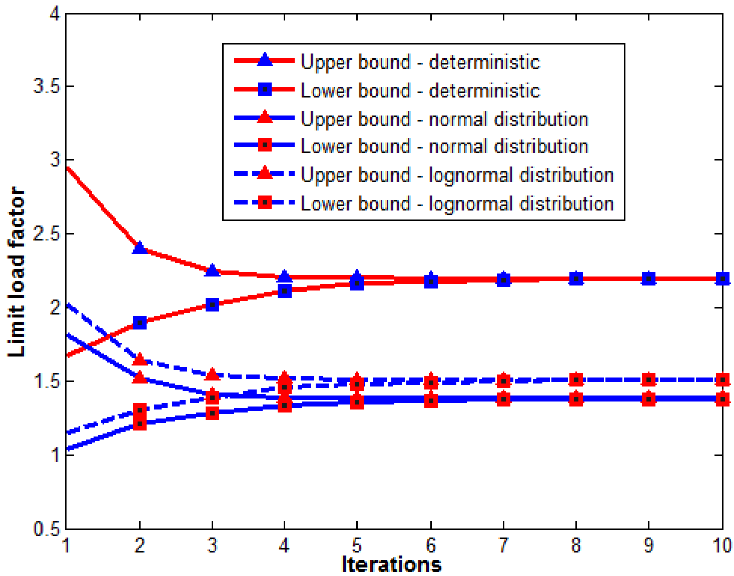

Fig. 4. Convergence of the limit load factor

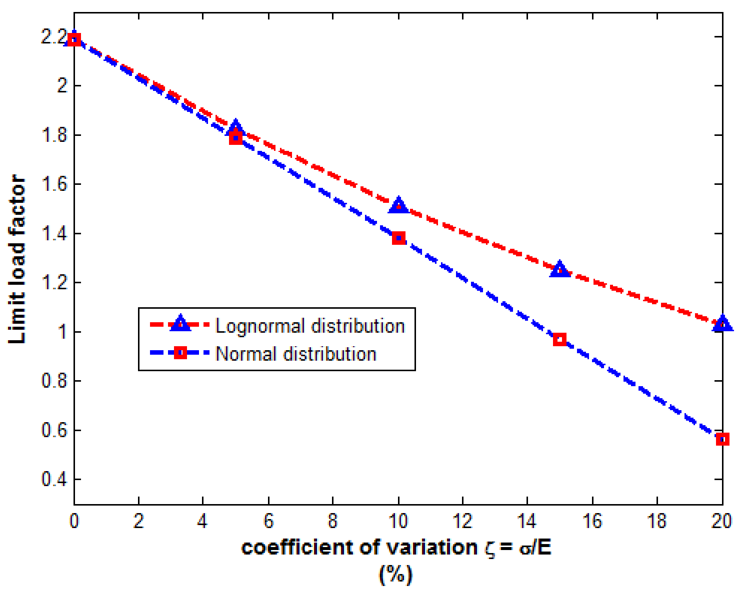

Fig. 5. Dependency of the limit load on the coefficient of variation

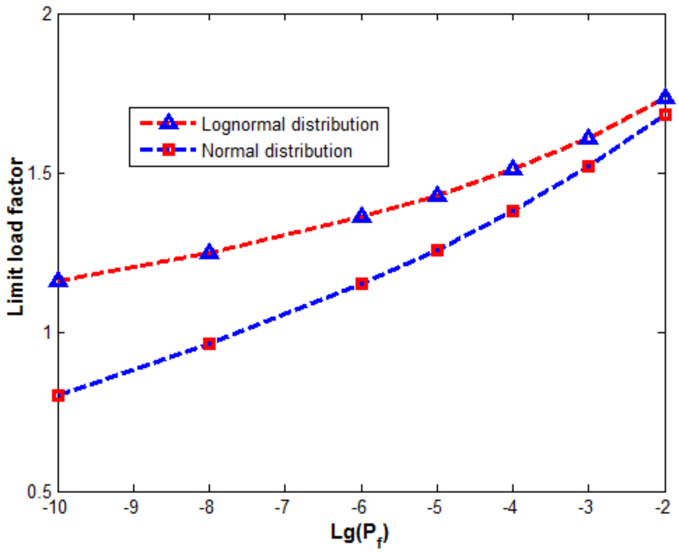

Fig. 6. Dependency of the limit load on the failure probability

4.2. Shakedown analysis of square plate with a hole

In the second example, we consider a plate with hole ($D/L = 0.2$), with yield stress σ_y, and subjected to two forces F_1, F_2 varying independently in a load domain

$$(F_1, F_2) \in \left[0; \sigma_y\right] \times \left[-\sigma_y; \sigma_y\right].$$

Due to symmetry, a quarter of the plate is modelled by 1200 T3 elements as shown in Fig. 7.

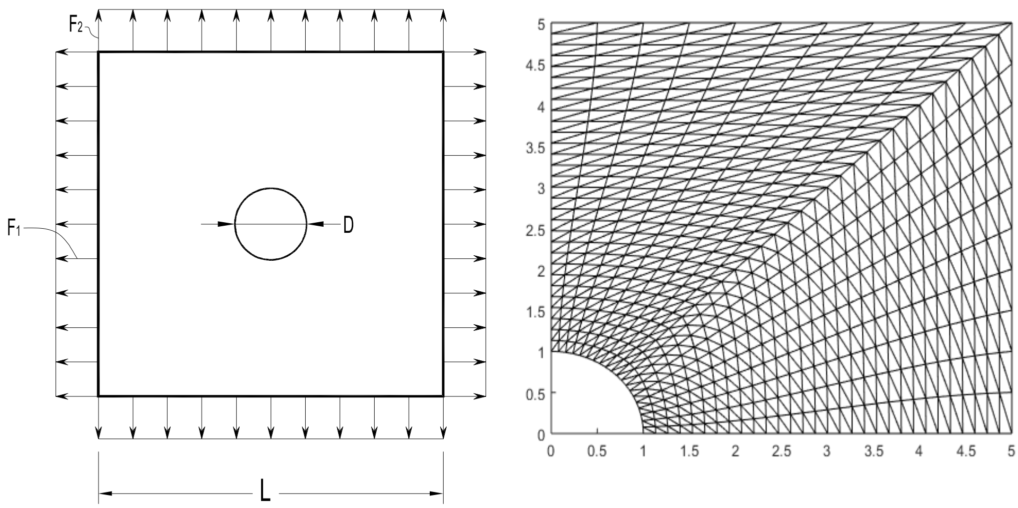

Fig. 7. Plate with hole and FE mesh

Table 2. Limit analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	Load model
numerical			
Belytschko [14]	—	0.780	deterministic
Genna [15]	—	0.793	
Garcea et al. [16]	0.902	0.806	
Tran et al. [17]	0.896	0.797	
Present	0.899	0.807	
	0.565	0.506	normal
	0.618	0.554	lognormal
exact			
Gaydon, McCrum [18]	—	0.800	deterministic
Present	—	0.502	normal
	—	0.549	lognormal

The chosen reliability level in this example is $\psi = 0.9999$ so that $\kappa = 3.719$. If the yield stress has the mean value $E\left[\sigma_y\right] = 1\,\mathrm{kN/m}$ and the standard deviation $\mathrm{Var}\left(\sigma_y\right) = (0.1\mu)^2 = (0.1\,\mathrm{kN/m})^2$, the limit and shakedown load factors for two load cases are compared with literature values in Tables 2-3. The tables also show the load factors for different distributions of random of strength. For normal and lognormal distributions, the exact stochastic limit load factors in Table 2 can be calculated from the exact deterministic limit load factor. Fig. 8 shows the convergence of the shakedown load.

Table 3. Shakedown analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	Load model
Belytschko [14]	0.431	0.571	deterministic
Genna [15]	0.478	0.653	
Garcea et al. [16]	0.438	0.604	
Tran et al. [17]	0.434	0.601	
Present	0.436	0.602	
	0.274	0.378	normal
	0.299	0.414	lognormal

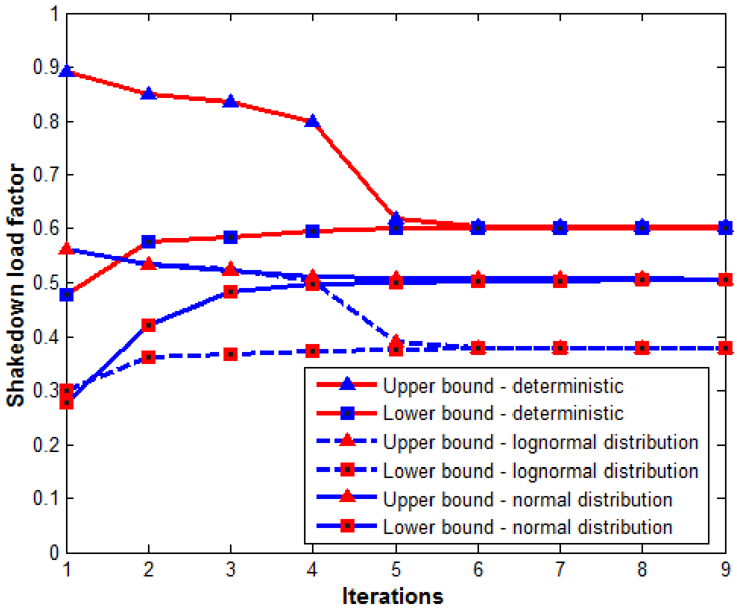

Fig. 8. The convergence of shakedown load for cases of distribution of strength

5. CONCLUSIONS

Modern structural design is done with respect to ultimate limit states of the structure such as plastic collapse, ratcheting and alternating plasticity. Admissible loads are calculated with respect to partial safety factors for actions and material [19]. The safety factors in design codes reflect experts' opinion about uncertainties and failure consequences.

This paper presents a technique of stochastic programming so called chance constrained programming to treat problem of shakedown analysis under random strength of material. Chance constrained programming leads to a reformulation of the deterministic problem so that the effort of the stochastic analysis is the same as a deterministic limit or shakedown analysis. Uncertainties can be quantified and a target failure probability chosen according to the failure consequences. Then a design load can calculated on the basis of the stochastic model and data of all uncertainties. Design code committees can decide on the target failure probabilities.

Probabilistic structural has the acceptance problem that the calculated failure probabilities are very sensitive. A small change of data or a different distribution type can change the failure probability by orders of magnitude. The stochastic programming approach is very robust. The calculated design loads change only in the order of changes in the input data. This behavior and the small numerical efforts make the method attractive for structural engineer.

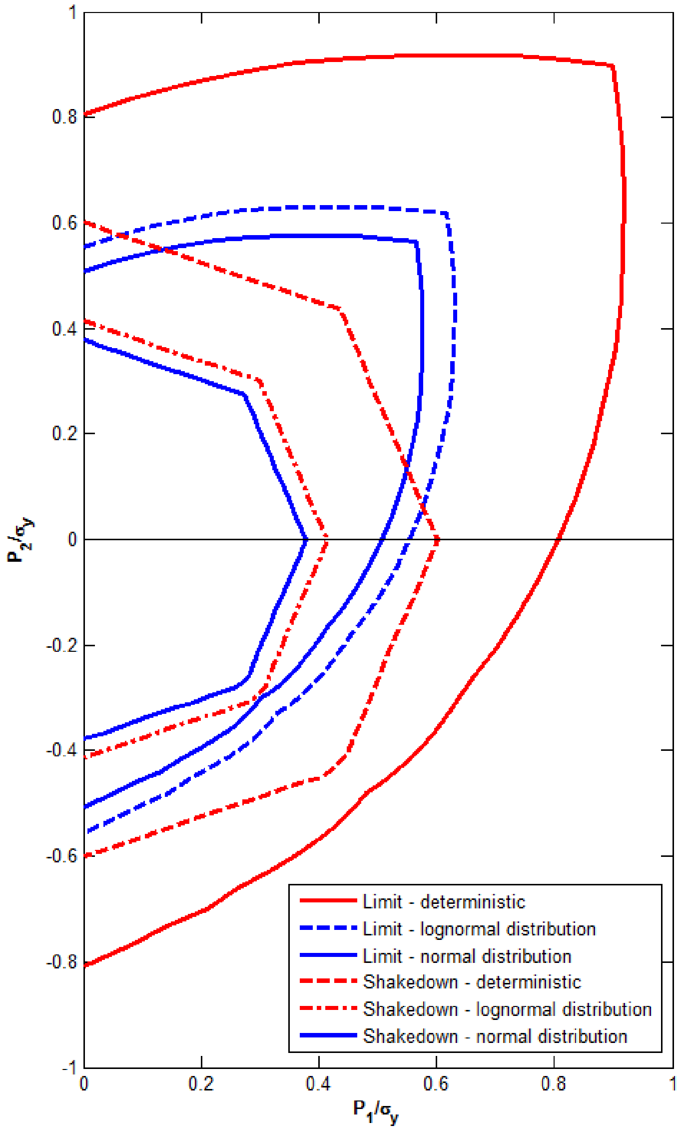

Fig. 9. Iteration diagram of square plate with central hole ($D/L = 0.2$)

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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