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THE CONTRIBUTIONS OF PROFESSOR NGUYEN VAN DAO IN THE FIELD OF NONLINEAR OSCILLATIONS

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Abstract. Nguyen Van Dao is a leading expert in the field of mechanics in Vietnam. In this paper, we present some contributions of Nguyen Van Dao to the theory of nonlinear oscillation. His outstanding studies are the interaction between self-excited, forced and parametric vibrations, and equally important there is the effect of dynamic absorbers in nonlinear systems.

Keywords: Nguyen Van Dao, mechanics in Vietnam, nonlinear oscillation.

Prof. Nguyen Van Dao (1937-2006)

1. INTRODUCTION

Professor Nguyen Van Dao is a leading scientist of Vietnam in the field of mechanics, an internationally recognized expert in the field of nonlinear vibrations. The scientific research area of Nguyen Van Dao is the nonlinear oscillations of dynamic systems. He published more than a hundred articles and several monographs.

To understand the importance of the nonlinear oscillation problems studied by Nguyen Van Dao, it is necessary to first explain some problems regarding the difference between the oscillations of a linear system and the oscillation of a nonlinear system.

1) The natural frequencies of a linear system depend only on the parameters (mass, stiffness) of the system, not on the initial conditions. The natural frequencies of a nonlinear system depend on the parameters (mass, stiffness) of the system and on the initial conditions.

2) For a linear forced vibration system subjected to harmonic excitation, the stable forced vibration of the system has the same frequency as the frequency of the exciting force. For a forced nonlinear vibrational system subjected to harmonic excitation, the forced steady vibration of the system can have a frequency that coincides with the frequency of the exciting force and can also have a frequency that is proportional to the frequency of the exciting force.

3) In the linear oscillation system, there are only 3 types of oscillations, including free, forced and parametric oscillation. In a nonlinear oscillation system, there are four types of oscillations: free, forced, parametric and self-excited one.

4) Considering the stability of a linear system, we have the concept of the stability of the whole system. However, there is no concept of stability of a non-linear system, only the concept of stability of each solution of the system.

Therefore, we cannot study the vibrations of nonlinear systems like linear systems due to the complexity of nonlinear systems. N. N. Bogoliubov and Yu. A. Mitropolskii investigated nonlinear vibrations based on the asymptotic method [1]. A. H. Nayfer and D. T. Mook studied nonlinear oscillators based on the method of multiple scales [2]. Nguyen Van Dao investigated nonlinear vibrations based on characteristic equations for fundamental properties of nonlinear systems, namely, Duffing equation for forced oscillations, Mathieu equation for non-linear parametric vibrations, and van der Pol equation for self-excited vibrations. The investigation results on the interaction of the three basic equation systems mentioned above have been shown a relatively comprehensive picture of the vibration phenomena occurring in nonlinear systems. Nguyen Van Dao published over 100 scientific articles and many monographs [3–9]. Next, we review some outstanding results from his research works.

2. DEVELOPMENT OF ASYMPTOTIC METHODS FOR ANALYSIS OF NONLINEAR OSCILLATIONS IN HIGH ORDER SYSTEMS

The asymptotic method was developed by Soviet scientists such as N. N. Bogoliubov, and Yu. A. Mitropolskii [1]. Nguyen Van Dao applied the asymptotic method to systematically study the vibrations of the third-, fourth- and N-order systems [10].

Using asymptotic method, Nguyen Van Dao investigated the forced periodic oscillation of the systems governed by N-order differential equation

$$\begin{aligned} &x^{(N)} + \alpha_1 x^{(N-1)} + ... + \alpha_{N-1}\dot{x} + \alpha_N x = \varepsilon F(x, \dot{x}, ..., x^{(N)}, \varepsilon), \\ &x^{(k)} = \frac{d^k x}{dt^k}, \end{aligned} \tag{1}$$

where ε is a small parameter, α_i are real constants, $F(x, \dot{x}, ..., x^{(N)}, \varepsilon)$ is the known function, which has enough derivatives with respect to its arguments for all their finite values. When $\varepsilon = 0$, Eq. (1) is degenerated to

$$x^{(N)} + \alpha_1 x^{(N-1)} + ... + \alpha_{N-1}\dot{x} + \alpha_N x = 0. \tag{2}$$

The behavior of the solution of this degenerative equation essentially depends on the roots of the characteristic equation of Eq. (2)

$$\lambda^{(N)} + \alpha_1 \lambda^{(N-1)} + ... + \alpha_{N-1}\lambda + \alpha_N = 0. \tag{3}$$

It is supposed that the characteristic equation has a pair of simple imaginary roots $\lambda = \pm i\Omega$ and its other roots have negative real parts with sufficiently great values. In this case, Eq. (2) has periodic solution

$$x = a\cos(\Omega t + \psi), \tag{4}$$

where a and ψ are arbitrary real constants.

By virtue of the continuous dependence of the solution on the parameter ε, Eq. (1) with sufficiently small ε, the solution of Eq. (1) can be found in the form

$$x = a\cos\varphi + \varepsilon u_2(a, \varphi) + \varepsilon^2 u_2(a, \varphi) + \dots, \tag{5}$$

where u_i do not contain $\sin\varphi$ and $\cos\varphi$. There functions are limited for all finite values a and real values φ, and are also periodic functions of φ with the period 2π, $\varphi = \Omega t + \psi$. The quantities a and ψ being the slowly varying functions of t can be determined by the differential equations

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + ... \\ \frac{d\psi}{dt} &= \varepsilon B_1(a) + \varepsilon^2 B_2(a) + ... \end{aligned} \tag{6}$$

For the forced oscillation of systems governed by N-order differential equation we consider the following equation

$$x^{(N)} + \alpha_1 x^{(N-1)} + ... + \alpha_{N-1}\dot{x} + \alpha_N x = \varepsilon F(x, \dot{x}, ..., x^{(N)}, \theta, \varepsilon). \tag{7}$$

Here the function F is periodic with respect to θ with period 2π, $\dfrac{d\theta}{dt} = \gamma =$ constant. The other assumptions for function F are the same as stated in the above paragraph. We consider first the case when the characteristic equation (3) has a pair of simple imaginary roots $\lambda = \pm i\Omega$ and the other roots have negative real parts with sufficiently great values. Moreover, it is supposed that there exists a resonance relation

$$\Omega = \frac{p}{q}\gamma + \varepsilon\eta, \tag{8}$$

where p and q are relatively prime.

We shall find the solution of the equation (7) in the form of an asymptotic series:

$$x = a\cos\left(\frac{p}{q}\theta + \psi\right) + \varepsilon u_2\left(a, \Phi, \theta\right) + \varepsilon^2 u_2\left(a, \Phi, \theta\right) + \ldots, \tag{9}$$

$$\Phi = \frac{p}{q}\theta + \psi, \tag{10}$$

where u_i do not contain $\sin\Phi$ and $\cos\Phi$. They are periodic functions in Φ and θ with period 2π, limited for finite values of a, and a, ψ are determined from the following equations

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a,\psi) + \varepsilon^2 A_2(a,\psi) + \ldots \\ \frac{d\psi}{dt} &= \varepsilon B_1(a,\psi) + \varepsilon^2 B_2(a,\psi) + \ldots \end{aligned} \tag{11}$$

Nguyen Van Dao studied in detail the vibrations of the third-order autonomous system, the third-order non-autonomous system in the case of $N = 3$ [11–13].

3. STABILITY OF NONLINEAR OSCILLATIONS

The theory of motion stability has been published extensively and systematically [14, 15]. While the problem of motion of relative equilibrium positions is quite simple, the stability of periodic solutions is a relatively complicated problem which was systematically investigated by Nguyen Van Dao. He proposed some important formulas to help us calculate the stability conditions of periodic solutions by a convenient way [16].

3.1. Autonomous Case

Many problems in Engineering and Physics lead to the weak nonlinear autonomous equation system

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}), \tag{12}$$

where ω is the natural frequency, f is a nonlinear function, ε is a small parameter.

If $\varepsilon = 0$, one has the harmonic oscillator

$$\ddot{x} + \omega^2 x = 0, \tag{13}$$

whose general solution is

$$x(t) = a\cos\psi, \quad \psi = \omega t + \theta.$$

The amplitude a is constant and the total phase ψ increases monotonically with t:

$$\frac{da}{dt} = 0, \quad \frac{d\psi}{dt} = \omega.$$

The presence of the nonlinear excitation ($\varepsilon \neq 0$) leads to the dependence of the momentary frequency $\dfrac{d\psi}{dt}$ on the amplitude and may give to a systematic increase or decrease of the amplitude. To apply the averaging method, we transform Eq. (12) into the Lagrange-Bogoliubov normal form by

$$\begin{aligned} x(t) &= a(t)\cos(\omega t + \theta(t)), \\ \dot{x}(t) &= -a(t)\omega\sin(\omega t + \theta(t)). \end{aligned} \tag{14}$$

The differentiation of the first equation in Eq. (14) with respect to time and a comparison of the result with the second equation in Eq. (14) yield

$$\dot{a}(t)\cos\psi - a(t)\dot{\theta}\sin\psi = 0, \quad \psi = \omega t + \theta. \tag{15}$$

Using Eq. (14), the differential equation (12) is reduced to the form

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega\sin\varphi)\sin\varphi d\varphi, \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega\sin\varphi)\cos\varphi d\varphi. \end{aligned} \tag{16}$$

The first expression of Eq. (16) can be rewritten as

$$\frac{da}{dt} = \Phi(a). \tag{17}$$

In Eq. (17) we have used the symbol

$$\Phi(a) = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega\sin\varphi)\sin\varphi d\varphi$$

The stationary root of Eq. (17) is determined by the nonlinear algebraic equation

$$\Phi(a_0) = 0. \tag{18}$$

Theorem 1. *The sufficient condition for the stationary solution of differential equation (15) to be asymptotically stable is*

$$\Phi'(a_0) < 0. \tag{19}$$

According to Lyapunov, the stability analysis methods are divided into 2 groups:
- Stability analysis by the first method;
- Stability analysis by the second method (Lyapunov functions).

Nguyen Van Dao used the first method to investigate the stability of nonlinear oscillations of autonomous systems and non-autonomous systems.

3.2. Non-autonomous Case

Consider the forced nonlinear oscillation system described by Duffing equation

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}) + \varepsilon E \sin \Omega t, \tag{20}$$

where ω is the natural frequency, f is a nonlinear function, ε is a small parameter. Assuming that there is a resonance relationship between Ω and ω as follows

$$\Omega^2 = \omega^2 + \varepsilon\sigma, \tag{21}$$

where σ is a parameter representing the difference between Ω and ω within the resonance region. Substitution of Eq. (21) into Eq. (20) yields

$$\ddot{x} + \Omega^2 x = \varepsilon [f(x, \dot{x}) + \sigma x + E \sin \Omega t] = \varepsilon g(x, \dot{x}, \Omega t). \tag{22}$$

Using the transformation

$$x = a\cos(\Omega t + \theta), \quad \dot{x} = -a\Omega \sin(\Omega t + \theta), \tag{23}$$

the Lagrange-Bogolubov normal form of Eq. (22) is obtained as

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{\Omega}[f(x, \dot{x}) + \sigma x + E \sin \Omega t)] \sin(\Omega t + \theta), \\ a\dot{\alpha} &= -\frac{\varepsilon}{\Omega}[f(x, \dot{x}) + \sigma x + E \sin \Omega t)] \cos(\Omega t + \theta). \end{aligned} \tag{24}$$

The average equation of Eq. (24) is

$$\begin{aligned} \dot{a} &= -\varepsilon \left[a\Phi(a, \Omega) + \frac{E}{2\Omega} \cos\theta \right], \\ \dot{\theta} &= -\varepsilon \left[\frac{\sigma}{2\Omega} + \psi(a, \Omega) - \frac{E}{2\Omega a} \sin\theta \right], \end{aligned} \tag{25}$$

where

$$\begin{aligned} \Phi(a, \Omega) &= \frac{1}{a\Omega} < f(x, \dot{x}) \sin \varphi >= \frac{1}{2\pi a\Omega} \int_0^{2\pi} f(a\cos\varphi, -a\Omega \sin\varphi) \sin \varphi d\varphi, \\ \psi(a, \Omega) &= \frac{1}{a\Omega} < f(x, \dot{x}) \cos \varphi >= \frac{1}{2\pi a\Omega} \int_0^{2\pi} f(a\cos\varphi, -a\Omega \sin\varphi) \cos \varphi d\varphi. \end{aligned} \tag{26}$$

Given $\dot{a} = 0, \dot{\theta} = 0$, the stationary solutions a_0, θ_0 of Eq. (26) can be derived as the following system of equations

$$\begin{aligned} & a_0\Phi(a_0, \Omega) + \frac{E}{2\Omega}\cos\theta_0 = 0, \\ & \frac{a_0\sigma}{2\Omega} + a_0\psi(a_0, \Omega) - \frac{E}{2\Omega}\sin\theta_0 = 0. \end{aligned} \tag{27}$$

Transforming Eq. (27) we obtain the function

$$W(a, \Omega) = a_0^2 \left\{ \Phi^2(a_0, \Omega) + \left[\frac{\sigma}{2\Omega} + \psi(a_0, \Omega) \right]^2 \right\} - \frac{E^2}{4\Omega^2} = 0. \tag{28}$$

To study the stability of the stationary solution, we consider the solutions adjacent to the stationary solution a_0, θ_0

$$a = a_0 + \delta a, \quad \theta = \theta_0 + \delta\theta. \tag{29}$$

This leads to a system of first-order approximations

$$\begin{aligned} \frac{d\delta a}{dt} &= a_{11}\delta a + a_{12}\delta\theta, \\ \frac{d\delta\theta}{dt} &= a_{21}\delta a + a_{22}\delta\theta. \end{aligned} \tag{30}$$

The characteristic equation of Eq. (30) is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{31}$$

Prof. Nguyen Van Dao studied stability based on the amplitude-frequency equation

$$W(a_0, \Omega) = a_0^2 \left\{ \Phi^2(a_0, \Omega) + \left[\frac{\sigma}{2\Omega} + \psi(a_0, \Omega) \right]^2 \right\} - \frac{E^2}{4\Omega^2} = 0. \tag{32}$$

He put in the function

$$P(a_0, \Omega) = \Phi(a_0, \Omega) + \frac{\partial}{\partial a_0} \left[a_0 \Phi(a_0, \Omega) \right]. \tag{33}$$

Conclusion: The stability conditions have the form

$$P(a_0, \Omega) > 0, \quad \frac{\partial W(a_0, \Omega)}{\partial a_0} > 0. \tag{34}$$

The first condition in Eq. (34) is usually satisfied. Thus, on the plane of amplitude and frequency (Ω, a_0), the resonance curve $W(a_0, \Omega) = 0$ is divided into two regions, namely, region $W(a_0, \Omega) > 0$ and region $W(a_0, \Omega) < 0$. Going from region $W(a_0, \Omega) < 0$ to region $W(a_0, \Omega) > 0$ the derivative

$$W(a_0, \Omega) = 0, \quad \frac{\partial W(a_0, \Omega)}{\partial a_0} > 0. \tag{35}$$

4. DYNAMIC ABSORBER FOR NONLINEAR SYSTEMS

The application of dynamic absorbers is intended to minimize mechanical vibrations,. first applied to linear systems [17, 18] and then for nonlinear systems. Currently, this problem is still a topical issue in technology. W. M. Mansour [19], W.R. Clendening and R.N. Dubey [20], A. Tondl [21–23], P. Hagedorn [24] published their works on the effect of the dynamic absorber for a self-excited system. In the PhD. Dissertation defended at the Moscow university in 1965, Nguyen Van Dao showed his research results on continuous oscillations by resonance of a nonlinear dynamical system with dampers under the influence of external forces of variable frequency and amplitude [25,26]. Nguyen Van Dao and Nguyen Van Dinh demonstrated the effect of dynamic absorbers for self-existed systems [27]. The content of this section is limited to the research results on the effects of dynamic absorbers in self-excited systems by Nguyen Van Dao and his colleagues in Hanoi, that is, according to our opinion, the most important part from the work done by him and his colleagues on the effect of dampers in nonlinear systems.

4.1. Dynamic absorber for self-excited systems with single degree of freedom

Self-excited vibration is a typical feature of nonlinear phenomena. Nguyen Van Dao focused his research on dynamic absorber effect on self-excited vibrations. He and his colleague Nguyen Van Dinh systematically investigated nonlinear absorbers such as dynamic absorbers, Lanchester absorbers, Voigt absorbers with the aim of reducing self-excited vibrations, forced vibrations and parametric vibrations in mechanical systems having one degree of freedom, many and infinite degrees of freedom.

The absorber is called a weak one if its mass m_2 and stiffness c_2 are small in comparison with the main mass m_1 and stiffness c_1.

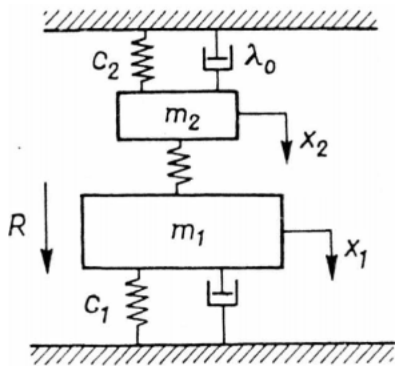


Fig. 1

The vibration equations for the system depicted in Fig. 1 are written in following form

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= \frac{\varepsilon}{m_1}[h_1\dot{x}_1 - h_3\dot{x}_1^3 - c_{12}(x_1 - x_2)], \\ \ddot{x}_2 + \omega_2^2 x_2 + \lambda\dot{x}_2 &= \frac{c_{12}}{m_2}x_1, \end{aligned} \tag{36}$$

where

$$\omega_1^2 = \frac{c_1}{m_1}, \quad \omega_2^2 = \frac{1}{m_2}(c_2 + c_{12}), \quad h_1 > 0, \quad h_2 > 0,$$

The first equation of Eq. (36) is quasi-linear while the second one is a linear equation. In the first approximation we shall find the solution of Eq. (36) in the form

$$\begin{aligned} &x_1 = a\cos\theta, \quad \dot{x}_1 = -a\omega_1\sin\theta, \quad \theta = \omega_1 t + \psi, \\ &x_2 = a(M\cos\theta + N\sin\theta), \quad \dot{x}_2 = a\omega_1(-M\sin\theta + N\cos\theta), \end{aligned} \tag{37}$$

where

$$M = \frac{c_{12}(\omega_2^2 - \omega_1^2)}{m_2[(\omega_2^2 - \omega_1^2)^2 + \omega_1^2\lambda^2]}, \quad N = \frac{c_{12}\omega_1\lambda}{m_2[(\omega_2^2 - \omega_1^2)^2 + \omega_1^2\lambda^2]},$$

and a, ψ satisfy the averaged equations as follows

$$\begin{aligned} \dot{a} &= \frac{\varepsilon a}{2m_1}\left\{h_1 - \frac{c_{12}^2\lambda}{m_2[(\omega_2^2-\omega_1^2)^2+\omega_1^2\lambda^2]} - \frac{3}{4}h_3\omega_1^2a^2\right\}, \\ \omega_1\dot{\psi} &= -\frac{c_{12}}{2m_1}\left\{\frac{c_{12}(\omega_2^2-\omega_1^2)^2}{m_2[(\omega_2^2-\omega_1^2)^2+\omega_1^2\lambda^2]} - 1\right\}. \end{aligned} \tag{38}$$

It is easy to verify that the equilibrium $a = 0$ is unstable and there exists a stationary self-excited vibration with the amplitude a determined by the following expression

$$\frac{3}{4}h_3\omega_1^2a^2 = h_1 - \frac{c_{12}^2\lambda}{m_2[(\omega_2^2-\omega_1^2)^2+\omega_1^2\lambda^2]}, \tag{39}$$

if the right side of Eq. (39) is positive.

Let us consider the so-called strong absorber when the parameters mass m_2 and stiffness c_2 are finite. In this case the motion equations are:

$$\begin{aligned} m_1\ddot{x}_1 + c_1x_1 + c_{12}(x_1-x_2) &= \varepsilon\left(h_1\dot{x}_1 - h_3\dot{x}_1^3\right), \\ m_2\ddot{x}_2 + c_2x_2 + c_{12}(x_2-x_1) &= -\varepsilon\lambda\dot{x}_2, \end{aligned} \tag{40}$$

and they are not separate when $\varepsilon = 0$ as Eq. (36). Using the transformation into the principal coordinates ξ_1, ξ_2:

$$x_1 = \xi_1 + \xi_2, \quad x_2 = d_1\xi_1 + d_2\xi_2, \tag{41}$$

where

$$\begin{aligned} d_i &= \frac{c_{12}}{(c_{12}+c_2-m_2\Omega_i^2)}, \quad (i=1,2) \\ \Omega_{1,2}^2 &= \frac{1}{2}\left\{\frac{c_1+c_{12}}{m_1} + \frac{c_2+c_{12}}{m_2} \pm \sqrt{\left(\frac{c_1+c_{12}}{m_1} - \frac{c_2+c_{12}}{m_2}\right)^2 + 4\frac{c_{12}^2}{m_1m_2}}\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \ddot{\xi}_1 + \Omega_1^2\xi_1 &= \frac{\varepsilon}{M_1}[f_1 + d_1f_2], \\ \ddot{\xi}_2 + \Omega_2^2\xi_2 &= \frac{\varepsilon}{M_2}[f_1 + d_2f_2], \end{aligned} \tag{42}$$

where

$$M_i = m_1 + d_i^2m_2 \ (i=1,2), \quad f_1 = h_1\dot{x}_1 - h_3\dot{x}_1^3, \quad f_2 = -\lambda\dot{x}_2.$$

Introducing the new variables a, b, ψ, ϕ by means of the formulas

$$\begin{aligned} \xi_1 &= a\cos\theta, \quad \dot{\xi}_1 = -a\Omega_1\sin\theta, \quad \theta = \Omega_1t + \psi, \\ \xi_2 &= b\cos\varphi, \quad \dot{\xi}_2 = -a\Omega_2\sin\varphi, \quad \varphi = \Omega_2t + \phi, \end{aligned} \tag{43}$$

and imposing the conditions

$$\dot{a}\cos\theta - a\dot{\psi}\sin\theta = 0, \quad \dot{b}\cos\varphi - b\dot{\phi}\sin\varphi = 0,$$

from Eq. (42) we have

$$\begin{aligned} \Omega_1 \dot{a} &= -\frac{\varepsilon}{M_1}(f_1 + d_1 f_2)\sin\theta, \quad \Omega_1 a\dot{\psi} = -\frac{\varepsilon}{M_1}(f_1 + d_1 f_2)\cos\theta, \\ \Omega_2 \dot{b} &= -\frac{\varepsilon}{M_2}(f_1 + d_2 f_2)\sin\varphi, \quad \Omega_2 b\dot{\phi} = -\frac{\varepsilon}{M_2}(f_1 + d_2 f_2)\cos\varphi. \end{aligned} \tag{44}$$

Since the new variables are slowly varying, Eq. (44) may be averaged over one cycle as follows

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon a}{2M_1}(-h_1 + d_1^2\lambda + \frac{3}{4}h_3\Omega_1^2 a^2 + \frac{3}{2}h_3\Omega_2^2 b^2), \\ \dot{b} &= -\frac{\varepsilon a}{2M_2}(-h_1 + d_2^2\lambda + \frac{3}{4}h_3\Omega_2^2 b^2 + \frac{3}{2}h_3\Omega_1^2 a^2), \\ \dot{\psi} &= 0, \quad \dot{\phi} = 0. \end{aligned} \tag{45}$$

From the result of the last equations of Eq. (45), it follows that $\psi = \text{const}, \phi = \text{const}$. It is easy to verify the following steady state regime of self-exited vibration determined by the conditions $\dot{a} = \dot{b} = 0$:

1. Equilibrium $\dot{a} = \dot{b} = 0$ which is stable if $h_1 - d_1^2\lambda < 0, h_1 - d_2^2\lambda < 0$.

2. Vibration of the first principal coordinate ξ_1 with frequency Ω_1 and amplitude a determined by

$$b = 0, \quad A = \frac{3}{4}h_3\Omega_1^2 a^2 = h_1 - d_1^2\lambda, \quad h_1 - d_1^2\lambda > 0. \tag{46}$$

This regime is stable if

$$\frac{3}{2}h_3\Omega_1^2 a^2 > h_1 - d_2^2\lambda, \tag{47}$$

or

$$A > \frac{1}{2}B.$$

3. Vibration of the second principal coordinate ξ_2 with frequency Ω_2 and amplitude b determined by

$$a = 0, \quad B = \frac{3}{4}h_3\Omega_2^2 b^2 = h_1 - d_2^2\lambda, \quad h_1 - d_2^2\lambda > 0. \tag{48}$$

This regime is stable if

$$\frac{3}{2}h_3\Omega_2^2 b^2 > h_1 - d_1^2\lambda, \tag{49}$$

or

$$B > \frac{1}{2}A.$$

4. Vibration of both two principal coordinates ξ_1, ξ_2 with two frequencies Ω_1, Ω_2 but this regime is always unstable.

4.2. Dynamic absorber for self-excited systems with several degree of freedom

To damp the vibration of masses m_1 and m_2 we use the absorber (m, k, λ). It is supposed that the absorber is attached to the first mass m_1 (Fig. 2).

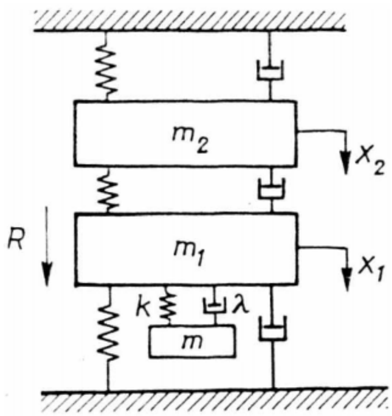


Fig. 2

The motion equations can now be expressed in the form

$$\begin{aligned} &m_1\ddot{x}_1 + (c_1 + c_{12} + k)x_1 - c_{12}x_2 - kz = \varepsilon\left[l_1\dot{x}_1 - l_3\dot{x}_1^3 - (h_1 + h_{12})\dot{x}_1 + h_{12}\dot{x}_2 - \lambda(\dot{x}_1 - \dot{z})\right], \\ &m_2\ddot{x}_2 - c_{12}x_1 + (c_{12} + c_2)x_2 = \varepsilon[h_{12}\dot{x}_1 - (h_{12} + h_2)\dot{x}_2], \\ &m\ddot{z} - k_1x_1 + kz = \varepsilon\lambda(\dot{x}_1 - \dot{z}). \end{aligned} \tag{50}$$

Denoting Ω_1 and $d_1^{(i)} = 1, d_2^{(i)}, d_3^{(i)}$ $(i = 1,2,3)$ the own frequencies and the distributed coefficients of the homogeneous system respectively and the transformation

$$x_i = d_i^{(1)}\xi_1 + d_i^{(i)}\xi_2 + d_i^{(3)}\xi_3, \quad (i = 1,\, 2,\, 3,\; x_3 = z). \tag{51}$$

Calculating is similar to paragraph 4.1, we have the following results:

1. The equilibrium $x_1 = \dot{x}_1 = 0$ which is stable if

$$\begin{aligned} &l_1 - H_i^* < 0, \quad (i = 1,2,3) \\ &H_i^* = h_1 + h_2d_2^{(i)2} + h_{12}(1 - d_2^{(i)})^2 + \lambda(1 - d_3^{(i)})^2 \end{aligned} \tag{52}$$

2. The harmonic vibration of vibration of the first principal coordinate ξ_1 with frequency Ω_1 and with amplitude a_1 is determined by

$$A_1 = \frac{3}{4}l_3\Omega_1^2a_1^2 = l_1 - H_1^*. \tag{53}$$

This vibration is stable if

$$A_1 > \frac{1}{2}(l_1 - H_j^*) = \frac{1}{2}A_j, \quad (j = 2,3) \tag{54}$$

3. The harmonic vibration of second principal coordinate ξ_2 with frequency Ω_2 and amplitude a_2 determined by

$$A_2 = \frac{3}{4} l_3 \Omega_2^2 a_2^2 = l_1 - H_2^*, \tag{55}$$

is stable if

$$A_2 > \frac{1}{2} A_j, \quad (j = 1, 3).$$

4. The harmonic vibration of the third principal coordinate ξ_3 with frequency Ω_3 and amplitude a_3 determined by

$$A_3 = \frac{3}{4} l_3 \Omega_3^2 a_3^2 = l_1 - H_3^*, \tag{56}$$

is stable if

$$A_3 > \frac{1}{2} A_j, \quad (j = 1, 2).$$

5. Simultaneous vibration of two or three principal coordinates is always unstable.

In the difference with the above paragraph, here the mass m and the stiffness k are small. The differential equations of motion becomes

$$\begin{aligned} & m_1 \ddot{x}_1 + (c_1 + c_{12} + k) x_1 - c_{12} x_2 - kz = \varepsilon f_1 \\ & \qquad = \varepsilon \left[l_1 \dot{x}_1 - l_3 \dot{x}_1^3 - (h_1 + h_{12}) \dot{x}_1 + h_{12} \dot{x}_2 - k(x_1 - z) - \lambda(\dot{x}_1 - \dot{z}) \right], \\ & m_2 \ddot{x}_2 - c_{12} x_1 + (c_{12} + c_2) x_2 = \varepsilon f_2 = \varepsilon [h_{12} \dot{x}_1 - (h_{12} + h_2) \dot{x}_2], \\ & \ddot{z} + \lambda_0 \dot{z} + \omega^2 z = \omega^2 x_1 + \lambda_0 \dot{x}_1, \end{aligned} \tag{57}$$

where

$$\omega^2 = \frac{k}{m}, \quad \lambda_0 = \frac{\lambda}{m}.$$

Transforming the subsystem (x_1, x_2) into the principal coordinates (ξ_1, ξ_2), we have analogously with the paragraph 4.1 the following equations

$$\begin{aligned} & \ddot{\xi}_1 + \Omega_1^2 \xi_1 = \frac{\varepsilon}{M_1} \left[f_1 + d_1 f_2 \right], \\ & \ddot{\xi}_2 + \Omega_2^2 \xi_2 = \frac{\varepsilon}{M_2} \left[f_1 + d_2 f_2 \right], \\ & \ddot{z} + \lambda_0 \dot{z} + \omega^2 z = \omega^2 (\xi_1 + \xi_2) + \lambda_0 (\dot{\xi}_1 + \dot{\xi}_2), \end{aligned} \tag{58}$$

where

$$\begin{aligned} & \xi_1 = a_1 \sin \varphi_1, \;\; \dot{\xi}_1 = a_1 \Omega_1 \cos \varphi_1, \;\; \xi_2 = a_2 \sin \varphi_2, \;\; \dot{\xi}_2 = a_2 \Omega_2 \cos \varphi_2, \\ & z = a_1 (P_1 \sin \varphi_1 + Q_1 \cos \varphi_1) + a_2 (P_2 \sin \varphi_2 + Q_2 \cos \varphi_2), \\ & \dot{z} = a_1 \Omega_1 (P_1 \cos \varphi_1 - Q_1 \sin \varphi_1) + a_2 \Omega_2 (P_2 \cos \varphi_2 - Q_2 \sin \varphi_2), \end{aligned} \tag{59}$$

and $a_1, \varphi_1, a_2, \varphi_2$ are new variables and Main references of this paragraph.

5. INTERACTION BETWEEN NONLINEAR OSCILLATING SYSTEMS

It is well-known that there is always an interaction of some kind between the nonlinear oscillations. N Minorsky stated that: "Perhaps the whole theory of nonlinear oscillations could be formed on the basis of interaction" [28]. The fundamental difference between linear and nonlinear systems is that in nonlinear systems there are always interactions, which are associated with the nonlinear nature of dynamical systems. In 2000 Prof. Nayfeh A. H. wrote the monograph "Nonlinear Interactions/Analytical Methods, Computational, and Experimental Methods" published by John Wiley & Sons in 2000 in New York [29]. This is a very famous book. In which, the study is quite detailed on internal resonance and combinatorial resonance. The main method that Nayfeh uses is the multiple scale method. Nguyen Van Dao used the other way, arguing that the interactions in nonlinear systems have deep roots in parametric excitation terms. Using the asymptotic method, Nguyen Van Dao focused his research on parametric excitation and the interaction between parametric excitation and forced oscillation, interaction between parameter oscillation and self-excited oscillation, interaction between forced oscillation and self-attack oscillation [7, 30, 31]. Nguyen Van Dao collected his research papers on parametric excitation, nonlinear vibrations and the interaction between parametric excitation and other types of excitations in nonlinear systems and defended his advanced dissertation in Poland in 1976.

5.1. Interaction between self-excited and forced oscillations

Based on the asymptotic method, Nguyen Van Dao studied forced oscillations of such a system, which could accomplish self-excitation oscillations in the absence of an external force. He considered a system described by the equation

$$\ddot{x} + \omega^2 x = \varepsilon[k(1 - \gamma x^2)\dot{x} + \beta x^3 + e \sin \nu t], \tag{60}$$

which is a classical Van der Pol equation with a small forcing term $\varepsilon e \sin \nu t$, where α, γ, e, ν are positive constants. It is supposed that Ω is close to unity, namely

$$\nu^2 = \omega^2 + \varepsilon\Delta, \tag{61}$$

where Δ is a detuning parameter and ε is a small positive one. Applying to (60) the asymptotic method and using the amplitude and phase variables (a, θ) given by

$$\begin{aligned} x &= a \cos(\nu t + \theta), \\ \dot{x} &= -a\nu \sin(\nu t + \theta). \end{aligned} \tag{62}$$

It follows that

$$\begin{aligned} \nu \dot{a} &= -\varepsilon[\Delta x + k(1 - \gamma x^2)\dot{x} + \beta x^3 + e \sin \nu t] \sin(\nu t + \theta), \\ \nu a\dot{\theta} &= -\varepsilon[\Delta x + k(1 - \gamma x^2)\dot{x} + \beta x^3 + e \sin \nu t] \sin(\nu t + \theta). \end{aligned} \tag{63}$$

Since a and θ are slowly varying functions of time, the change in their values during a period $T = 2\pi/\nu$ is very small. Hence, in the first approximation one may replace

Eq. (63) by their time – averages over $(t, t+T)$ by assuming a and θ to be constant:

$$\begin{aligned} \nu\dot{a} &= \frac{\varepsilon}{2}[ak\nu(1-\frac{1}{4}\gamma a^2) - e\cos\theta], \\ \nu a\dot{\theta} &= \frac{\varepsilon}{2}(-\Delta a - \frac{3}{4}\beta a^3 + e\sin\theta). \end{aligned} \tag{64}$$

The steady-state equations are

$$\begin{aligned} a_0 k\nu(1 - \frac{1}{4}\gamma a_0^2 &= e\cos\theta_0, \\ \Delta a_0 + \frac{3}{4}\beta a_0^3 &= e\sin\theta_0. \end{aligned} \tag{65}$$

By eliminating the phase θ_0 from these equations we obtain the following equation

$$A[(1-A)^2 + \sigma^2] = E^2, \tag{66}$$

where

$$A = \frac{\gamma}{4}a_0^2 = \frac{a_0^2}{a_*^2}, \quad E^2 = \frac{\gamma}{4k^2\omega^2}e^2, \quad \sigma = \frac{\omega}{k}[\frac{\alpha^2-1}{\varepsilon} + \frac{3\beta}{4\omega^2}a_0^2], \quad \alpha = \frac{\nu}{\omega}, \tag{67}$$

$a_* = 2/\sqrt{\gamma}$ is the amplitude of the purely self-excited Van der Pol oscillator. Below only the behavior of forced oscillations with the frequency ν which in close to ω will be considered.

The oscillation described by Eq. (60) with stationary amplitude a_0 and phase amplitude of the solution of Eq. (60) when $e = 0$. The graph in Fig. 3 is the branching diagram of the equation (64) in the plane (Ω, a) with $\varepsilon = 1.0, \omega = 1.5, \mu = 0.3$.

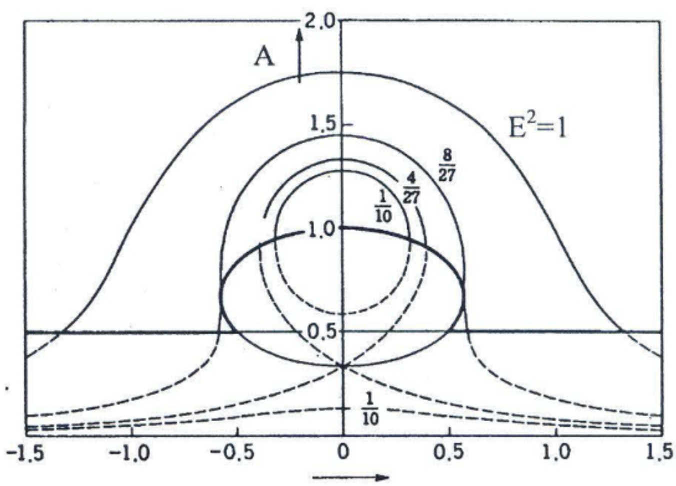


Fig. 3. Amplitude curves for the Duffing - Van der Pol oscillator (β) with various values of the external excitation

5.2. Interaction between self-excited and parametric oscillations

Nguyen Van Dao investigated the oscillation system described by the system

$$\ddot{x} + \omega^2 x + \varepsilon R(\dot{x}) + \varepsilon \alpha x \cos \gamma t = 0, \tag{68}$$

where ω, α and γ are constants and ε is a small positive parameter. The "negative" friction function $R(\dot{x})$ is assumed to be of the form:

$$R(\dot{x}) = -h_1 \dot{x} + h_3 \dot{x}^3, \tag{69}$$

where h_1 and h_3 are positive constants. When $\alpha = 0$ Eq. (68) describes a self-excited oscillator and when $h_1 = h_3 = 0$ it describes a parametric oscillator. Each of them considered separately are self-sus-parametric oscillator.

It is assumed that there exists a resonance relation

$$\omega^2 = \frac{\gamma^2}{4} + \varepsilon \Delta. \tag{70}$$

The solution of Eq. (68) is found in the following form

$$x = a \cos \theta, \quad \dot{x} = -\frac{a}{2} \gamma \sin \theta, \tag{71}$$

with the additional condition

$$\dot{a} \cos \theta - a \dot{\psi} \sin \theta = 0, \tag{72}$$

where $\theta = \dfrac{\gamma}{2} t + \psi$. Substituting Eq. (71) into Eq. (68) and solving with respect to $\dot{a}$ and $\dot{\psi}$ with the half of Eq. (72) we obtain

$$\begin{aligned} \dot{a} &= \frac{2\varepsilon}{\gamma} (\Delta x + R(\dot{x}) + \alpha x \cos \gamma t) \sin \theta, \\ a\dot{\psi} &= \frac{2\varepsilon}{\gamma} (\Delta x + R(\dot{x}) + \alpha x \cos \gamma t) \cos \theta. \end{aligned} \tag{73}$$

Since a and ψ are slowly varying in time, the right-hand sides of Eq. (73) can be replaced in the first approximation by their mean values. It follows that

$$\begin{aligned} \dot{a} &= \frac{\varepsilon}{\gamma} \left(\frac{1}{2} h_1 \gamma a - \frac{3}{32} h_3 \gamma^3 a^3 + \frac{\alpha}{2} a \sin 2\psi \right), \\ a\dot{\psi} &= \frac{\varepsilon}{\gamma} \left(\Delta a + \frac{\alpha}{2} a \cos 2\psi \right). \end{aligned} \tag{74}$$

The stationary solution ($a_0 \neq 0$) of Eq. (74) is determined by the following equation

$$W\left(a_0^2, \eta^2\right) = 0 \tag{75}$$

where

$$\begin{aligned} &\eta^2 = \gamma^2 / 4\omega^2, \\ &W\left(a_0^2,\, \eta^2\right) = \left(a_0^2 - \frac{4h_1}{3h_3\omega^2} \right)^2 + \frac{16\left(\eta^2 - 1\right)^2}{9\varepsilon^2 \omega^2 h_3^2} - \frac{4\alpha^2}{9\omega^6 h_3^2}. \end{aligned}$$

In the (a_0^2, η^2)- plane the expression (75) represents an ellipse (Fig. 4) with the center at

$$\eta_0^2 = 1, \quad a_0^2 = \frac{4h_1}{3h_3\omega^2}, \tag{76}$$

and with two semiaxes

$$l = \frac{\varepsilon\,|\alpha|}{2\omega^2}, \quad L = \frac{2\,|\alpha|}{3\omega^3 h_3}. \tag{77}$$

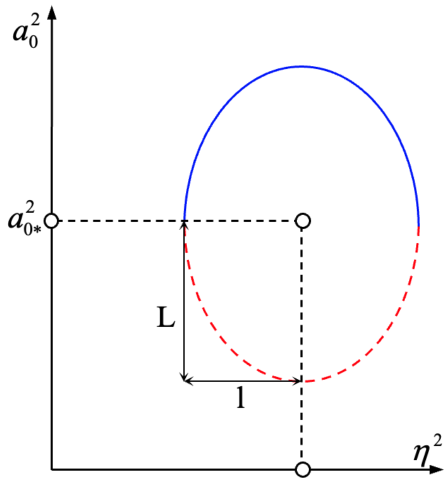


Fig. 4

5.3. Interaction between nonlinear parametric and forced oscillations

The interaction between forced and parametric excitations is described by the following equation

$$\ddot{x} + \omega^2 x = \varepsilon[\Delta x - h\dot{x} - \gamma x^3 + 2px^3\cos 2\omega t + r\cos(\omega t - \delta)], \tag{78}$$

where $\varepsilon > 0$ is the small parameter; $h \geq 0$ is the damping coefficient; $\gamma > 0, p > 0, r > 0, \omega > 0$ are the constant parameters; $\varepsilon\Delta = \omega^2 - 1$ is the detuning parameter, where the natural frequency is equal to unity; and $\delta \geq 0$ is the phase shift between two excitations.

Introducing new variables, a and ψ instead of $x$and $\dot{x}$ as follows,

$$x = a\cos\theta, \quad \dot{x} = -a\omega\sin\theta, \quad \theta = \omega t + \psi, \tag{79}$$

we have a system of two equations which is fully equivaled to Eq. (75)

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega}F\sin\theta, \quad a\frac{d\psi}{dt} = -\frac{\varepsilon}{\omega}F\cos\theta, \tag{80}$$

where

$$F = \Delta x - h\dot{x} - \gamma x^3 + 2px^3\cos 2\omega t + r\cos(\omega t - \delta). \tag{81}$$

Using the asymptotic method to calculate the oscillation described by Eq. (80), we can determine the resonance curves [6]. The resonance curve has three branches and is presented in Figs. 5 and 6 for the parameters $r = 0.01, p = 0.1, \gamma = 0.25$, and $\omega^2 = 1.1$. When

h increases, the upper branch 1 moves up and the two lower branches 2 and 3 are tied and then separated as branches 4 and 5, see Fig. 5 for $h = 0.01$ and Fig. 6 for $h = 0.027$.

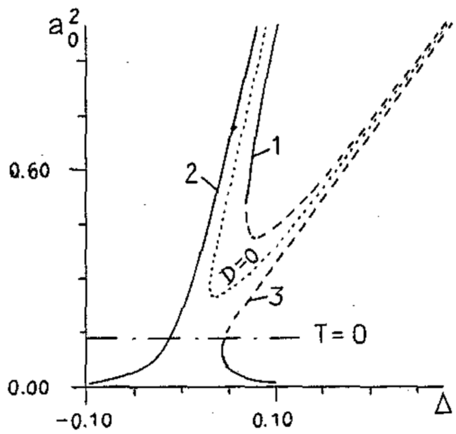


Fig. 5. Resonance curve

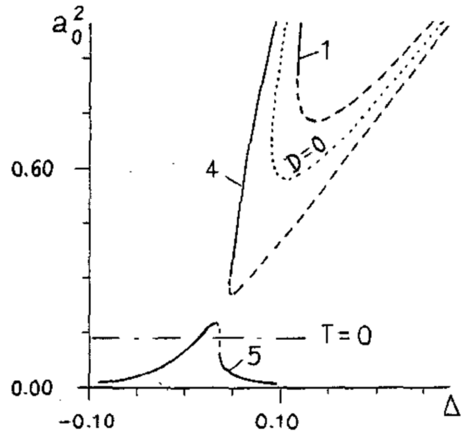


Fig. 6. Resonance curve

5.4. Van der Pol oscillator under parametric and forced excitations

The system under consideration is described by the equation

$$\ddot{x} + \omega^2 x = \varepsilon \left\{ \Delta x - \gamma x^3 + h(1 - kx^2)\dot{x} + 2px \cos 2\omega t + e \cos(\omega t + \sigma) \right\}, \tag{82}$$

where $h > 0$ and $k > 0$ are coefficients characterizing the self-excitation of a pure Van der Pol oscillator, $2p > 0$ is the intensity of the parametric excitation, $e > 0$ is the intensity of the forced excitation, and $\sigma, 0 \leq \sigma \leq 2\pi$, is the phase shift between the parametric and forced excitation.

Case of weak parametric excitation $(p^2 < h^2)$

In Fig. 7, the resonance curves 0, 1, 2, 3, 4, 5 correspond to the linear case $\gamma = 0$, for $e = 0; 0.015; 0.017; 0.050; 0.100; 0.120$, respectively. The curve 0 is a critical oval. The curve 1 has two branches: branch C' lies near abscissa axis, branch C'' lies higher and consists of two cycles, one of them C_1'' is outside and the other C_2'' is inside the critical oval. There cycles are connected to one another at the critical note I_* on the critical oval.

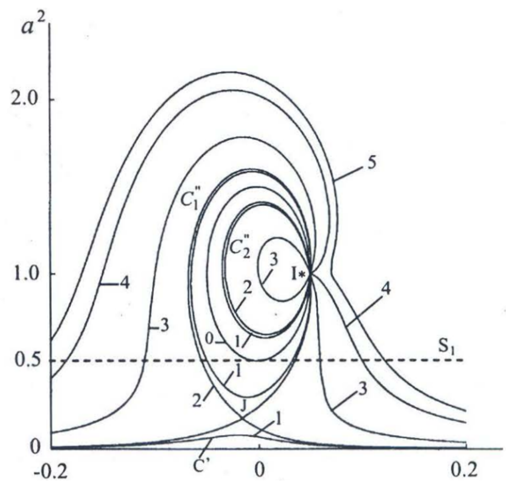


Fig. 7. Resonance curves for $e = 0$ (curve 0), $e = 0.115$ (curve 1), $e = 0.0177$ (curve 2), $e = 0.050$ (curve 3), $e = 0.100$ (curve 4), $e = 0.120$ (curve 5)

6. CHAOTIC MOTION IN THE INTERACTION NONLINEAR SYSTEM

In recent years, the study of chaotic behaviors and strange attractors in deterministic nonlinear systems has undoubtedly developed into one of the main topics in the study of nonlinear phenomena in dynamical systems performed by engineers and applied scientists [32–35]. There is no precise definition for a chaotic solution because it cannot be represented through standard mathematical functions. However, a chaotic solution is an aperiodic solution, which is endowed with some special identifiable characteristic. From a practical point of view, chaos can be defined study as a bounded steady state behavior, chaotic attractors are complicated geometrical objects that possess fractal dimensions. The spectrum of signal has a continuous broadband character. The fundamental characteristic of a chaotic system is its sensitivity to the conditions [36–38]. For determining the chaotic or regular motions of dynamic systems, one often use the following methods:

(a) Poincare sections
(b) Fourier distribution of frequency spectra
(c) Fractal dimension of chaotic attractor
(d) Lyapunov exponents
(e) Invariant probability distribution of attractor

Nguyen Van Dao and his colleagues at the Institute of Mechanics used the above mentioned methods to calculate chaotic motion of fundamental equations from the theory of nonlinear vibrations. His research group tried to determine the chaotic motion in the strong nonlinear Van der Pol system, in the strong nonlinear Duffing system, and in the strong nonlinear Mathieu system. Chaotic motion in a nonlinear forced oscillator with self-excitation and chaotic motion in a Van der Pol oscillator under the parametric and forced excitation are studied.

Here, some research results of him and his colleagues on nonlinear dynamics and chaos motion in the interaction nonlinear systems are briefly introduced.

6.1. Chaotic motion in a strong nonlinear Van der Pol oscillator under the forced excitation

Nguyen Van Dao considered firstly a system described by a strong nonlinear differential equation as follows

$$\ddot{x} + \omega^2 x = k(1 - \gamma x^2)\dot{x} + \beta x^3 + e \sin \nu t. \tag{83}$$

Poincare sections. He calculated a concrete case for the parameters $\omega^2 = 0.7, k = 1, \gamma = 0.6, \beta = -1, e = 5$, and $\nu = 0.783$. The aperiodic appearance of x (see Fig. 8) suggests that the system under consideration is chaotic.

Much more insight can be gained from a Poincare section (Fig. 9) consisting of stroboscopic points at instants $t = n(2\pi/0.782), n = 0, 1, 2, ...$ of the orbit of the system (83) in the space $(x, \dot{x})$. Fig. 9 shows the next 10000 points after the transition decays about the first 1000 periods.

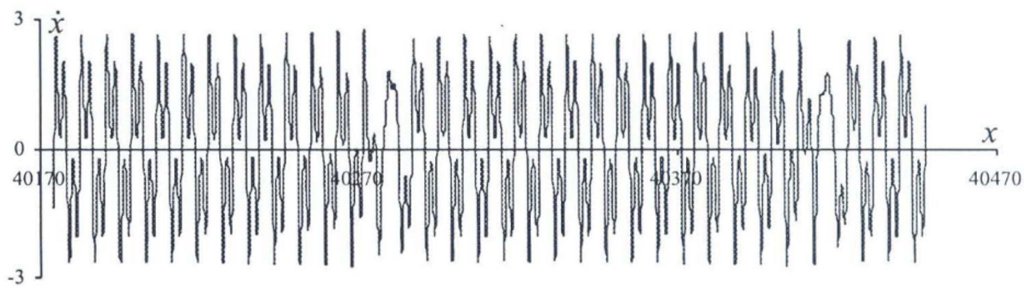


Fig. 8. Aperiodic appearance of $x(t)(\nu = 0.782)$

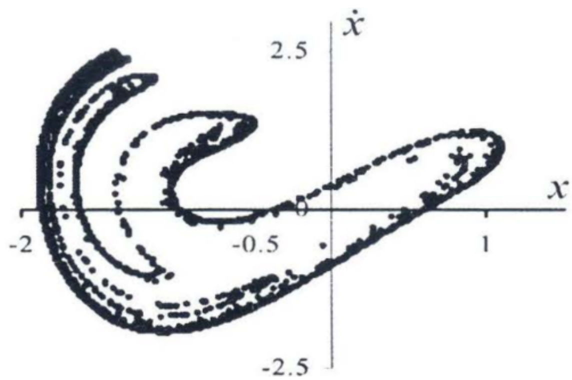


Fig. 9. Poincare section for $\nu = 0.782$

Lyapunov exponent. To evaluate the largest Lyapunov exponent in the case $\omega^2 = 0.7, k = 1, \gamma = 0.6, \beta = -1, e = 4.825$, and $\nu = 0.837$, we represent Eq. (83) in the following form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -0.7x_1 + (1 - 0.6x_1^2)x_2 - x_1^3 + 4.825\sin(0.837z), \\ \dot{z} &= 1. \end{aligned} \tag{84}$$

Let $\mathbf{u} = (x_1, x_2, z)^T$ is a three dimensions vector and $\mathbf{u}^* = \mathbf{u}^*(t, \mathbf{u}_0)$ is a reference trajectory of the system according to Eq. (84), where $\mathbf{u}_0$ is the initial condition. The variational equation corresponding to this reference trajectory is

$$\dot{\boldsymbol{\eta}} = \mathbf{A}\boldsymbol{\eta}, \tag{85}$$

where $\boldsymbol{\eta} = \mathbf{u} - \mathbf{u}^*$ and the matrix $\mathbf{A}$ depends on $\mathbf{u}^*$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{86}$$

where

$$a_{11} = 0, \quad a_{21} = -0.7 - 1.2x_1^* x_2^* - 3(x_1^*)^2, \quad a_{31} = 0,$$
$$a_{12} = 1, \quad a_{22} = 1 - 0.6(x_1^*)^2, \quad a_{32} = 0,$$
$$a_{13} = 0, \quad a_{23} = 4.0384\cos(0.837z^*), \quad a_{33} = 0.$$

The time evolution of the Lyapunov exponent is presented in Fig. 10. The largest exponent value is a positive number $\lambda \approx 0.0698 > 0$, which shows the chaotic character of the motion of the system according to Eq. (84).

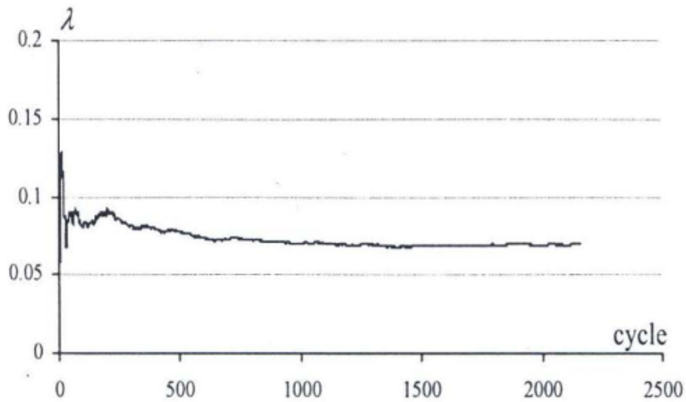


Fig. 10. Time evolution of the largest Lyapunov exponent (one cycle $= 2\pi/\nu, \nu = 0.837$)

6.2. Chaotic motion in a strong nonlinear Van der Pol oscillator under the parametric and forced excitations

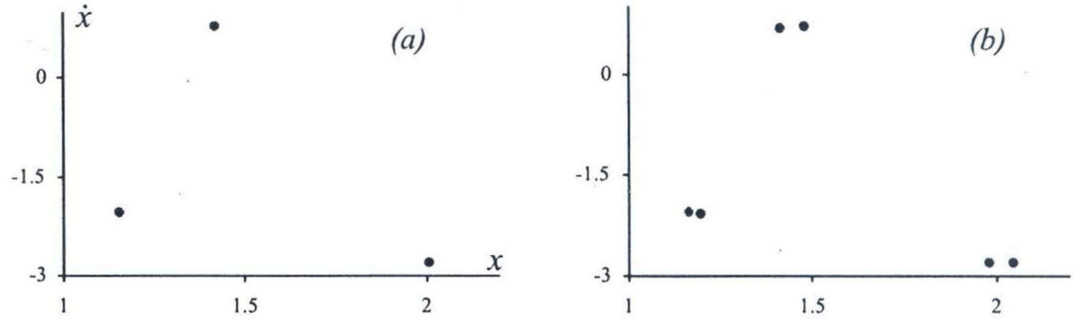
We will consider the following differential equation

$$\ddot{x} + \omega^2 x = \Delta x - \gamma x^3 + h(1 - kx^2)\dot{x} + 2px\cos 2\omega t + e\cos(\omega t + \sigma). \tag{87}$$

We fix the parameters: $\omega = 0.83, \Delta = 0.01, \gamma = 1, k = 0.6, p = 0.001, \sigma = 0$ and use e as a control parameter. With different values of e, solution of Eq. (87) can be regular or chaotic. To identify the regular or chaotic character of solution, we can use various methods, such as consideration of the sign of the largest Lyapunov exponent or building the Poincare sections.

Poincare sections. To construct a Poincare section of an orbit, we use the period $T = 2\pi/\omega$ of the external excitation force. Then, the Poincare section acts like a stroboscope, freezing the components of the motion commensurate with the period T. If we have a collection of n discrete points on the Poincare section, the corresponding motion is periodic with the period nT. For example, for $e = 5.09$, the Poincare section consists of three points (Fig. 11(a)), the motion is periodic with the period $3T$; for $e = 5.116$, the Poincare section consists of six points (Fig. 11(b)), the motion is periodic with the period $6T$. When the Poincare section does not consist of finite number of discrete points, the motion is aperiodic, it may be chaotic.

The aperiodic attractor and its power spectrum realized at $e = 5.15$ are illustrated in Fig. 12. To verify that the motion realized at $e = 5.15$ is chaotic, we need to show the sensitivity to initial conditions on this attractor. Fig. 13 illustrates the variation of the separation



(a) $e = 5.09$ (b) $e = 5.116$

Fig. 11. Poincare section

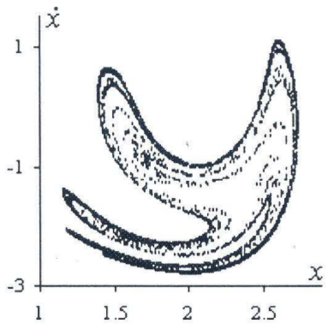


Fig. 12. Poincare section realized at $e = 5.15$

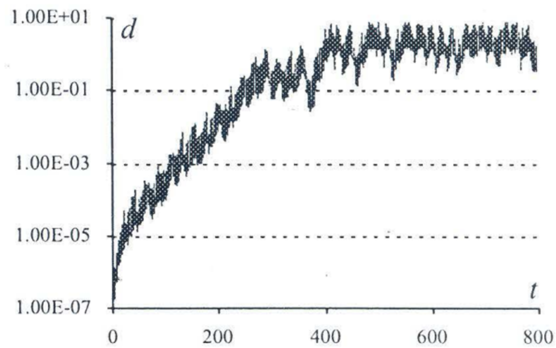


Fig. 13. Sensitivity to initial condition at $e = 5.15$

Lyapunov exponent. To evaluate the largest Lyapunov exponent in the case of $\omega = 0.83, \Delta = 0.01, \gamma = 1, h = 1, k = 0.6, p = 0.001, \sigma = 0$ and $e = 5.15$, we represent Eq. (83) in the following form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -0.6889x_1 + 0.01x_1 - x_1^3 + (1 - 0.6)x_2 + 0.002x_1 \cos 2z + 5.15 \cos z, \\ \dot{z} &= 0.83. \end{aligned} \tag{88}$$

Let $\mathbf{u} = (x_1, x_2, z)^T$ is a three dimensions vector and $\mathbf{u}^* = \mathbf{u}^*(t, \mathbf{u}_0)$ is a reference trajectory of the system (88), where $\mathbf{u}_0$ is the initial condition. The variational equation corresponding to this reference trajectory is

$$\dot{\boldsymbol{\eta}} = \mathbf{A}\boldsymbol{\eta}, \tag{89}$$

where $\boldsymbol{\eta} = \mathbf{u} - \mathbf{u}^*$ and the matrix $\mathbf{A}$ depends on $\mathbf{u}^*$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{90}$$

where

$$a_{11} = 0, \quad a_{21} = -0.6789 - 3(x_1^*)^2 - 1.2x_1^* x_2^* + 0.002\cos 2z^*, \quad a_{31} = 0,$$
$$a_{12} = 1, \quad a_{22} = 1 - 0.6(x_1^*)^2, \quad a_{32} = o,$$
$$a_{13} = 0, \quad a_{23} = -0.004x_1^* \sin 2z^* - 5.15\sin 2z^*, \quad a_{33} = 0.$$

The time evolution of the largest Lyapunov exponent is presented in Fig. 14. The largest Lyapunov exponent value is a positive number $\lambda \approx 0.062$, which shows the chaotic character of the motion of the system according to Eq. (87).

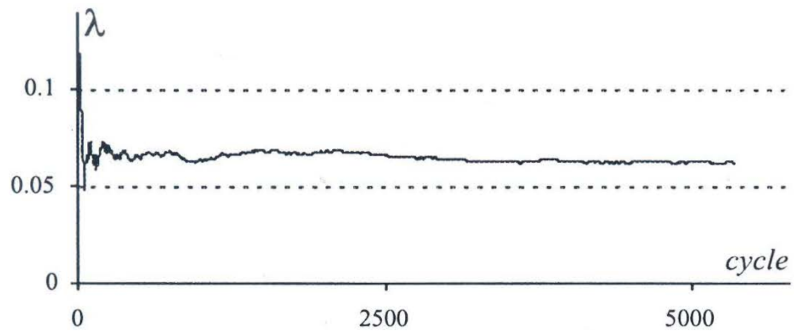


Fig. 14. Time evolution of the largest Lyapunov exponent (one cycle $= 2\pi/\omega, \omega = 0.83$)

7. CONCLUSIONS

Nguyen Van Dao chosen a particular way of thinking when studying nonlinear vibrations. He selected three typical equations for forced parametric, and self-excited vibrations: the Duffing equation, the nonlinear Mathieu equation, and the Van der Pol equation. Using the asymptotic method, he discovered important properties of these oscillating systems. Studying the regular oscillation of the Duffing system, he found a variety of resonance modes: main resonance, harmonic minor resonance, super-harmonic super resonance. When he studied the chaotic oscillations of the Duffing equation, he found the chaotic oscillations of this system. He studied the interaction between these three basic types of equations. The results of studying the interaction of three basic types of vibrations give us a relatively comprehensive picture of nonlinear oscillations.

The contribution of Prof. Nguyen Van Dao in the field of nonlinear oscillation is highly valued by international mechanics research community. Prof. Yu. A. Mitropolskii, a world-renowned researcher, believed that Nguyen Van Dao created a school of nonlinear oscillation in Vietnam.

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