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# SOME APPROXIMATE BUCKLING SOLUTIONS OF TRIPLE-WALLED CARBON NANOTUBE

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**Abstract.** The present investigation analyses the critical buckling studies of triple-walled carbon nanotube using the Euler–Bernoulli model. The present study deals with three different boundary conditions, namely, simply-simply, clamped-clamped, and clamped-simply supported carbon nanotube. Using Bubnov–Galerkin and Petrov–Galerkin methods, the continuum model estimates the critical buckling load. The main advantage of these two approximate methods is to obtain a quick and valid result. The first and second Euler critical buckling loads decrease with the increase of length to outer diameter ratio for boundary conditions like simply-simply, clamped-clamped, and clamped-simply supported. Interestingly, the increase in the length to outer diameter ratio results in the rise in third Euler critical buckling for all three different boundary conditions. These two approximate methods provide reliable buckling load estimation using suitable polynomials.

*Keywords*: triple-walled carbon nanotube, Bubnov–Galerkin method, Petrov–Galerkin method, buckling load.

## 1. INTRODUCTION

In 1991, Iijima [1] experimentally showed the carbon nanotube structure. Since then, carbon nanotubes become popular among researchers to explore the mechanical, structural [2], and electrical properties [3,4]. Buckling [5,6] of structural members plays a crucial role in understanding the elastic and inelastic behavior of the bar, beam, plate, and shell. Batra and Sears [7] used continuum models to analyze the multi-walled carbon nanostructures. The molecular simulations [8,9] provide reliable results in predicting the mechanical/structural behaviors of nanostructures. But due to the substantial computational cost required to carry out the simulations using supercomputers, the continuum models serve as an alternative approach. Elishakoff and Pentaras [10] investigated the buckling analysis of double-walled carbon nanotube using three different boundary conditions, namely simply-simply, clamped-clamped, and clamped-hinged support using Bubnov–Glaerlkin and Petrov–Galerkin methods. Kitipornchai et al. [11] investigated the

buckling studies of triple-walled carbon nanotube with elastic medium using polar coordinates model. Guo et al. [12] studied the critical strain behavior of carbon nanotubes using an atomic-scale finite element approach. Lu et al. [13] studied the effect of the slenderness ratio in the buckling analysis of multi-walled carbon nanotubes. Rahmani and Antonov [14] used the polar coordinates model for the buckling studies of single and multi-walled carbon nanotubes by using the finite element approach. Coşkun [15] computed the buckling loads of columns using various numerical methods. Pentaras and Elishakoff [16] estimated the vibrational frequencies for triple-walled nanotube using Euler–Bernoulli's continuum model. In their research investigation, the estimation of frequencies happens using approximate methods like Bubnov–Galerkin and Petrov–Galerkin method in a reliable manner.

Ru [17] investigated the critical buckling study of multi-walled carbon nanotube using the van der Waals force effect. Later, Ru [18] studied the axial buckling effect of double-walled carbon nanotubes. Further, Ru [19] examined the influence of elastic medium support over critical buckling in the double-walled carbon nanotubes. Senthilkumar [20] used a semi-analytical method to study the single-walled carbon nanotube using Euler–Bernoulli's model with nonlocal effects. Recently, Malikan et al. [21] investigated the buckling behavior of non-concentric double-walled carbon nanotubes. It is often easy to model the simply-simply supported boundary conditions because the analytical solutions are readily available. But other boundary conditions like clamped-clamped, clamped-simply supported require numerical solutions for the buckling solutions of triple-walled carbon nanotube. Elishakoff et al. [22] proposed the polynomials to estimate critical buckling loads of double-walled carbon nanotubes for simply-simply, clamped-clamped and clamped-simply supported boundary conditions using Bubnov–Galerkin and Petrov–Galerkin methods. The present investigation extends it to triple-walled carbon nanotubes. As per the author's knowledge, the buckling studies of triple-walled carbon nanotube with the Euler–Bernoulli model for different boundary conditions are not available in the literature. This work aims to fill this void using quick and reliable numerical solutions.

## 2. MATHEMATICAL MODEL OF TRIPLE-WALLED NANOTUBE

Elishakoff and Pentaras [10,23] proposed the governing equations of double-walled carbon nanotube with buckling behaviour using $\sigma_{x2}^{E} \equiv P^{E}/(A_1 + A_2)$. In the present study, the buckling behavoiur of third nanotube induced by considering the stress as $\sigma_{x3}^{E} \equiv P^{E}/(A_1 + A_2 + A_3)$. So the mathematical model for triple-walled carbon nanotube with buckling behaviour can be expressed as,

$$EI_1 \frac{\partial^4 w_1}{\partial x^4} + P^E \frac{A_1}{A_{T3}} \frac{\partial^2 w_1}{\partial x^2} = c_{12}(w_2 - w_1), \tag{1}$$

$$EI_2 \frac{\partial^4 w_2}{\partial x^4} + P^E \frac{A_2}{A_{T3}} \frac{\partial^2 w_2}{\partial x^2} = c_{23}(w_3 - w_2) - c_{12}(w_2 - w_1), \tag{2}$$

$$EI_3 \frac{\partial^4 w_3}{\partial x^4} + P^E \frac{A_3}{A_{T3}} \frac{\partial^2 w_3}{\partial x^2} = -c_{23}(w_3 - w_2), \tag{3}$$

where $w_1$, $w_2$, and $w_3$ are the transverse displacements of the innermost carbon nanotube, middle carbon nanotube, and outer carbon nanotube, respectively. $I_1$, $I_2$, and $I_3$ are the moment of inertia of the innermost carbon nanotube, middle carbon nanotube, and outer carbon nanotubes. $A_1$, $A_2$, and $A_3$ are the innermost carbon nanotube, middle carbon nanotube, and outer carbon nanotubes' cross-sectional area. $E$, $P^E$, $L$, $c_{12}$, and $c_{23}$ are Young's modulus, Euler's buckling load of triple-walled carbon nanotube, length of the nanotube, and the van der Waals interaction coefficients. Here $A_{T3}$ is defined as $A_{T3} = (A_1 + A_2 + A_3)$. It is interesting to note that when $c_{23} = 0$, $A_3 = 0$, $I_3 = 0$, and $w_3 = 0$, the present model leads to the buckling model of Elishakoff and Pentaras [10] for the double-walled carbon nanotube. Using a non-dimensional approach, the governing equation of triple-walled carbon nanotube takes form as,

$$\frac{d^4\overline{W}_1}{d\overline{X}^4} = \beta_1(\overline{W}_2 - \overline{W}_1) - \left(\alpha^E_{twn}\right)\frac{d^2\overline{W}_1}{d\overline{X}^2}, \tag{4}$$

$$\frac{d^4\overline{W}_2}{d\overline{X}^4} = \beta_2(\overline{W}_3 - \overline{W}_2) + \left(\frac{\beta_1}{\delta_1}\right)(\overline{W}_1 - \overline{W}_2) - \left(\alpha^E_{twn}\right)\left(\frac{\varepsilon_1}{\delta_1}\right)\frac{d^2\overline{W}_2}{d\overline{X}^2}, \tag{5}$$

$$\frac{d^4\overline{W}_3}{d\overline{X}^4} = \beta_2\left(\frac{\delta_1}{\delta_2}\right)(\overline{W}_2 - \overline{W}_3) - \left(\alpha^E_{twn}\right)\left(\frac{\varepsilon_2}{\delta_2}\right)\frac{d^2\overline{W}_3}{d\overline{X}^2}, \tag{6}$$

where

$$w_1(\overline{X}, t) = \overline{W}_1(\overline{X})e^{i\omega_{twn}t}, \quad w_2(\overline{X}, t) = \overline{W}_2(\overline{X})e^{i\omega_{twn}t}, \tag{7}$$

$$w_3(\overline{X}, t) = \overline{W}_3(\overline{X})e^{i\omega_{twn}t}, \quad \overline{X} = \frac{x}{L}, \tag{8}$$

$$\alpha^E_{twn} = \frac{P^E A_1 L^2}{EI_1(A_1 + A_2 + A_3)}, \quad \beta_1 = \frac{c_{12}L^4}{EI_1}, \quad \beta_2 = \frac{c_{23}L^4}{EI_2}, \tag{9}$$

$$\varepsilon_1 = \frac{A_2}{A_1}, \quad \varepsilon_2 = \frac{A_3}{A_1}, \quad \delta_1 = \frac{I_2}{I_1}, \quad \delta_2 = \frac{I_3}{I_1}. \tag{10}$$

## 3. SIMPLY-SIMPLY SUPPORTED

For the triple-walled carbon nanotube with simply-simply supported boundary conditions at both ends, the non-dimensional form converts as,

$$\overline{W}_1\Big|_{\overline{X}=0} = 0, \quad \frac{d^2\overline{W}_1}{d\overline{X}^2}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_2\Big|_{\overline{X}=0} = 0, \tag{11}$$

$$\frac{d^2\overline{W}_2}{d\overline{X}^2}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_3\Big|_{\overline{X}=0} = 0, \quad \frac{d^2\overline{W}_3}{d\overline{X}^2}\Big|_{\overline{X}=0} = 0, \tag{12}$$

$$\overline{W}_1\Big|_{\overline{X}=1} = 0, \quad \frac{d^2\overline{W}_1}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_2\Big|_{\overline{X}=1} = 0, \tag{13}$$

$$\frac{d^2\overline{W}_2}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_3\Big|_{\overline{X}=1} = 0, \quad \frac{d^2\overline{W}_3}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0. \tag{14}$$

By ignoring the third nanotube's effect as $w_3 = 0$, the present boundary conditions convert to double-walled nanotube's criteria [23].

## 4. CLAMPED-CLAMPED SUPPORTED

The non-dimensional form of clamped-clamped boundary conditions at both ends for triple-walled carbon nanotube is,

$$\overline{W}_1\Big|_{\overline{X}=0} = 0, \quad \frac{d\overline{W}_1}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_2\Big|_{\overline{X}=0} = 0, \tag{15}$$

$$\frac{d\overline{W}_2}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_3\Big|_{\overline{X}=0} = 0, \quad \frac{d\overline{W}_3}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \tag{16}$$

$$\overline{W}_1\Big|_{\overline{X}=1} = 0, \quad \frac{d\overline{W}_1}{d\overline{X}}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_2\Big|_{\overline{X}=1} = 0, \tag{17}$$

$$\frac{d\overline{W}_2}{d\overline{X}}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_3\Big|_{\overline{X}=1} = 0, \quad \frac{d\overline{W}_3}{d\overline{X}}\Big|_{\overline{X}=1} = 0. \tag{18}$$

## 5. CLAMPED-SIMPLY SUPPORTED

The triple-walled carbon nanotube with clamped-simply supported boundary conditions satisfy the buckling behavior's following non-dimensional boundary conditions.

$$\overline{W}_1\Big|_{\overline{X}=0} = 0, \quad \frac{d\overline{W}_1}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_2\Big|_{\overline{X}=0} = 0, \tag{19}$$

$$\frac{d\overline{W}_2}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \quad \overline{W}_3\Big|_{\overline{X}=0} = 0, \quad \frac{d\overline{W}_3}{d\overline{X}}\Big|_{\overline{X}=0} = 0, \tag{20}$$

$$\overline{W}_1\Big|_{\overline{X}=1} = 0, \quad \frac{d^2\overline{W}_1}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_2\Big|_{\overline{X}=1} = 0, \tag{21}$$

$$\frac{d^2\overline{W}_2}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0, \quad \overline{W}_3\Big|_{\overline{X}=1} = 0, \quad \frac{d^2\overline{W}_3}{d\overline{X}^2}\Big|_{\overline{X}=1} = 0. \tag{22}$$

## 6. APPROXIMATE METHODS

Since exact solutions are available for triple-walled carbon nanotube with simply-simply supported boundary conditions as $\overline{W}_1(\overline{X}) = Y_1 \sin(\pi\overline{X})$, $\overline{W}_2(\overline{X}) = Y_2 \sin(\pi\overline{X})$ [22] and $\overline{W}_3(\overline{X}) = Y_3 \sin(\pi\overline{X})$, it results in quick analytical solutions. Based on the suitable polynomial usage, the Bubnov–Galerkin method is classified here as Bubnov-Galerkin-1 (B-G-1) and Bubnov-Galerkin-2 (B-G-2). In the present analysis, two approximate methods, namely Bubnov–Galerkin and Petrov–Galerkin, estimate critical buckling loads. Since it is a triple-walled carbon nanotube, the results yield three critical buckling loads. Three different boundary conditions, namely simply-simply supported, clamped-clamped, and clamped-simply supported, are considered in the present study.

In a generic approach, the Bubnov–Galerkin Method or Petrov–Galerkin Method with weighted residual formulation using polynomial functions for any boundary conditions of triple-walled carbon nanotube takes the form as,

$$\int_0^1 \begin{vmatrix} A_{11}(\overline{X}) & A_{12}(\overline{X}) & A_{13}(\overline{X}) \\ A_{21}(\overline{X}) & A_{22}(\overline{X}) & A_{23}(\overline{X}) \\ A_{31}(\overline{X}) & A_{32}(\overline{X}) & A_{33}(\overline{X}) \end{vmatrix} \psi(\overline{X}) d\overline{X} = 0, \tag{23}$$

$$A_{11}(\overline{X}) = \frac{d^4\overline{W}_1(\overline{X})}{d\overline{X}^4} + \beta_1 \overline{W}_1(\overline{X}) + \left(\alpha_{twn}^E\right) \frac{d^2\overline{W}_1(\overline{X})}{d\overline{X}^2}, \tag{24}$$

$$A_{12}(\overline{X}) = -\beta_1 \overline{W}_2(\overline{X}), \quad A_{33}(\overline{X}) = 0, \tag{25}$$

$$A_{21}(\overline{X}) = -\left(\frac{\beta_1}{\delta_1}\right) \overline{W}_1(\overline{X}), \tag{26}$$

$$A_{22}(\overline{X}) = \frac{d^4\overline{W}_2(\overline{X})}{d\overline{X}^4} + \left(\frac{\beta_1}{\delta_1}\right) \overline{W}_2(\overline{X}) + \left(\alpha_{twn}^E\right) \left(\frac{\varepsilon_1}{\delta_1}\right) \frac{d^2\overline{W}_2(\overline{X})}{d\overline{X}^2}, \tag{27}$$

$$A_{23}(\overline{X}) = -\beta_2 \overline{W}_3(\overline{X}), \quad A_{31}(\overline{X}) = 0, \quad A_{32}(\overline{X}) = -\beta_2 \left(\frac{\delta_1}{\delta_2}\right) \overline{W}_2(\overline{X}), \tag{28}$$

$$A_{33}(\overline{X}) = \frac{d^4\overline{W}_3(\overline{X})}{d\overline{X}^4} + \beta_2 \left(\frac{\delta_1}{\delta_2}\right) \overline{W}_3(\overline{X}) + \left(\alpha_{twn}^E\right) \left(\frac{\varepsilon_2}{\delta_2}\right) \frac{d^2\overline{W}_3(\overline{X})}{d\overline{X}^2}. \tag{29}$$

Eq. (23) further reduces to the following form by utilising Eq. (24) to Eq. (29),

$$A_{twn}\left(\alpha_{twn}^E\right)^3 + B_{twn}\left(\alpha_{twn}^E\right)^2 + C_{twn}\left(\alpha_{twn}^E\right) + D_{twn} = 0. \tag{30}$$

Eq. (30) estimates three different values of $\alpha_{twn}^E$ and using Eq. (9), the Euler critical buckling loads for the innermost carbon nanotube, middle carbon nanotube, and outer nanotube are estimated.

## 7. SIMPLY-SIMPLY SUPPORTED AT BOTH ENDS

The present work uses the following polynomial functions for the first time by satisfying the triple-walled carbon nanotube's boundary conditions as simply-simply supported at both ends, and the Bubnov-Galerkin-1 uses the below polynomials as,

$$\overline{W}_1(\overline{X}) = D_1 \left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}\right), \tag{31}$$

$$\overline{W}_2(\overline{X}) = D_2 \left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}\right), \tag{32}$$

$$\overline{W}_3(\overline{X}) = D_3 \left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}\right), \tag{33}$$

$$\overline{\psi}_{ss}^{(4)}(\overline{X}) = \left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}\right). \tag{34}$$

It is interesting to note that the polynomials $\overline{W}_1(\overline{X}) = D_1(-3\overline{X}^5 + 10\overline{X}^3 - 7\overline{X})$, $\overline{W}_2(\overline{X}) = D_2(-3\overline{X}^5 + 10\overline{X}^3 - 7\overline{X})$, and $\overline{W}_3(\overline{X}) = D_3(-3\overline{X}^5 + 10\overline{X}^3 - 7\overline{X})$ produce the same results as previous polynomials results [Eq. (31) to Eq. (34)] using Bubnov-Galerkin-1. Also, the polynomials $\overline{W}_3 = 0$, $c_{23} = 0$, $\overline{W}_1(-\overline{X})$ and $\overline{W}_2(-\overline{X})$ yields the results of double-walled carbon nanotube proposed by Elishakoff et al [22]. For the Bubnov-Galerkin-2, polynomials take form as $\overline{W}_1(\overline{X}) = D_1(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$, $\overline{W}_2(\overline{X}) = D_2(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$, and $\overline{W}_3(\overline{X}) = D_3(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$. The Petrov–Galerkin (P-G) method evaluates the three critical buckling loads using $\overline{W}_1(\overline{X}) = D_1(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$, $\overline{W}_2(\overline{X}) = D_2(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$, and $\overline{W}_3(\overline{X}) = D_3(-\overline{X}^4 + 2\overline{X}^3 - \overline{X})$ polynomials. The weight function assumes in the form of $\overline{\psi}_{ss}^{(3)}(\overline{X}) = -\overline{X}^6 + 5\overline{X}^3 - 4\overline{X}$. If the effect of the third carbon nanotube attains as $\overline{W}_3 = 0$, $\overline{W}_1(-\overline{X})$, $\overline{W}_2(-\overline{X})$, and $c_{23} = 0$, the results estimate the critical buckling loads of double-walled carbon nanotube [22].

### 7.1. Explicit exact critical buckling load of simply-simply supported at both ends

Using the trigonometric functions $\overline{W}_1(\overline{X}) = Y_1 \sin(\pi\overline{X})$, $\overline{W}_2(\overline{X}) = Y_2 \sin(\pi\overline{X})$ [22] and $\overline{W}_3(\overline{X}) = Y_3 \sin(\pi\overline{X})$, the explicit expression of Exact solution is in form of Eq. (30) by following steps from Eq. (23) to Eq. (29) results,

$$A_{twn}^{Exact} = -\left(\frac{\pi^6\epsilon_1\epsilon_2}{8\delta_1\delta_2}\right), \tag{35}$$

$$B_{twn}^{Exact} = \left(\frac{\pi^8\epsilon_1}{8\delta_1}\right) + \left(\frac{[\pi^4\beta_2(\epsilon_1+\epsilon_2)]}{8\delta_2}\right) + \left(\frac{\pi^8\epsilon_2}{8\delta_2}\right) + \left(\frac{[\pi^4\beta_1\epsilon_2(1+\epsilon_1)]}{8\delta_1\delta_2}\right) + \left(\frac{\pi^8\epsilon_1\epsilon_2}{8\delta_1\delta_2}\right), \tag{36}$$

$$C_{twn}^{Exact} = -\left(\frac{\pi^{10}\epsilon_2}{8\delta_2}\right) - \left(\frac{[\pi^6\beta_1\epsilon_2] + (\pi^6\beta_2[\epsilon_1+\epsilon_2+\delta_1])}{8\delta_2}\right) - \left(\frac{[\pi^2\beta_1\beta_2(1+\epsilon_1+\epsilon_2)]}{8\delta_2}\right) - \frac{\pi^6\beta_1\epsilon_2}{8\delta_1\delta_2} - \left(\frac{\pi^{10}\epsilon_1}{8\delta_1}\right) - \left(\frac{[\pi^6\beta_1(1+\epsilon_1)]}{8\delta_1}\right) - \left(\frac{\pi^{10}}{8}\right) - \left(\frac{\pi^6\beta_2}{8}\right), \tag{37}$$

$$D_{twn}^{Exact} = \left(\frac{\pi^8\beta_1}{8\delta_1}\right) + \left(\frac{\pi^8\beta_2\delta_1}{8\delta_2}\right) + \left(\frac{[\pi^4\beta_1\beta_2(1+\delta_1)]}{8\delta_2}\right) + \left(\frac{\pi^{12}}{8}\right) + \left(\frac{[\pi^8(\beta_1+\beta_2)]}{8}\right) + \left(\frac{\pi^4\beta_1\beta_2}{8}\right). \tag{38}$$

The three different values of $\alpha_{twn}^{E}$ from Eq. (30) compute the innermost, middle, and outer nanotube's exact Euler critical buckling load for the triple-walled carbon nanotube using Eq. (9).

### 7.2. Explicit critical buckling load based on Bubnov-Galerkin-1 method of simply-simply supported at both ends

With the help of Eq. (31) to Eq. (34) for Bubnov-Galerkin-1 method, as mentioned earlier, Euler critical buckling loads for innermost, middle, and outer nanotubes result from solving Eq. (30) using Eq. (9). The relevent coefficients of Eq. (30) is in the form of,

$$A_{twn}^{BG-1} = -\left(\frac{4913\epsilon_1\epsilon_2}{128024064000\delta_1\delta_2}\right), \tag{39}$$

$$\begin{aligned} B_{twn}^{BG-1} = &\left(\frac{289\epsilon_1}{762048000\delta_1}\right) + \left(\frac{[8959\beta_2(\epsilon_1+\epsilon_2)]}{2304433152000\delta_2}\right) + \left(\frac{289\epsilon_2}{762048000\delta_2}\right) \\ &+ \left(\frac{[8959\beta_1\epsilon_2(1+\epsilon_1)]}{2304433152000\delta_1\delta_2}\right) + \left(\frac{289\epsilon_1\epsilon_2}{762048000\delta_1\delta_2}\right), \end{aligned} \tag{40}$$

$$\begin{aligned} C_{twn}^{BG-1} = &-\left(\frac{17\epsilon_2}{4536000\delta_2}\right) - \left(\frac{[527\beta_1\epsilon_2] + (527\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{13716864000\delta_2}\right) \\ &- \left(\frac{[16337\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{41479796736000\delta_2}\right) - \frac{527\beta_1\epsilon_2}{13716864000\delta_1\delta_2} \\ &- \left(\frac{17\epsilon_1}{4536000\delta_1}\right) - \left(\frac{[527\beta_1\,(1+\epsilon_1)]}{13716864000\delta_1}\right) - \left(\frac{17}{4536000}\right) - \left(\frac{527\beta_2}{13716864000}\right), \end{aligned} \tag{41}$$

$$\begin{aligned} D_{twn}^{BG-1} = &\left(\frac{31\beta_1}{81648000\delta_1}\right) + \left(\frac{31\beta_2\delta_1}{81648000\delta_2}\right) + \left(\frac{[961\beta_1\beta_2\,(1+\delta_1)]}{246903552000\delta_2}\right) + \left(\frac{1}{27000}\right) \\ &+ \left(\frac{[31\,(\beta_1+\beta_2)]}{81648000}\right) + \left(\frac{961\beta_1\beta_2}{246903552000}\right). \end{aligned} \tag{42}$$

### 7.3. Explicit critical buckling load based on Bubnov-Galerkin-2 method of simply-simply supported at both ends

The Bubnov-Galerkin-2 method with appropriate polynomials discussed earlier results in the following explicit expression, and it is in the form of Eq. (30),

$$A_{twn}^{BG-2} = -\left(\frac{4913\epsilon_1\epsilon_2}{42875\delta_1\delta_2}\right), \tag{43}$$

$$\begin{aligned} B_{twn}^{BG-2} = &\left(\frac{6936\epsilon_1}{6125\delta_1}\right) + \left(\frac{[8959\beta_2(\epsilon_1+\epsilon_2)]}{771750\delta_2}\right) + \left(\frac{6936\epsilon_2}{6125\delta_2}\right) \\ &+ \left(\frac{[8959\beta_1\epsilon_2(1+\epsilon_1)]}{771750\delta_1\delta_2}\right) + \left(\frac{6936\epsilon_1\epsilon_2}{6125\delta_1\delta_2}\right), \end{aligned} \tag{44}$$

$$\begin{aligned} C_{twn}^{BG-2} = &-\left(\frac{9792\epsilon_2}{875\delta_2}\right) - \left(\frac{[2108\beta_1\epsilon_2] + (2108\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{18375\delta_2}\right) \\ &- \left(\frac{[16337\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{13891500\delta_2}\right) - \frac{2108\beta_1\epsilon_2}{18375\delta_1\delta_2} - \left(\frac{9792\epsilon_1}{875\delta_1}\right) \\ &- \left(\frac{[2108\beta_1\,(1+\epsilon_1)]}{18375\delta_1}\right) - \left(\frac{9792}{875}\right) - \left(\frac{2108\beta_2}{18375}\right), \end{aligned} \tag{45}$$

$$D_{twn}^{BG-2} = \left(\frac{992\beta_1}{875\delta_1}\right) + \left(\frac{992\beta_2\delta_1}{875\delta_2}\right) + \left(\frac{[1922\beta_1\beta_2\,(1+\delta_1)]}{165375\delta_2}\right) + \left(\frac{13824}{125}\right) + \left(\frac{[992\,(\beta_1+\beta_2)]}{875}\right) + \left(\frac{1922\beta_1\beta_2}{165375}\right). \tag{46}$$

Eq. (9) helps in the determination of Euler's critical buckling load of Bubnov-Galerkin-2 by solving Eq. (30) for innermost, middle and outer nanotube using the above Eq. (43) to Eq. (46).

### 7.4. Explicit critical buckling load based on Petrov-Galerkin-1 method of simply-simply supported at both ends

The solution of Eq. (30) derived from the Petrov-Galerkin-1 method by utilizing the relevant polynomials as discussed previously yields the three different Euler critical buckling loads of triple-walled nanotube with the help of Eq. (9). The explicit expression is in the form of,

$$A_{twn}^{PG-1} = -\left(\frac{2197\epsilon_1\epsilon_2}{216\delta_1\delta_2}\right), \tag{47}$$

$$B_{twn}^{PG-1} = \left(\frac{4225\epsilon_1}{42\delta_1}\right) + \left(\frac{[28561\beta_2(\epsilon_1+\epsilon_2)]}{27720\delta_2}\right) + \left(\frac{4225\epsilon_2}{42\delta_2}\right) + \left(\frac{[28561\beta_1\epsilon_2(1+\epsilon_1)]}{27720\delta_1\delta_2}\right) + \left(\frac{4225\epsilon_1\epsilon_2}{42\delta_1\delta_2}\right), \tag{48}$$

$$C_{twn}^{PG-1} = -\left(\frac{48750\epsilon_2}{49\delta_2}\right) - \left(\frac{[10985\beta_1\epsilon_2] + (10985\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{1078\delta_2}\right) - \left(\frac{[371293\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{3557400\delta_2}\right) - \frac{10985\beta_1\epsilon_2}{1078\delta_1\delta_2} - \left(\frac{48750\epsilon_1}{49\delta_1}\right) - \left(\frac{[10985\beta_1\,(1+\epsilon_1)]}{1078\delta_1}\right) - \left(\frac{48750}{49}\right) - \left(\frac{10985\beta_2}{1078}\right), \tag{49}$$

$$D_{twn}^{PG-1} = \left(\frac{380250\beta_1}{3773\delta_1}\right) + \left(\frac{380250\beta_2\delta_1}{3773\delta_2}\right) + \left(\frac{[85683\beta_1\beta_2\,(1+\delta_1)]}{83006\delta_2}\right) + \left(\frac{3375000}{343}\right) + \left(\frac{[380250\,(\beta_1+\beta_2)]}{3773}\right) + \left(\frac{85683\beta_1\beta_2}{83006}\right). \tag{50}$$

## 8. CLAMPED AT BOTH ENDS

Since the exact solution is not available for clamped-clamped boundary conditions, the Bubnov-Galerkin-1, Bubnov-Galerkin-2, and Petrov–Galerkin methods evaluate the critical buckling loads based on different polynomials for triple-walled carbon nanotube. The Bubnov-Galerkin-1 method computes the critical buckling loads of triple-walled carbon nanotube for clamped-clamped boundary conditions using the Filonenko-Borodich trigonometric polynomials $\overline{W}_1(\overline{X}) = Y_1\left[1-\cos\left(2\pi\overline{X}\right)\right]$, $\overline{W}_2(\overline{X}) = Y_2\left[1-\cos\left(2\pi\overline{X}\right)\right]$ [22], and $\overline{W}_3(\overline{X}) = Y_3\left[1-\cos\left(2\pi\overline{X}\right)\right]$. Using the polynomials as

$$\overline{W}_1(\overline{X}) = D_1\left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}^2\right), \tag{51}$$

$$\overline{W}_2(\overline{X}) = D_2\left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}^2\right), \tag{52}$$

$$\overline{W}_3(\overline{X}) = D_3\left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}^2\right), \tag{53}$$

$$\overline{\psi}^{(4)}_{cc}(\overline{X}) = \left(\frac{1}{12}\overline{X}^4 - \frac{1}{6}\overline{X}^3 + \frac{1}{12}\overline{X}^2\right), \tag{54}$$

the Bubnov-Galerkin-2 method estimates the critical buckling loads for the triple-walled carbon nanotube. It is interesting to observe that the polynomials $\overline{W}_1(\overline{X}) = D_1(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$, $\overline{W}_2(\overline{X}) = D_2(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$, and $\overline{W}_3(\overline{X}) = D_3(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$, used in the Bubnov-Galerkin-2 estimates the same results as compared with earlier polynomials [Eq. (51) to Eq. (54)]. In evaluating critical buckling loads for triple-walled carbon nanotubes using the Petrov–Galerkin method, the polynomials assume in the form of $\overline{W}_1(\overline{X}) = D_1(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$, $\overline{W}_2(\overline{X}) = D_2(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$, and $\overline{W}_3(\overline{X}) = D_3(-\overline{X}^4 + 2\overline{X}^3 - \overline{X}^2)$. The weight function takes the form of $\overline{\psi}^{(2)}_{cc}(\overline{X}) = 1 - \cos\left(2\pi\overline{X}\right)$ [22]. In the value of $\overline{W}_3 = 0$, $\overline{W}_1(-\overline{X})$, $\overline{W}_2(-\overline{X})$, and $c_{23} = 0$, the critical buckling load yields the results of a double-walled carbon nanotube [22].

### 8.1. Explicit critical buckling load based on Bubnov-Galerkin-1 method of clamped at both ends

Using the Filonenko–Borodich trigonometric polynomials discussed earlier, the Bubnov-Galerkin-1 method results in the following explicit form by utilizing the steps Eq. (24) to Eq. (30),

$$A^{BG-1}_{twn} = -\left(\frac{8\pi^6\epsilon_1\epsilon_2}{\delta_1\delta_2}\right), \tag{55}$$

$$\begin{aligned} B^{BG-1}_{twn} = {} & \left(\frac{32\pi^8\epsilon_1}{\delta_1}\right) + \left(\frac{\left[6\pi^4\beta_2(\epsilon_1+\epsilon_2)\right]}{\delta_2}\right) + \left(\frac{32\pi^8\epsilon_2}{\delta_2}\right) \\ & + \left(\frac{\left[6\pi^4\beta_1\epsilon_2(1+\epsilon_1)\right]}{\delta_1\delta_2}\right) + \left(\frac{32\pi^8\epsilon_1\epsilon_2}{\delta_1\delta_2}\right), \end{aligned} \tag{56}$$

$$\begin{aligned} C^{BG-1}_{twn} = {} & -\left(\frac{128\pi^{10}\epsilon_2}{\delta_2}\right) - \left(\frac{\left[24\pi^6\beta_1\epsilon_2\right] + \left(24\pi^6\beta_2\left[\epsilon_1+\epsilon_2+\delta_1\right]\right)}{\delta_2}\right) \\ & - \left(\frac{\left[9\pi^2\beta_1\beta_2\left(1+\epsilon_1+\epsilon_2\right)\right]}{\delta_2}\right) - \frac{24\pi^6\beta_1\epsilon_2}{\delta_1\delta_2} - \left(\frac{128\pi^{10}\epsilon_1}{\delta_1}\right) \\ & - \left(\frac{\left[24\pi^6\beta_1\left(1+\epsilon_1\right)\right]}{\delta_1}\right) - \left(\frac{128\pi^{10}}{1}\right) - \left(\frac{24\pi^6\beta_2}{1}\right), \end{aligned} \tag{57}$$

$$D_{twn}^{BG-1} = \left(\frac{96\pi^8\beta_1}{\delta_1}\right) + \left(\frac{96\pi^8\beta_2\delta_1}{\delta_2}\right) + \left(\frac{[18\pi^4\beta_1\beta_2\,(1+\delta_1)]}{\delta_2}\right) + \left(\frac{512\pi^{12}}{1}\right) + \left(\frac{[96\pi^8\,(\beta_1+\beta_2)]}{1}\right) + \left(\frac{18\pi^4\beta_1\beta_2}{1}\right). \tag{58}$$

The three different Euler's critical buckling loads for Bubnov-Galerkin-1 results are estimated by substituting the above expressions in Eq. (30) and Eq. (9).

### 8.2. Explicit critical buckling load based on Bubnov-Galerkin-2 method of clamped at both ends

The explicit expression of Eq. (30) arranges in the following form with the usage of Eq. (51) to Eq. (54) by adhereing the procedures from Eq. (23) to Eq. (29).

$$A_{twn}^{BG-2} = -\left(\frac{432081216000\epsilon_1\epsilon_2}{\delta_1\delta_2}\right), \tag{59}$$

$$B_{twn}^{BG-2} = \left(\frac{\epsilon_1}{10287648000\delta_1}\right) + \left(\frac{[\beta_2(\epsilon_1+\epsilon_2)]}{5184974592000\delta_2}\right) + \left(\frac{\epsilon_2}{10287648000\delta_2}\right) + \left(\frac{[\beta_1\epsilon_2(1+\epsilon_1)]}{5184974592000\delta_1\delta_2}\right) + \left(\frac{\epsilon_1\epsilon_2}{10287648000\delta_1\delta_2}\right), \tag{60}$$

$$C_{twn}^{BG-2} = -\left(\frac{\epsilon_2}{244944000\delta_2}\right) - \left(\frac{[\beta_1\epsilon_2] + (\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{123451776000\delta_2}\right) - \left(\frac{[\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{622196951040000\delta_2}\right) - \frac{\beta_1\epsilon_2}{123451776000\delta_1\delta_2} - \left(\frac{\epsilon_1}{244944000\delta_1}\right) - \left(\frac{[\beta_1\,(1+\epsilon_1)]}{123451776000\delta_1}\right) - \left(\frac{1}{244944000}\right) - \left(\frac{\beta_2}{123451776000}\right), \tag{61}$$

$$D_{twn}^{BG-2} = \left(\frac{\beta_1}{2939328000\delta_1}\right) + \left(\frac{\beta_2\delta_1}{2939328000\delta_2}\right) + \left(\frac{[\beta_1\beta_2\,(1+\delta_1)]}{14814213120000\delta_2}\right) + \left(\frac{1}{5832000}\right) + \left(\frac{[(\beta_1+\beta_2)]}{2939328000}\right) + \left(\frac{\beta_1\beta_2}{1481421312000}\right). \tag{62}$$

From the above coefficients, the evaluation of Euler's critical buckling loads of innermost, middle, and outer nanotube happens with the help of Eq. (30) and Eq. (9).

### 8.3. Explicit critical buckling load based on Petrov-Galerkin-1 method of clamped at both ends

The determination of the innermost, middle, and outer nanotube's Euler critical buckling loads is computed from the Petrov-Galerkin-1 method with the help of the appropriate polynomials discussed previously. The explicit expression for the coefficients is in the form of Eq. (30) as follows,

$$A_{twn}^{PG-1} = +\frac{216\epsilon_1\epsilon_2}{\pi^6\delta_1\delta_2}, \tag{63}$$

$$B_{twn}^{PG-1} = -\left(\frac{864\epsilon_1}{\pi^4\delta_1}\right) + \left(-\frac{54\beta_2\epsilon_1}{\pi^8\delta_2} - \frac{6\beta_2\epsilon_1}{5\pi^4\delta_2} - \frac{54\beta_2\epsilon_2}{\pi^8\delta_2} - \frac{6\beta_2\epsilon_2}{5\pi^4\delta_2}\right) - \left(\frac{864\epsilon_2}{\pi^4\delta_2}\right) + \left(-\frac{54\beta_1\epsilon_2}{\pi^8\delta_1\delta_2} - \frac{6\beta_1\epsilon_2}{5\pi^4\delta_1\delta_2} - \frac{54\beta_1\epsilon_1\epsilon_2}{\pi^8\delta_1\delta_2} - \frac{6\beta_1\epsilon_1\epsilon_2}{5\pi^4\delta_1\delta_2}\right) - \left(\frac{864\epsilon_1\epsilon_2}{\pi^4\delta_1\delta_2}\right), \tag{64}$$

$$C_{twn}^{PG-1} = +\left(\frac{3456\epsilon_2}{\pi^2\delta_2}\right) + \left(\frac{216\beta_1\epsilon_2}{\pi^6\delta_2} + \frac{24\beta_1\epsilon_2}{5\pi^2\delta_2} + \frac{24\beta_2\epsilon_1}{5\pi^2\delta_2} + \frac{216\beta_2\epsilon_1}{\pi^6\delta_2} + \frac{24\beta_2\epsilon_2}{5\pi^2\delta_2}\right) + \left(\frac{216\beta_2\epsilon_2}{\pi^6\delta_2} + \frac{24\beta_2\delta_1}{5\pi^2\delta_2} + \frac{216\beta_2\delta_1}{\pi^6\delta_2}\right) + \left(\frac{27\beta_1\beta_2}{2\pi^{10}\delta_2} + \frac{\beta_1\beta_2}{150\pi^2\delta_2} + \frac{3\beta_1\beta_2}{5\pi^6\delta_2} + \frac{27\beta_1\beta_2\epsilon_1}{2\pi^{10}\delta_2}\right) + \left(\frac{\beta_1\beta_2\epsilon_1}{150\pi^2\delta_2} + \frac{3\beta_1\beta_2\epsilon_1}{5\pi^6\delta_2} + \frac{27\beta_1\beta_2\epsilon_2}{2\pi^{10}\delta_2} + \frac{\beta_1\beta_2\epsilon_2}{150\pi^2\delta_2} + \frac{3\beta_1\beta_2\epsilon_2}{5\pi^6\delta_2}\right) + \left(\frac{216\beta_1\epsilon_2}{\pi^6\delta_1\delta_2} + \frac{24\beta_1\epsilon_2}{5\pi^2\delta_1\delta_2}\right) + \left(\frac{3456\epsilon_1}{\pi^2\delta_1}\right) + \left(\frac{216\beta_1}{\pi^6\delta_1} + \frac{24\beta_1}{5\pi^2\delta_1} + \frac{216\beta_1\epsilon_1}{\pi^6\delta_1} + \frac{24\beta_1\epsilon_1}{5\pi^2\delta_1}\right) + \left(\frac{3456}{\pi^2}\right) + \left(\frac{24\beta_2}{5\pi^2} + \frac{216\beta_2}{\pi^6}\right), \tag{65}$$

$$D_{twn}^{PG-1} = \left(-\frac{96\beta_1}{5\delta_1} - \frac{864\beta_1}{\pi^4\delta_1}\right) + \left(-\frac{96\beta_2\delta_1}{5\delta_2} - \frac{864\beta_2\delta_1}{\pi^4\delta_2}\right) + \left(-\frac{2\beta_1\beta_2}{75\delta_2} - \frac{12\beta_1\beta_2}{5\pi^4\delta_2}\right) + \left(-\frac{54\beta_1\beta_2}{\pi^8\delta_2} - \frac{2\beta_1\beta_2\delta_1}{75\delta_2} - \frac{12\beta_1\beta_2\delta_1}{5\pi^4\delta_2} - \frac{54\beta_1\beta_2\delta_1}{\pi^8\delta_2}\right) + (-13824) + \left(-\frac{96\beta_1}{5} - \frac{864\beta_1}{\pi^4}\right) + \left(-\frac{96\beta_2}{5} - \frac{864\beta_2}{\pi^4}\right) + \left(-\frac{2\beta_1\beta_2}{75} - \frac{12\beta_1\beta_2}{5\pi^4} - \frac{54\beta_1\beta_2}{\pi^8}\right). \tag{66}$$

Further, the three different Euler's critical loads for innermost, middle, and outer nanotube are computed using the above expression using Eq. (30) and Eq. (9).

## 9. CLAMPED-SIMPLY SUPPORTED

The Bubnov-Galerkin-1 method uses the polynomials as

$$\overline{W}_1(\overline{X}) = D_1\left(\frac{1}{12}\overline{X}^4 - \frac{5}{24}\overline{X}^3 + \frac{1}{8}\overline{X}^2\right), \tag{67}$$

$$\overline{W}_2(\overline{X}) = D_2\left(\frac{1}{12}\overline{X}^4 - \frac{5}{24}\overline{X}^3 + \frac{1}{8}\overline{X}^2\right), \tag{68}$$

$$\overline{W}_3(\overline{X}) = D_3\left(\frac{1}{12}\overline{X}^4 - \frac{5}{24}\overline{X}^3 + \frac{1}{8}\overline{X}^2\right), \tag{69}$$

$$\overline{\psi}_{cs}^{(4)}(\overline{X}) = \left(\frac{1}{12}\overline{X}^4 - \frac{5}{24}\overline{X}^3 + \frac{1}{8}\overline{X}^2\right), \tag{70}$$

for the clamped-simply supported boundary conditions to evaluate the critical buckling loads of triple-walled carbon nanotube. Incidentally, the following polynomials

$\overline{W}_1(\overline{X}) = D_1(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$, $\overline{W}_2(\overline{X}) = D_2(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$, and $\overline{W}_3(\overline{X}) = D_3(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$ satisfy the triple-walled carbon nanotube with clamped-simply supported boundary condition computes the same critical buckling load results estimated by Eq. (67) to Eq. (70). With the help of three polynomials as $\overline{W}_1(\overline{X}) = D_1(5X^5 - 16X^4 + 14X^3 - 3X)$, $\overline{W}_2(\overline{X}) = D_2(5X^5 - 16X^4 + 14X^3 - 3X)$, and $\overline{W}_3(\overline{X}) = D_3(5X^5 - 16X^4 + 14X^3 - 3X)$, the Bubnov-Galkerin-2 method helps to estimate the three critical buckling loads. The Petrov–Galerkin method computes the critical buckling loads of triple-walled carbon nanotube utilizing the three polynomials as $\overline{W}_1(\overline{X}) = D_1(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$, $\overline{W}_2(\overline{X}) = D_2(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$, and $\overline{W}_3(\overline{X}) = D_3(-2\overline{X}^4 + 3\overline{X}^3 - \overline{X})$. Also, the weight function considers in the form of $\psi_{cs}^{(3)}(\overline{X}) = (5X^5 - 16X^4 + 14X^3 - 3X)$. If the polynomials reach the value of $\overline{W}_3 = 0$, $\overline{W}_1(-\overline{X})$, $\overline{W}_2(-\overline{X})$, and $c_{23} = 0$ yield the results of double-walled carbon nanotube's critical buckling load results [22].

### 9.1. Explicit critical buckling load based on Bubnov-Galerkin-1 method of clamped-simply supported

By following the steps from Eq. (23) to Eq. (29) with the substitution of Eq. (67) to Eq. (70) appropriately, the explicit expression takes the form of (30) and the coefficients are,

$$A_{twn}^{BG-1} = -\left(\frac{\epsilon_1\epsilon_2}{4741632000\delta_1\delta_2}\right), \tag{71}$$

$$\begin{aligned} B_{twn}^{BG-1} = &\left(\frac{\epsilon_1}{225792000\delta_1}\right) + \left(\frac{[19\beta_2(\epsilon_1+\epsilon_2)]}{10241925120000\delta_2}\right) + \left(\frac{\epsilon_2}{225792000\delta_2}\right) \\ &+ \left(\frac{[19\beta_1\epsilon_2(1+\epsilon_1)]}{10241925120000\delta_1\delta_2}\right) + \left(\frac{\epsilon_1\epsilon_2}{225792000\delta_1\delta_2}\right), \end{aligned} \tag{72}$$

$$\begin{aligned} C_{twn}^{BG-1} = &-\left(\frac{\epsilon_2}{10752000\delta_2}\right) - \left(\frac{[19\beta_1\epsilon_2] + (19\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{48771072000\delta_2}\right) \\ &- \left(\frac{[361\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{221225582592000\delta_2}\right) - \frac{19\beta_1\epsilon_2}{48771072000\delta_1\delta_2} - \left(\frac{\epsilon_1}{10752000\delta_1}\right) \\ &- \left(\frac{[19\beta_1\,(1+\epsilon_1)]}{48771072000\delta_1}\right) - \left(\frac{1}{10752000}\right) - \left(\frac{19\beta_2}{48771072000}\right), \end{aligned} \tag{73}$$

$$\begin{aligned} D_{twn}^{BG-1} = &\left(\frac{19\beta_1}{2322432000\delta_1}\right) + \left(\frac{\beta_2\delta_1}{2322432000\delta_2}\right) + \left(\frac{[361\beta_1\beta_2\,(1+\delta_1)]}{10534551552000\delta_2}\right) \\ &+ \left(\frac{1}{512000}\right) + \left(\frac{[19\,(\beta_1+\beta_2)]}{2322432000}\right) + \left(\frac{\beta_1\beta_2}{10534551552000}\right). \end{aligned} \tag{74}$$

The above expression is evaluated for the three different Euler's critical buckling for innermost, middle, and outer nanotubes, respectively, using Eq. (30) and Eq. (9).

### 9.2. Explicit critical buckling load based on Bubnov Galerkin-2 method of clamped-simply supported

For the explicit expression of the Bubnov-Galerkin-2 method with relevant polynomials discussed earlier, the coefficients come in the form of the following methodology adopted from Eq. (23) to Eq. (29),

$$A_{twn}^{BG-2} = -\left(\frac{425259008\epsilon_1\epsilon_2}{31255875\delta_1\delta_2}\right), \tag{75}$$

$$\begin{aligned} B_{twn}^{BG-2} = &\left(\frac{36192256\epsilon_1}{128625\delta_1}\right) + \left(\frac{[4524032\beta_2(\epsilon_1+\epsilon_2)]}{4244625\delta_2}\right) + \left(\frac{36192256\epsilon_2}{128625\delta_2}\right) \\ &+ \left(\frac{[4524032\beta_1\epsilon_2(1+\epsilon_1)]}{4244625\delta_1\delta_2}\right) + \left(\frac{36192256\epsilon_1\epsilon_2}{128625\delta_1\delta_2}\right), \end{aligned} \tag{76}$$

$$\begin{aligned} C_{twn}^{BG-2} = &-\left(\frac{249495552\epsilon_2}{42875\delta_2}\right) - \left(\frac{[10395648\beta_1\epsilon_2] + (10395648\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{471625\delta_2}\right) \\ &- \left(\frac{[433152\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{5187875\delta_2}\right) - \frac{10395648\beta_1\epsilon_2}{471625\delta_1\delta_2} - \left(\frac{249495552\epsilon_1}{42875\delta_1}\right) \\ &- \left(\frac{[10395648\beta_1\,(1+\epsilon_1)]}{471625\delta_1}\right) - \left(\frac{249495552}{42875}\right) - \left(\frac{10395648\beta_2}{471625}\right), \end{aligned} \tag{77}$$

$$\begin{aligned} D_{twn}^{BG-2} = &\left(\frac{214990848\beta_1}{471625\delta_1}\right) + \left(\frac{214990848\beta_2\delta_1}{471625\delta_2}\right) + \left(\frac{[8957952\beta_1\beta_2\,(1+\delta_1)]}{5187875\delta_2}\right) \\ &+ \left(\frac{5159780352}{42875}\right) + \left(\frac{[214990848\,(\beta_1+\beta_2)]}{471625}\right) + \left(\frac{8957952\beta_1\beta_2}{5187875}\right). \end{aligned} \tag{78}$$

The Euler's critical buckling loads for innermost, middle, and outer nanotube for Bubnov-Galerkin-2 methods are computed using the above expression with the help of Eq. (30) and Eq. (9).

### 9.3. Explicit critical buckling load based on Petrov-Galerkin-1 method of clamped-simply supported

The Petrov-Galerkin-1 Method with appropriate polynomials discussed previously by computing the Eq. (23) to Eq. (30), the explicit expression is in the following form,

$$A_{twn}^{PG-1} = -\left(\frac{29791\epsilon_1\epsilon_2}{42875\delta_1\delta_2}\right), \tag{79}$$

$$\begin{aligned} B_{twn}^{PG-1} = &\left(\frac{84568\epsilon_1}{6125\delta_1}\right) + \left(\frac{[45167\beta_2(\epsilon_1+\epsilon_2)]}{771750\delta_2}\right) + \left(\frac{84568\epsilon_2}{6125\delta_2}\right) \\ &+ \left(\frac{[45167\beta_1\epsilon_2(1+\epsilon_1)]}{771750\delta_1\delta_2}\right) + \left(\frac{84568\epsilon_1\epsilon_2}{6125\delta_1\delta_2}\right), \end{aligned} \tag{80}$$

$$C_{twn}^{PG-1} = -\left(\frac{240064\epsilon_2}{875\delta_2}\right) - \left(\frac{[64108\beta_1\epsilon_2] + (64108\beta_2\,[\epsilon_1+\epsilon_2+\delta_1])}{55125\delta_2}\right) - \left(\frac{[68479\beta_1\beta_2\,(1+\epsilon_1+\epsilon_2)]}{13891500\delta_2}\right) - \frac{64108\beta_1\epsilon_2}{55125\delta_1\delta_2} - \left(\frac{240064\epsilon_1}{875\delta_1}\right) - \left(\frac{[64108\beta_1\,(1+\epsilon_1)]}{55125\delta_1}\right) - \left(\frac{240064}{875}\right) - \left(\frac{64108\beta_2}{55125}\right), \tag{81}$$

$$D_{twn}^{PG-1} = \left(\frac{181984\beta_1}{7875\delta_1}\right) + \left(\frac{181984\beta_2\delta_1}{7875\delta_2}\right) + \left(\frac{[48598\beta_1\beta_2\,(1+\delta_1)]}{496125\delta_2}\right) + \left(\frac{681472}{125}\right) + \left(\frac{[181984\,(\beta_1+\beta_2)]}{7875}\right) + \left(\frac{48598\beta_1\beta_2}{496125}\right). \tag{82}$$

So the coefficients of the above expressions are solved to estimate the Euler's critical buckling loads of nanotube for innermost, middle, and outer nanotube with the help of Eq. (30) and Eq. (9).

## 10. DISCUSSIONS

The classification of nanotubes takes place like short nanotubes $(L/D < 10)$ [24] or sufficiently long nanotubes $(10 < L/D < 50)$ [24] or more. The outer diameter of a multi-walled carbon nanotube ranges from 2 nm to 100 nm [25]. Also, the length of multi-walled carbon nanotube varies from 0.2 μm to several μm [25] with experimental observation. It corresponds to the $L/D$ ranges from 100, 250, 1000, or higher for the multi-walled carbon nanotubes. So the present continuum model analysis considers the triple-walled carbon nanotube as a straight nanotube structure, although its length-outer diameter ratio can be high. But, the nanotube is in a curve shape when it is long (like hair). Based on the chiral indices $(n,m)$ [26], the carbon nanotube is in the form of the armchair $(n,n)$, or zigzag $(n,0)$, or chiral nanotube $(n\neq m)$. For the single-walled carbon nanotube, the diameter in terms of the chiral vector is defined by $D_{swn} = \sqrt{3}a_{C-C}\dfrac{\sqrt{n^2+nm+m^2}}{\pi}$ [27]. For the triple-walled carbon nanotube, the chiral indices take from as $(n,m)(n,m)(n,m)$. In particular, the chirality of triple-walled nanotube $(15,16),(14,17)$, and $(13,18)$ yields the diameter $D_{twn} = 14$ Å. Similarly, the diameter of $D_{twn} = 21$ Å with chirality $(15,16),(14,17)$ and $(13,18)$ is arrived. So the different diameter of the nanotube estimates many chiral possibilities of triple-walled carbon nanotubes. But the continuum model used in the present buckling analysis doesn't consider the nanotube's chirality effect.

The van der Waals interaction coefficient [28] of the nanotube is defined by, $c_{(N)(N+1)} = \dfrac{320\times(2R_N)\ \text{erg/cm}^2}{0.16a_{C-C}^2}$, $a_{C-C} = 0.142$ nm, $N = 1,2$. The first and second adjacent concentric nanotubes using the distributed springs all along the circumference [28] are modeled by van der Waals interaction coefficient $c_{12}$. Similarly, the second and third adjacent nanotube [28] experiences the effect of van der Waals interaction coefficient $c_{23}$. The

present analysis calculates the moment of inertia and cross-sectional areas using the wall thickness of the nanotube suggested by Sears and Batra [29]. The moment of inertia [29] of the three nanotubes such as $I_1$, $I_3$, and $I_3$ is estimated using the expression in terms of the wall thickness of the nanotube $t_h$ as, $I_i(t_h) = \frac{\pi}{4}\left[\left(R_i + \frac{t_h}{2}\right)^4 - \left(R_i - \frac{t_h}{2}\right)^4\right], i = 1,2,3$. Similarly, the cross-sectional areas [29] of nanotubes $A_1$, $A_2$, and $A_3$ are determined by $A_i(t_h) = \pi\left[\left(R_i + \frac{t_h}{2}\right)^2 - \left(R_i - \frac{t_h}{2}\right)^2\right], i = 1,2,3$. The mechanical properties of the triple-walled nanotube are $E = 1.0$ TPa, $R_1 = 0.35$ nm, $R_2 = 0.70$ nm, $R_3 = 1.05$ nm, $D_o = 2.10$ nm, $t_h = 0.34$ nm, $\varepsilon_2/\varepsilon_1 = 1.5$, $\delta_2/\delta_1 = 3.27057$, $\beta_2/\beta_1 = 0.291771$, $c_{12} = 69.43$ GPa and $c_{23} = 138.86$ GPa [16]. In the present investigation, the Bubnov–Galerkin and Petrov–Galerkin methods estimate the Euler's critical buckling of triple-walled carbon nanotube. For the triple-walled carbon nanotubes with simply-simply supported boundary conditions, the accurate estimation of the Euler critical buckling is possible using exact solutions. The Bubnov–Galerkin Method is named Bubnov-Galerkin-1 and Bubnov-Galerkin-2 based on different polynomials in the present analysis. In addition to these methods, the Petrov–Galerkin method determines the critical buckling load. Developing the analytical solutions for triple-walled nanotube supported with the clamped-clamped and clamped-hinged is cumbersome. So to obtain a quick and reasonably reliable estimation, the Bubnov–Galerkin and Petrov–Galerkin method determine the critical buckling load. All the polynomials [22] used in these methods satisfy the respective boundary conditions.

The Bubnov–Galerkin and Petrov–Galerkin method yield three Euler critical buckling loads corresponding to respective boundary conditions. Since three polynomials satisfy the mathematical model, the numerical methods estimate three critical loads of buckling alone. Here comes the interesting question of which numerical method estimates better results than analytical solutions for simply-simply supported triple-walled carbon nanotube. Table 1 to Table 3 list three critical buckling loads of triple-walled carbon nanotube for simply-simply supported boundary conditions. Among the three results like Bubnov-Galerkin-1, Bubnov-Galerkin-2, and Petrov-Galerkin-1, the Bubnov-Galerkin-2 is comparable with exact solution results. So for the triple-walled carbon nanotube with simply-simply supported boundary conditions, the Bubnov-Galerkin-2 predicts closure results reliably. From Table 4 to Table 6, it is evident that the exact solution is not listed for the clamped-clamped support of triple-walled nanotube. It is because finding the exact solutions for triple-walled nanotube with clamped-clamped support is a tough task. These three methods like Bubnov-Galerkin-1, Bubnov-Galerkin-2, and Petrov-Galerkin-1, predict reliable critical buckling loads for clamped-clamped support. Similarly, for the boundary of clamped-simply supported, Table 7 to Table 9 show the critical buckling loads. Again, Bubnov-Galerkin-1, Bubnov-Galerkin-2, and Petrov-Galerkin-1 methods estimate the critical buckling loads for triple-walled carbon nanotube with clamped-simply support in a reliable manner.

*Table 1.* First Euler critical buckling loan $P_1^E$ in nN, simply-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 38.832065 | 26.974931 | 19.820904 | 15.176341 | 11.991587 | 9.713373 |
| B-G-2 | 38.375548 | 26.657690 | 19.587761 | 14.997816 | 11.850520 | 9.599103 |
| P-G | 38.405651 | 26.678608 | 19.603134 | 15.009587 | 11.859821 | 9.606637 |
| Exact | 38.326074 | 26.623311 | 19.562497 | 14.978470 | 11.835233 | 9.586720 |

*Table 2.* Second Euler critical buckling loan $P_2^E$ in nN, simply-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 16194.0547 | 23300.8321 | 31704.1697 | 41402.6500 | 52395.5968 | 64682.6542 |
| B-G-2 | 16241.3171 | 23369.1092 | 31797.2302 | 41524.2793 | 52549.5883 | 64872.8057 |
| P-G | 16239.9778 | 23367.1660 | 31794.5769 | 41520.8084 | 52545.1919 | 64867.3755 |
| Exact | 16243.5254 | 23372.3128 | 31801.6045 | 41530.0015 | 52556.8364 | 64881.7581 |

*Table 3.* Third Euler critical buckling loan $P_3^E$ in nN, simply-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 42621.7820 | 61354.5922 | 83498.2987 | 109051.3209 | 138012.9044 | 170382.6518 |
| B-G-2 | 42746.7303 | 61534.7623 | 83743.6729 | 109371.9001 | 138418.6983 | 170883.6749 |
| P-G | 42743.1726 | 61529.6231 | 83736.6684 | 109362.7454 | 138407.1079 | 170869.3630 |
| Exact | 42752.5958 | 61543.2351 | 83755.2208 | 109386.9928 | 138437.8066 | 170907.2701 |

*Table 4.* First Euler critical buckling loan $P_1^E$ in nN, clamped-clamped supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 152.918292 | 106.364244 | 78.198875 | 59.890951 | 47.329625 | 38.340873 |
| B-G-2 | 162.700689 | 113.163038 | 83.195627 | 63.717229 | 50.353125 | 40.790033 |
| P-G | 152.943073 | 106.372515 | 78.202150 | 59.892420 | 47.330349 | 38.341258 |

*Table 5.* Second Euler critical buckling loan $P_2^E$ in nN, clamped-clamped supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 12262.9236 | 17584.8768 | 23892.0496 | 31178.7615 | 39442.3216 | 48681.3185 |
| B-G-2 | 13444.5567 | 19281.7298 | 26198.9167 | 34190.0774 | 43252.3500 | 53384.2332 |
| P-G | 12930.5037 | 18546.2099 | 25200.5360 | 32887.8068 | 41605.3328 | 51351.7031 |

*Table 6*. Third Euler critical buckling loan $P_3^E$ in nN, clamped-clamped supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 32153.9447 | 46219.5201 | 62862.0174 | 82075.1518 | 103855.9331 | 128202.7896 |
| B-G-2 | 35257.1342 | 50682.8712 | 68934.0773 | 90004.0682 | 113889.6632 | 140589.1903 |
| P-G | 33912.5702 | 48751.9504 | 66308.9392 | 86577.2548 | 109553.9076 | 135237.3262 |

*Table 7*. First Euler critical buckling loan $P_1^E$ in nN, clamped-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 81.479486 | 56.624659 | 41.614903 | 31.866280 | 25.180343 | 20.397025 |
| B-G-2 | 80.229103 | 55.760059 | 40.980855 | 31.381271 | 24.797308 | 20.086851 |
| P-G | 77.100166 | 53.580757 | 39.377714 | 30.153116 | 23.826599 | 19.300431 |

*Table 8*. Second Euler critical buckling loan $P_2^E$ in nN, clamped-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 14133.0340 | 20312.4462 | 27624.6933 | 36066.7837 | 45637.2936 | 56335.4746 |
| B-G-2 | 12591.3063 | 18092.9441 | 24604.0484 | 32121.6703 | 40644.4070 | 50171.5212 |
| P-G | 13532.5655 | 19449.8759 | 26451.8655 | 34535.7039 | 43700.0438 | 53944.1772 |

*Table 9*. Third Euler critical buckling loan $P_3^E$ in nN, clamped-simply supported TWCNT

| $L/D_o$ | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|
| B-G-1 | 37150.7365 | 53453.4092 | 72730.5728 | 94978.9001 | 120196.8047 | 148383.4518 |
| B-G-2 | 33090.5900 | 47607.4570 | 64773.9678 | 84586.8444 | 107044.5242 | 132146.1848 |
| P-G | 35573.2171 | 51184.1287 | 69643.1987 | 90947.2790 | 115094.8684 | 142085.1771 |

Fig. 1 shows the effect of $L/D_o$ ratio over the first Euler buckling load $P_1^E$ for three different boundary conditions of the triple-walled carbon nanotube. For the increase in the $L/D_o$ ratio, the buckling load $P_1^E$ decreases. It implies that for a shorter length of the nanotubes, the first critical load $P_1^E$ is predominant compared to triple-walled nanotubes with longer lengths. This phenomenon happens in all three boundary conditions, namely clamped-clamped, clamped-simply and simply-simply supported. From Fig. 2 the second critical buckling load $P_2$ increases along with $L/D_o$ ratio increment. It happens because the Euler's critical load depends on the length of the triple-walled nanotube. But the critical buckling load $P_2^E$ behaves entirely different from $P_1^E$ for three different boundary conditions. The same behaviour (Fig. 3) occurs for the third Euler critical buckling load $P_3^E$ of triple-walled carbon nanotube for simply-simply, clamped-simply, and clamped-clamped support. Fig. 4 to Fig. 6 reveal the behavior of Euler's

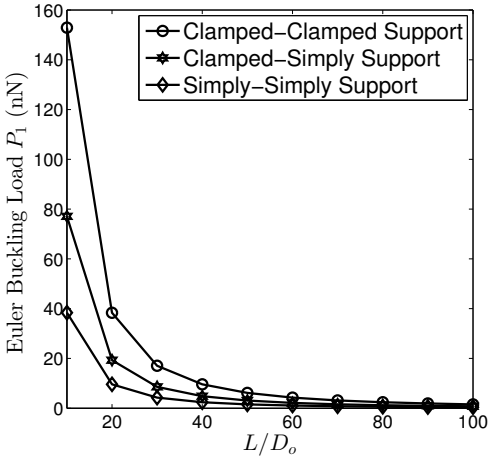


*Fig. 1.* Euler's first critical buckling loads for different boundary conditions vary with an aspect ratio

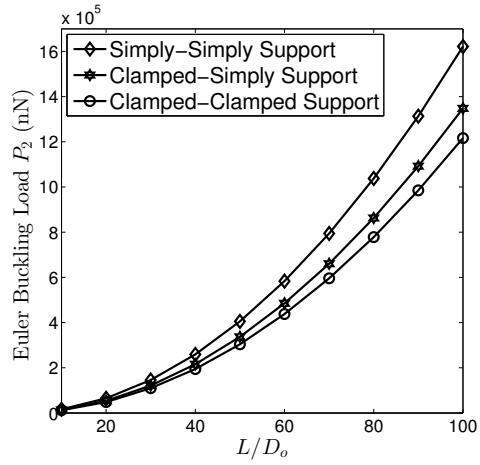


*Fig. 2.* Euler's second critical buckling loads for different boundary conditions vary with an aspect ratio

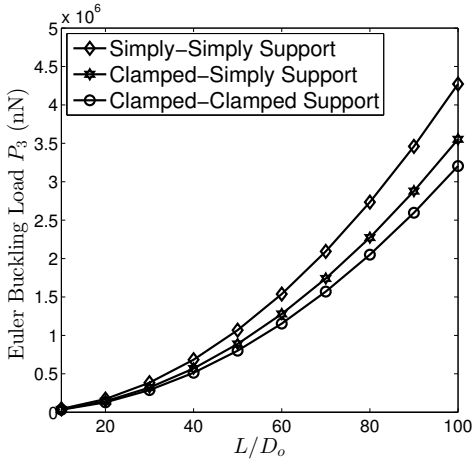


*Fig. 3.* Euler's third critical buckling loads for different boundary conditions vary with an aspect ratio

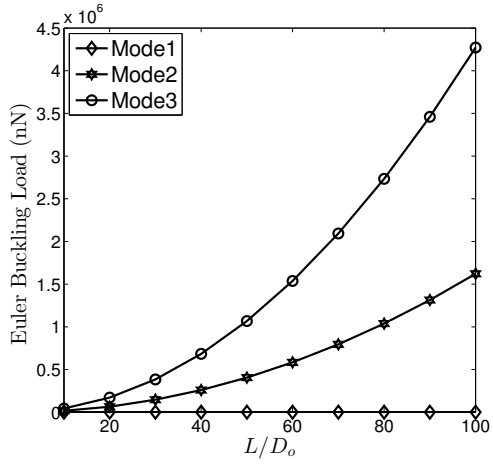


*Fig. 4.* First three modes of Euler's critical buckling loads for simply-simply boundary conditions vary with an aspect ratio

critical buckling load over the increase in $L/D_o$ ratio of the first, second, and third mode for a triple-walled carbon nanotube. The first mode interprets the fundamental buckling mode, which is the smallest buckling load among the three buckling loads. Interestingly, the fundamental Euler critical buckling mode decreases with the increase in the $L/D_o$ ratio. This phenomenon happens in all three boundary conditions: Simply-simply, clamped-clamped, and clamped-simply support. Further, the higher modes like second and third buckling modes increase with the increase in the $L/D_o$ ratio for all the

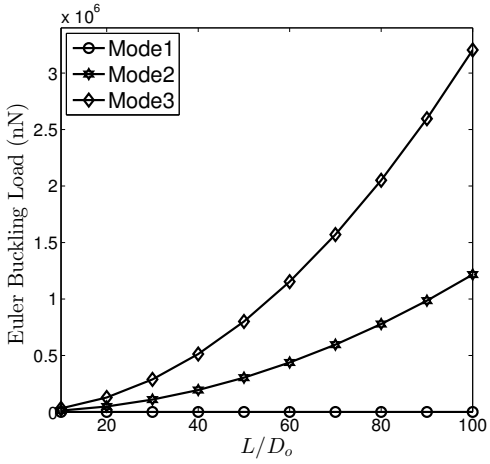


*Fig. 5.* First three modes of Euler's critical buckling loads for clamped-clamped boundary conditions vary with an aspect ratio

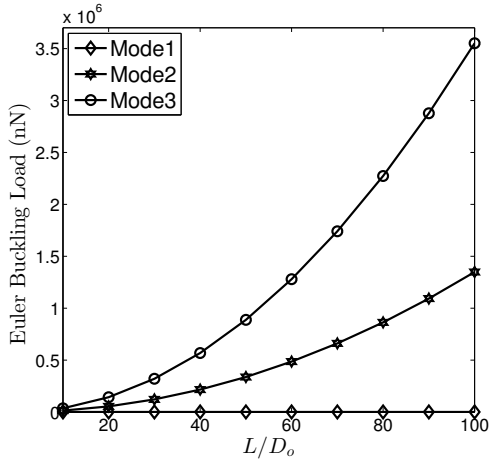


*Fig. 6.* First three modes of Euler's critical buckling loads for clamped-simply boundary conditions vary with an aspect ratio

boundary conditions like Simply-simply, Clamped-clamped, and Clamped-simply support. The higher modes behave entirely differently from the fundamental buckling mode for all three boundary conditions.

## 11. CONCLUSIONS

The present investigation estimates the Euler's critical buckling load using the mathematical model of a triple-walled carbon nanotube with a buckling effect. The Bubnov–Galerkin and Petrov–Galerkin methods determine the critical load of triple-walled nanotube quickly in a reliable manner for three different boundary conditions; namely, simply-simply, clamped-clamped, and clamped-simply supported. Two different numerical methods, namely Bubnov–Galerkin and Petrov–Galerkin method, determine the critical buckling load of triple-walled nanotube quickly in a reliable manner. For the increment of length from shorter to longer nanotube, the buckling load $P_1^E$ decreases with the increment in length for three different boundary conditions, respectively. But the critical buckling loads of triple-walled nanotube called $P_2^E$ and $P_3^E$ increase for longer nanotubes in comparison with short nanotubes. So the first critical buckling load $P_1^E$ behaves entirely different from the second and third critical buckling load, namely $P_2^E$ and $P_3^E$. The two approximate methods, namely, Bubnov–Galerkin and Petrov–Galerkin method, are useful in obtaining quick and reliable results for triple-walled carbon nanotubes.

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