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ADAPTIVE SINGULAR EVOLUTIVE INTERPOLATED KALMAN FILTER AND ITS APPLICATION TO DATA ASSIMILATION IN 2D WATER POLLUTION MODEL

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Abstract. This study promotes a new algorithm for estimating the water pollution propagation with the primary goal of providing more reliable and high quality estimates to decision makers. To date, the widely used variational method suffers from the large computational burden which limits its application in practice. Moreover, this method, considering the initial state as a control variable, is very sensitive in specifying initial error, especially for unstable dynamical systems. The adaptive filter (AF), proposed in this paper, is aimed at overcoming these two drawbacks in the variational method: by its nature, the AF is sequential (no large batch assimilation window used) and stable even for unstable dynamics, with the gain parameters as control variables. The AF, developed in this paper, is an adaptive version of the Singular Evolutive Interpolated Kalman Filter (SEIKF). One of the new versions of this AF is that it uses a time-varying structure of the gain of SEIKF. To deal with the uncertainty of the system parameters and of the noise covariance, the proposed adaptive SEIKF (ASEIKF) makes use of the covariance of reduced rank iterated during assimilation process and of some pertinent gain parameters tuned adaptively to yield the minimum prediction error for the system output. The computational burden in implementation of the ASEIKF is reduced drastically due to applying the optimization tool known as a simultaneous perturbation stochastic approximation algorithm, which requires only two integrations of the numerical model. No iterative loop is required at each assimilation instant as usually happens with the standard gradient descent optimization algorithms. Data assimilation experiment, carried out by the SEIKF and ASEIKF, is implemented for the Thanh Nhan Lake in Hanoi and the performance comparison between the ASEIKF and SEIKF is given to show the high effectiveness of the proposed ASEIKF.

Keywords: adaptive filtering, Singular Evolutive Interpolated Kalman Filter, data assimilation, 2D water pollution model.

1. INTRODUCTION

Kalman filter (KF) [1] is a very useful and efficient algorithm for computing the optimal state estimate of the dynamic system (DS) that uses measurements observed over time. The KF is optimal only if the filtering problem at hand is linear and all the system parameters and statistics of the entering random variables (initial condition, model and observation errors, ...) are known exactly. Inaccuracies of the system parameters and of the noise statistics can lead to a poor performance or even to divergence of the KF.

For data assimilation (DA) problems in geophysical systems (GEO-PhyS), described by systems of partial differential equations, there are two great difficulties, one (D1) concerns a high dimension (HD) of the discretized numerical models and the other (D2) - uncertainties of the system parameters and noise statistics.

There are many approaches to overcome D1 which all are based on a model reduction technique. Due to very high dimensions of the error covariance matrices (ECM) in the KF, in the Ensemble KF (EnKF) [2], for example, the ECM is approximated and represented by an ensemble of model states of the size n_e. The KF formalism then is applied to construct the EnKF. As the integration of the numerical model is very expensive in high dimensional GEO-PhySs, in practice the ensemble size n_e is limited to be of the order $O(100)$ which is much less than the dimension of the original system dynamics (typically of order $10^7 - 10^8$). It means that a possible maximal rank of the ECM is only equal to n_e. This can lead to missing a lot of the important directions of the system state and to instability of the EnKF. However, due to the simplicity of this concept and relative ease of its implementation (no derivation of a tangent linear operator (TL) or adjoint equations (AE) and no integration backward in time), the EnKF finds its popularity in the solution of DA problems.

Another approach, known as a Singular Evolutive Extended Kalman filter (SEEK), is proposed in [3]. The idea, behind this filter, is to view the ECM as singular with a low rank $n_e << n$. The correction in the filter is projected onto certain directions forming a linear subspace of dimension n_e (Empirical Orthogonal Functions). The other version known as a SEIKF (Singular Evolutive Interpolated Kalman Filter) [4] replaces the linearization in the SEEK by an interpolation which could reduce the errors for the large deviations. The SEIKF operates in three steps: resampling, forecasting, and assimilation. Unlike the EnKF, where the members of the model ensemble are operated by forecasting and assimilation, in the SEIKF, the members of the model ensemble are selected in the main orthogonal directions of a functional space described by an approximation to the ECM. This enhanced sampling strategy, embedded into the resampling step, and possibly improves the filter stability and delivers rapid convergence. Thus, with the resampling step, the SEIKF is partly trying to overcome the D2, to deal with the uncertainties in the model error statistics. However, no any theoretical proof of stability of the SEIKF has been established.

The adaptive filter (AF), proposed in [5], is designed for overcoming the two difficulties D1 and D2 mentioned above. Here, the correction is found in a linear subspace spanned by the most growing directions of the system dynamics (all unstable and neutrally stable directions). The choice of such directions is crucial since it allows to ensure

a stability of the designed filter. Mention that the number n_e of such directions is usually much less than the dimension of the system state in the GEO-PhySs, i.e. $n_e << n$. In terms of filter' stability, the different parameterized stabilizing structures of the filter gain are given in [6].

It is well known that the noise covariance uncertainties in the KF enter through the associated non-linear matrix Riccati equations. It makes a lot of problems concerning the identifiability of the unknown noise covariance as well as analysis of stability of the KF. As the final objective of the identification of covariance uncertainty is to approach as precisely as possible to the optimal filter gain, in an adaptive filter a direct searching the optimal gain is formulated with the objective to minimize the MPE (mean prediction error) of the system output. Introducing a parameterized stabilizing gain structure, the AF avoids the divergence and instability problems in the filter.

Mention that in the AF the control variables are chosen from some pertinent elements of the filter gain and they are turned during the assimilation process in order to minimize the MPE of the innovation, i.e. of the signal between the predicted output of the filter and the observation. As the filter adapts its gain parameters, the mean square error (MSE) of the system output converges to its minimal value. The filter output is then said to match as closely as possible to the desired signal - the observation.

In this paper the data assimilation for estimation of the water pollution, considered in [7], will be solved on the basis of the SEIKF and its adaptive version, i.e. adaptive SEIKF (ASEIKF).

The main objective of this paper is to demonstrate that by considering the SEIKF as a non-adaptive version one can improve considerably its performance by introducing the adaptive mechanism developed in [8]. Mention that in the SEIKF, the ECM is approximated using the Riccati-like matrix equations, but done in a reduced space. The pertinent parameters in the filter gain are tuned to better balance the uncertainty in the predicted estimate and that of the innovation vector - the difference between the measurement and its predicted estimate. As the SEIKF is approximate due to the introduced reduced-space in the Riccati-like equations, the adaptation mechanism, proposed in this paper, is aimed at compensating the error introduced by reduced-order approximation. In this paper, first some theoretical aspects of the AF as well as the SEIK are outlined. The 2D hydraulic and pollution models are described and will be used to solve the transport problem related to pollution substances. We remark that the linear 2D water pollution problem has been studied by the semi-group methods in [9] and a unique existence of the solution has been also given. The nonlinear 2D-Imech water pollution model has been studied in [7]. This model allows to simulate the transport of pollution substance and thus can be used to estimate the pollution level if the initial values for the model equations are known and the model parameters are adequately specified. Since in practice the initial values are unknown or poorly given, one simple way is to use their estimates is to simulate a long run of the model and compute its average values and covariances. As these values are (not optimal) only the estimates for the initial state and the model is always imperfect with the parameters not adequately specified, the error level becomes larger and larger as time progresses. Data assimilation techniques allow to combine the available measurements with the numerical model to produce more precise estimates for

the system state when the assimilation process advances in time. By applying the AF approach [8], an associated optimization problem will be solved for minimizing the MPE of the system output. As a result one obtains the optimal parameters of the gain. The algorithm is greatly simplified if the SPSA algorithm is used [10]. Mention that the traditional approach to adaptive KF is dealing with direct estimating the unknown model and observational covariance matrices Q, R (see [11]).

2. FORMULATION OF THE 2D WATER POLLUTION PROBLEM

The study on a water pollution in this paper is based on the 2D surface flow model which is useful to understand and predict the flow. In order to predict and simulate the system behavior, a mathematical model with the initial and boundary conditions is established using the Saint-Venant partial differential equations (cf., [12])

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} = -\frac{gu(u^2+v^2)^{1/2}}{K_x^2 h^{4/3}} - g\frac{\partial z_b}{\partial x} \quad \text{in } \Omega, \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} = -\frac{gv(u^2+v^2)^{1/2}}{K_y^2 h^{4/3}} - g\frac{\partial z_b}{\partial y} \quad \text{in } \Omega. \tag{3}$$

In (1)–(3) Ω, divided by n_Ω elements, is a bounded domain of $\mathbb{R}^2$ with the boundary $\partial\Omega$, z_b is the bottom elevation, $h = z - z_b$ is the water depth, and z is the free surface elevation. Further, u is the average velocity in the x direction, v is the average velocity in the y direction, g is the gravity acceleration, K_x and K_y are the Strickler coefficients in the x and y directions, respectively.

The pollutant concentration satisfies the equation (see [13])

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} - \eta\Delta C = KC + S \quad \text{in } \Omega, \tag{4}$$

where $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, $C = C(x,y,t)$ is the pollutant concentration at the time instant t which is an element of the Hilbert space $\mathcal{R}_C = L^2(\Omega)$, with the initial condition $C(0) = V \in \mathcal{R}_C$, K is the conversion coefficient, $S = S(x,y)$ is the pollution function source in fluid, η is the diffusion coefficient.

The initial conditions for $Z = (h,u,v)^T$ and C are

$$Z|_{t=0} = (h(x,y,0), u(x,y,0), v(x,y,0))^T = U, \;\; C(x,y,0) = V,$$

and the boundary conditions:
(i) $\mathbf{U}\cdot\vec{n} = \bar{\mathbf{U}}_{in}(t) \in H^1(\Gamma_1)$, $C(x,y,t) = \bar{C}_{in}(t)$ on the inflow boundary Γ_1;
(ii) $h(x,y,t) = \bar{h}(t) \in H^1(\Gamma_2)$, $\dfrac{\partial C}{\partial \vec{n}} = 0$ on the outflow boundary Γ_2;
(iii) $\mathbf{U}\cdot\vec{n} = 0$, $\dfrac{\partial C}{\partial \vec{n}} = 0$ on the solid wall S_{w}, where $\mathbf{U} = (u(x,y,t), v(x,y,t))$, $\partial\Omega =$

$\Gamma_1 \cup \Gamma_2 \cup S_w$ is the boundary of the domain Ω, $\vec{n} = (n_x, n_y)$ is the unit normal vector to $\partial\Omega$.

Eqs. (1)–(4) with the boundary and initial conditions are rewritten as follows (cf. [14])

$$\begin{cases} \dfrac{\partial Z}{\partial t} + \dfrac{\partial \mathbf{A}(Z)}{\partial x} + \dfrac{\partial \mathbf{B}(Z)}{\partial y} = F(Z) & \text{in } \Omega \\ n_x u + n_y v = \bar{\mathbf{U}}_{in} & \text{on } \Gamma_1 \\ n_x u + n_y v = 0 & \text{on } S_W \\ h = \bar{h} & \text{on } \Gamma_2 \\ Z(0) = U \end{cases} \tag{5}$$

$$\begin{cases} \dfrac{\partial C}{\partial t} + u\dfrac{\partial C}{\partial x} + v\dfrac{\partial C}{\partial y} - \eta \triangle C = KC + S & \text{in } \Omega \\ C = \bar{C}_{in} & \text{on } \Gamma_1 \\ \dfrac{\partial C}{\partial \vec{n}} = 0 & \text{on } \Gamma_2 \bigcup S_W \\ C(0) = V \end{cases} \tag{6}$$

where

$$\mathbf{A}(Z) = \begin{pmatrix} uh \\ \frac{1}{2}u^2 + gh \\ uv \end{pmatrix}, \ \mathbf{B}(Z) = \begin{pmatrix} vh \\ uv \\ \frac{1}{2}v^2 + gh \end{pmatrix}, \ F(Z) = \begin{pmatrix} 0 \\ -gu\dfrac{\sqrt{u^2+v^2}}{K_x^2 h^{4/3}} + u\dfrac{\partial v}{\partial y} - g\dfrac{\partial z_b}{\partial x} \\ -gv\dfrac{\sqrt{u^2+v^2}}{K_y^2 h^{4/3}} + v\dfrac{\partial u}{\partial x} - g\dfrac{\partial z_b}{\partial y} \end{pmatrix}.$$

The problem we are interested in is to retrieve the fields from the observations $Y = (Z_{obs}, C_{obs}) \in \mathcal{R}^p_{Z,C}$.

3. STATE-SPACE NUMERICAL DYNAMIC MODEL

The most advantage of the well-known Kalman filter (KF) is that it is written for the input-output systems in a state-space form. This allows the solution of the KF to be presented directly in a recursive (in time) algorithmic procedure, easy to realize in modern computers. In this paper, we will follow a state-space approach to represent a numerical solution of the pollution model which is in accordance with the formalism of the SEIKF and the AF.

By applying temporal and spacial difference schemes to the continuous equations (5), (6) on obtains a discrete numerical model with the state $X(k)$ at time instant $k := t_k$ composed from all values of the variables (Z, C). The solution of the numerical model can be represented in a discrete state-space form where $(k+1 := t_k + \Delta T$ symbolizes the next assimilation instant and ΔT is the time distance between two moments of arrival of two successive observations). Mention usually $\Delta T = n_a \delta t$ where δt is a sampling model

time step chosen to ensure a precision and stability of the numerical scheme used, n_a is a positive integer number.

$$X(k+1) = \Phi X(k) + w(k), \quad k = 0, 1, \ldots \tag{7}$$

where $X(k) = (Z(k), C(k)) \in \mathcal{R}^n_{Z,C}$. The vector $w(k)$ represents a model error (difference between the solution of the continuous and discrete models), Φ is (may be nonlinear) function modeling the transition of the system state from the time instant k to $k+1$. It is supposed that at each moment $k+1$ we are given the observations

$$Y(k+1) = HX(k+1) + v(k+1), \quad k = 0, 1, \ldots \tag{8}$$

where $H : \mathcal{R}^n_{Z,C} \Rightarrow \mathcal{R}^p_{Z,C}$ is some (possibly nonlinear) function and $v(k) \in \mathcal{R}^p_{Z,C}$ is the measurement noise. Mention that the operator H may be non-linear. In this paper, the vector $Y(t+1) \in \mathcal{R}^p_{Z,C}$ consists of observations available only at certain points in the area of interest, hence H is a known linear operator. It is assumed that the vectors $w(k), v(k)$, are independent random variables of zero mean and covariance matrix $\mathbf{Q}(k)$ and $\mathbf{R}(k)$ respectively. The objective of this paper is to estimate, as precisely as possible, the state $X(k+1)$ of the system based on the ensemble of measurements $Y[0 : k+1] := [Y(\tau), t_0 \leq \tau \leq t_k + 1]$.

4. ADAPTIVE FILTER

Theoretically the KF is the optimal linear filter in cases where (i) the model matches the real system perfectly, (ii) the entering noise is "white" (uncorrelated), and (iii) the covariances of the noise are known exactly. If the second condition is, generally speaking, may be satisfied in practice (for the correlated noise, there exist the methods to it whitening), contrary, the first (i) and the third condition (iii) practically never hold. This leads to a non-optimality of the estimates produced by the KF.

The second disadvantage of the KF concerns its stability. As the transition matrix of the KF depends on the solutions of the matrix non-linear algebraic Riccati equations (AREs), a stability of the KF depends on the solutions of the AREs. However, analysis of the solution of the ARE is very complicated and there is no guarantee that the resulting filter is stable. Moreover, we do not know exactly the system parameters and the noise statistics.

Mention that all the drawbacks of the KF are also those of the SEIKF. Moreover, in the SEIKF, a subspace of the covariance matrix is introduced which makes the problem of analysis of the filter stability more difficult.

The AF in [6] is designed to overcome the above difficulties. A parameterized structure of the AF is chosen a-priori in such a way to ensure a stability of the filter which does not depend on the solution the ARE. Hence the AF in some sense is free from the spec- ification of the solution of the ARE. Its optimality is always achieved by adjusting some pertinent parameters of the filter gain. Mention that the objective function in the AF is quite different from what in the KF: If the KF is optimal in the probabilistic sense (the minimum mean-square-error (MMSE) - MMS of the difference between the state estimate and the true system state), in the AF the minimization is done in the space of realisations (MMS of the difference between the observation and the prediction of the filter output,

i.e. of the innovation). Since the true system state in never known, its produced estimate may be non-optimal if we do not know, for example, the noise covariances since then the objective function is unknown or poorly defined. As for the AF, one replaces a minimization in the probability space by minimization in the space of realizations by assuming the ergodicity: the average behavior of the system can be deduced from the trajectory of a point. It means that a sufficiently large collection of random samples from a process can represent the average statistical properties of the entire process. Using this property, in the AF, the optimization is done on the basis the stochastic optimization (SA) algorithm which uses the samples of the objective function: meanwhile the samples of the difference between observation and prediction of the filter output are always known and calculated at each assimilation instant.

Let $\hat{X}(k) \in \mathcal{R}^n_{Z,C}$ be a posteriori state estimate at the time instant k given $Y[0:k]$. Introduce the following class of filters for solving the filtering problem (7)-(8)

$$\hat{X}(k+1) = \Phi\hat{X}(k) + K(k+1)\zeta(k+1), \tag{9}$$

where Φ is the transition matrix,

$$\zeta(k+1) = Y(k+1) - H\hat{X}(k+1/t), \tag{10}$$

here $\hat{X}(t+1/t) = \Phi\hat{X}(t)$, H is a known observation operator, $\hat{X}(k+1/t)$ is a one step predictor for $X(k+1)$, $K(k+1)$ is an $(n \times p)$ gain matrix.

It is well known that under standard ergodic conditions, relative the joint process $(X(k), Y(k))$, after some transition period the gain $K(k)$ becomes a constant $K(\infty)$. In [6] it is proposed to search an optimal gain K as a solution to the following optimization problem

$$\begin{cases} J[K] \to min_K \\ J[K] = E[\Psi(\zeta(k)] \\ \Psi(\zeta(k) =< \zeta(k), \zeta(k) >_\Sigma = \zeta^T(k)\Sigma\zeta(k) \end{cases} \tag{11}$$

In (11) $E(.)$ denotes the mathematical expectation, ζ^T is the transpose of ζ, Σ is a weight matrix which is symmetric, non-negative, definite. The AF thus is optimal in the MPE (Minimum Prediction Error) sense since $\zeta(t)$ is a one step prediction error for the system outputs. For simplicity, in the further, let $\Sigma = I$ where I is the identity operator.

Due to the stochastic character of the gain estimate, instability of the resulting filter may happen during the optimization process. To avoid such instability, it is proposed to choose a parameterized structure for the gain, i.e. $K := K(\theta)$. The optimal filter is obtained by solving the following optimization problem

$$\begin{aligned} &J[K] \to \min_\theta E[\Psi(\zeta(k))], \Psi(\zeta(k)) := ||\zeta(k)||^2, \\ &\zeta(k) = Y(k) - H\Phi\hat{X}(k-1) = Y(k) - H\Phi[\hat{X}(k-1/k-2) + K(\theta)\zeta(k-1)]. \end{aligned} \tag{12}$$

In (12), $\Psi(\zeta(k))$ is a sample cost function. As seen from the equation for $\zeta(k)$, the value of $\zeta(k)$ is always known since $Y(k)$ and $H\Phi\hat{X}(k-1)$ are available. By introducing the objective function $J[K]$ (12) one wants to minimize the variance of the innovation process $\zeta(k)$. Parameterizing $K = K(\theta)$ by some vector of parameters θ, the objective function

(12) is a function of θ which is chosen for ensuring a stability of the filter. For example, the filter gain can be chosen in the form (see [8]),

$$K_{af}(k;\theta) = P_r \Theta K_e(k), \tag{13}$$

where P_r is a matrix of dimensions $n \times n_e$, n_e is the number of all unstable and neutrally stable eigenvectors (or singular vectors) of the transition matrix Φ, $\Theta = \text{diag}\,(\theta_1, \ldots, \theta_{n_e})$ - diagonal matrix with the diagonal elements $\theta_i \in (1-\epsilon_i, 1+\epsilon_i)$, $\epsilon_i \in (0,1)$ is a function of the absolute value of the i^{th} eigenvalue, $\epsilon_i \to 0$ as $|\lambda_i| \to \infty$, $\epsilon_i \to 1$ as $|\lambda_i| \to 1$.

Comment 4.1. In (13), $K_e(k)$ has the meaning of the gain of the filter in a reduced space $R[L_e], L_e \in R^{n_e \times n_e}, n_e < n$. More precisely, if instead of the full system state $X(k)$ one wants to estimate only a reduced state $X_e(k)$ (of the less dimension). For example, we are interested in estimating only some components of $X(k)$, ($X_e(k) = L_e X(k)$), then one can construct a reduced-order filter having the gain $K_e(k)$ for producing the estimate $\hat{X}_e(k)$ and reconstruct the estimate for the full state $\hat{X}(k)$ as $\hat{X}_e(k) = L_e^+ \hat{X}_e(k)$ where L_e^+ is the pseudo-inverse of L_e. For more details, see [5]. For the SEIKF, the matrix $K_e(k)$ is equivalent to the matrix $G_e(k)$ in the formulas (28)–(29) in the next section.

Applying the SPSA optimization algorithm yields the following recursive equation for estimating θ [10]

$$\theta(k+1) = \theta(k) - G(k+1)\nabla_{\theta(k)}\Psi(\zeta(k+1)), \quad k = 0, 1, 2 \ldots \tag{14}$$

In (14), $\nabla_{\theta(k)}\Psi$ is the gradient of the sample cost function Ψ with respect to (w.r.t.) θ evaluated at the point $\theta(k)$; $G(k)$ is a factor ensuring convergence of the algorithm (it may be a matrix or scalar variable). It is seen that we replace the problem of minimization in probabilistic space (average) by minimization along the system trajectory if we assume that the system possesses the ergodicity property. This simplifies too much the solution of the considered optimization problem.

Substituting $\theta(k)$, computed from (14), into the gain $K_{af}(t;\theta(k))$ in (13), one obtains the following equation for the AF (see [6])

$$\hat{X}(k+1) = \Phi\hat{X}(k) + K_{af}(t;\theta(k))\zeta(k+1) = \Phi\hat{X}(k) + P_r\Theta(k)K_e(k)\zeta(k+1). \tag{15}$$

5. ADAPTIVE SEIK FILTER (ASEIKF)

In this section we describe the AF whose non-adaptive version is based on the SEIKF [4]. As reviewed in the Introduction, the main characteristics of the SEIKF concerns about the resampling procedure : the members of the model ensemble are selected in the main orthogonal directions of a functional space, described by an approximation to the ECM.

5.1. SEIKF algorithm

Let the numerical model, resulting from discretization of the system (7), be represented as

$$X_k = \mathcal{F}_{k-1}(X_{k-1}) + w_k, \tag{16}$$

where the vector $X_k \in \mathcal{R}^n_{Z,C}$ and $w_k \in \mathcal{R}^n_{Z,C}$ are the state and model error at the time instant T_k; $\mathcal{F}_{k-1}(\cdot)$ is a (generally nonlinear) function modeling the transition of the system state from time T_{k-1} to time T_k. The function $\mathcal{F}_{k-1}(\cdot)$ can include other external known

variables (not expressed here). In (16) the model error w_k is a temporally uncorrelated sequence of zero mean and covariance Q_k.

At each time instant T_k we are given the observation

$$Y_k = \mathcal{H}_k(X_k) + v_k. \tag{17}$$

It is supposed that the measurements are available only at certain points of the area. Thus, Y_k is composed of the measures of pollutant concentration at certain mesh points.

Comment 5.1. The time instant $T_{k+1} = T_k + \Delta T_k$ corresponds to the assimilation instant (arrival of observation) which is different to the time instant $t_{k'+1} = t'_k + \delta t$ where δt denotes a sampling time step in derivative approximation.

The AF based on the SEIKF (Singular Evolutive Interpolated Kalman filter) is denoted as ASEIKF. The original SEIKF consists of three stages (cf., [7]): resampling, forecast and correction, preceded by the initialization.

Consider the filtering problem for $k = 0, 1, 2, \ldots, N$. At $k = 0$, let $\hat{X}_0$ be the system estimate and P_0 be its error covariance matrix (ECM) for the initial state X_0. These statistics can be obtained by using a sequence of historical data or, more practically, by a preliminary simulation study: a long sequence of states $\tilde{X}_1, \ldots \tilde{X}_N$ is generated by solving Eqs. (9). The statistics $\hat{X}_0$, P_0 are estimated by

$$\hat{X}_0 = (1/N) \sum_{j=1}^{N} \tilde{X}_j, \tilde{P}_0 = (1/N) \sum_{j=1}^{N} (\tilde{X}_j - \hat{X}_0)(\tilde{X}_j - \hat{X}_0)^T, \tag{18}$$

here x^T denotes the transpose of x.

Originally the SEIKF has been proposed for solving the data assimilation problems in the systems of very high dimensions ($n \approx O(10^7) - O(10^8)$ for many geophysical systems). As the KF, for such systems, the SEIKF cannot be applied due to its prohibitive cost if the full covariance space is used. This difficulty is overcome by reducing the dimension of the ECM by its projection onto a low dimensional subspace.

Consider the main three stages in the SEIKF. At the assimilation instant T_k, let P_k be the prediction ECM. Let $k = 0$.

- *Sampling stage*. For a given k, let P_k be of the rank $r << n$ where r is the rank of P_k. Suppose that the true ECM of the system forecast is $\tilde{P}_k$. Consider the eigen-decomposition of $\tilde{P}_k$,

  $$\tilde{P}_k = \sum_{j=1}^{n} \lambda_j Z_j Z_j^T,$$

  where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\tilde{P}_k$, arranged in decreasing order, and $Z_1, \ldots, Z_n$ are the corresponding (normalized) eigenvectors. The sequence of eigenvalues usually decreases to zero very fast so that one may retain only the first r terms in the above sum, yielding an approximate ECM of the rank r for $\tilde{P}_k$, i.e.

  $$P_k = \sum_{j=1}^{r} \lambda_j Z_j Z_j^T.$$

The matrix P_k can be expressed in the factorized form $P_k = L_k \mathbf{U}_k L_k^T$ where

$$P_k = L_k \mathbf{U}_k L_k^T, L_k = [Z_1 \quad \cdots \quad Z_r], \tag{19}$$

and $\mathbf{U}_k$ is the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_r$.

- *Forecast stage*. This stage is aimed at forecasting the system state and computing the forecast error, based on the model equation and the knowledge of previous system state. At the time instant T_{k-1}, assume that we are given the estimate $\hat{X}_{k-1}$. Let $\mathbf{\Omega}_{k-1}$ be a $r \times (r+1)$ matrix such that its row vectors are orthonormal and have components summing to zero. For the algorithm for constructing a such matrix , see [4]. With $\Omega_{k-1}^{(i)}$ - the i^{th} column of $\mathbf{\Omega}_{k-1}$, one defines

$$X_{k-1}^i = \hat{X}_{k-1} + \sqrt{r} Ł_{k-1} \mathbf{U}_{k-1}^{-1/2} \Omega_{k-1}^{(i)}. \tag{20}$$

The set $\{X_{k-1}^i, i = 1, \ldots, r+1\}$ may be viewed as samplings of the estimated distribution for the system state at the time instant T_{k-1}, as far as the second order statistics are concerned. The samplings of the forecast for the system state at the next time instant T_k are

$$X_{k/k-1}^i = \mathcal{F}_{k-1}(X_{k-1}^i). \tag{21}$$

From (21) we have

$$\hat{X}_{k/k-1} = \frac{1}{r+1} \sum_{i=1}^{r+1} X_{k/k-1}^i = \frac{1}{r+1} \sum_{i=1}^{r+1} \mathcal{F}_{k-1}(X_{k-1}^i). \tag{22}$$

In the further, for simplicity, we introduce the notation

$$\hat{X}_{k/k-1} := \hat{\mathcal{F}}_{k-1}(\hat{X}_{k-1}). \tag{23}$$

- *Correction stage*. This stage takes into account the information, provided by the observation, to obtain a new more accurate estimate of the state vector X_k and update the associated ECM.

Let $\mathbf{T}$ be a $((r+1) \times r)$ matrix with orthonormal and zero column sums. Introduce [1]

$$L_k = \begin{pmatrix} X_{k/k-1}^1 & \cdots X_{k/k-1}^{r+1} \end{pmatrix} \mathbf{T}, \tag{24}$$

$$\mathbf{U}_{k/k-1} = (r\mathbf{T}^T\mathbf{T})^{-1} + (L_k^T L_k)^{-1} (L_k^T \bar{\mathbf{Q}}_k L_k)(L_k^T L_k)^{-1}. \tag{25}$$

Here, $\bar{\mathbf{Q}}_k$ is the cumulative model ECM, at the moment T_k. In the experiment in the next Section, $\mathcal{H}_k$ is a linear operator $\mathcal{H}_k(X_k) = \mathbf{H}_k X_k$. Using the matrix identity, the above right hand side is invertible if $\mathbf{U}_{k-}$ is assumed to be non-singular,

$$\mathbf{U}_k^{-1} = \mathbf{U}_{k/k-1}^{-1} + Ł_k^T \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k Ł_k, \tag{26}$$

[1] One example of $\mathbf{T}$ is defined by the formula (40) in [7].

$\mathbf{R}_k$ is the measurement ECM at the time instant k. The formula for the correction stage in the SEIKF has the following form

$$\begin{aligned}\hat{X}_k &= \hat{X}_{k/k-1} + K_{seik}(k)(Y_k - \mathbf{H}_k\hat{X}_{k/k-1}) \\ &= \hat{\mathcal{F}}_{k-1}(\hat{X}_{k-1}) + K_{seik}(k)(Y_k - \mathbf{H}_k\hat{\mathcal{F}}_{k-1}(\hat{X}_{k-1})),\end{aligned} \tag{27}$$

where

$$K_{seik}(k) = \mathbf{G}_k = \mathbf{L}_k\mathbf{U}_k\mathbf{L}_k^T\mathbf{H}_k^T\mathbf{R}_k^{-1} = \mathbf{L}_k\mathbf{G}_e(k), \tag{28}$$

$$\mathbf{G}_e(k) := \mathbf{U}_k\mathbf{L}_k^T\mathbf{H}_k^T\mathbf{R}_k^{-1}. \tag{29}$$

For $k < N$ the procedure returns to the sampling stage subject to $k := k + 1$. Otherwise, stop.

Mention that the matrix $\mathbf{G}_e(k)$ has a meaning of the gain of a reduced-order filter if we want to construct a filter for estimating the reduced-order state $x_e(k)$ ($x_e(k) \in R^{n_e}$).

5.2. ASEIKF

According to the AF in [6], based on the SEIKF, described above, one can write out here the ASEIKF. In the ASEIKF, the gain is of the following form

$$\mathbf{K}_{af}(k;\theta(k)) = \mathbf{P}_r(k)\Theta(k)\mathbf{G}_e(k), \tag{30}$$

$$\mathbf{P}_r(k) = \mathbf{L}_k, \tag{31}$$

where $\Theta_e(k)$ is the diagonal matrix with diagonal elements $\theta_1, \ldots, \theta_{n_e}$. The diagonal elements $\theta_1, \ldots, \theta_{n_e}$ are positive, $1 - \epsilon_i \leq \theta_i \leq 1 + \epsilon_i$ where $\epsilon_i \in (0, 1)$ is a function of the absolute value of the λ_i - i^{th} eigenvalue (or singular value) of the transition matrix Φ (for a linear dynamical system).

From (31) it is seen that the matrix $\mathbf{L}_k$ plays the role of the projecting operator P_r in the AF (Section 4). In the ASEIKF, $\theta := (\theta_1, \ldots, \theta_{n_e})^T$ will be updated by the algorithm in Subsection 5.2 to minimize the MPE of the system outputs. This allows to improve the performance of the SEIKF algorithm. Mention that we will apply two versions of the ASEIKF, one is ASEIKF(0) with the gain $K_{af0} := \mathbf{K}_{af}(0;\theta(k))$, and another is ASEIKF(k) with $K_{afk} := \mathbf{K}_{af}(k;\theta(k))$. The gain K_{af0} in the ASEIKF(0) is based on the constant initial gain $K_{seik}(0)$ calculated in the SEIKF at T_0 whereas in the ASEIKF(k) is based on the time-varying gain $K_{seik}(k)$ of the SEIKF.

Comment 5.2. It is interesting to reveal that from the structure of the SEIKF it is found from (30)–(31) that the operator $\mathbf{L}_k$ is identified to the projecting operator $\mathbf{P}_r(k)$. As the columns of $\mathbf{L}_k$ are the main directions of the sample ECM, the linear space, spanned by the columns of $\mathbf{L}_k$, in some sense is close to (but not the same) the subspace spanned by all unstable and neutrally stable eigenvectors (most growing directions) of the system dynamics. It is clear that, from the stability point of view of the SEIKF, by its construction, there is no guarantee of stability of the SEIKF, contrary to that established for the AF.

5.3. ASEIKF algorithm

The algorithm for the ASEIK is written as follows:

- Step 0: $k = 1$;

- Step 1: Calculate L_0 by the equation (19), $\mathbf{U}_0$ is the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_r$ of matrix P_0 defined by the formula (18);
- Step 2: Calculate X^i_{k-1} by the formula (20);
- Step 3: Calculate $X^i_{k/k-1}$ by the formula (21);
- Step 4: Calculate $\hat{X}_{k/k-1}$ using (22);
- Step 5: Calculate L_k using (24);
- Step 6: Calculate $\mathbf{U}_{k/k-1}$ using (25);
- Step 7: Calculate U_k^{-1} using (26) and then obtain the inverse matrix U_k;
- Step 8: Calculate the gain by (30), $K(k;\theta(k)) = L_k\Theta(k)\mathbf{G}_e(k)$ where $\Theta(k)$ is diagonal with the diagonal elements $\theta_i(k), i = 1, \ldots, n_e$ where $\theta_i(k)$ is the i^{th} component of $\theta(k)$ and $\theta(k)$ is updated by SPSA algorithm (see [10]):
  - Step 8.1: $\theta(k+1) = \theta(k) - \alpha_k g(\theta(k))$;
  - Step 8.2: $g(\theta(k)) = [g_1(\theta(k), \ldots, g_{n_e}(\theta(k)))]^T$;
  - Step 8.3: $g_i(\theta(k)) = \dfrac{\Psi(\zeta, \theta(k) + c_k\delta_k) - \Psi(\zeta, \theta(k) - c_k\delta_k)}{2c_k\delta_k}$;

  where $\delta_k = (\delta_{k_1}, \ldots, \delta_{k_{n_e}})^T$, δ_{k_i} can be chosen as random variable having the symmetric Bernoulli distribution assuming two values 1 and -1 with the same probability $1/2$.
- Step 9: Update $\hat{X}_k$ by the formula (27);
- Step 10: Set $k := k + 1$;
- Step 11: if $k < N_{step}$ go to step 2, Else: Stop.

Comment 5.3. The positive scalar sequences α_k, c_k must satisfy certain conditions.

6. NUMERICAL EXPERIMENT ON 2D WATER POLLUTION PROBLEM BASED ON SEIKF AND ASEIKF

The 2D water pollution (5), (6) model is solved by applying a cell centered finite volume method (see [12]), accompanied by an explicit scheme in time. The two filters SEIKF and ASEIKF in Subsection 5.2 are implemented to estimate the concentration of the Thanh Nhan lake in Hanoi, Vietnam.

6.1. Configuration of the Thanh Nhan lake

The Thanh Nhan Lake of Hanoi has the surface area of about 8.1 ha and the water volume of about 162000 m^3. For the hydro properties of the lake, see [7]. In this paper, the indicator BOD_5, showing the oxygen quantity needed for bacterium in oxygen reactions of organic substances in water, is used. The geographical data are divided into two groups, one contains the boundary points and the other - the points in the inside area. The data set is determined by the unstructured grid with 1964 triangular cells and 1058 nodes. Fig. 1(a) shows the unstructured grid. The substance enters into the lake uniquely by the inflow gate Γ_1 (called gate-in, in the inflow boundary). The data, concerning discharges and substance contents in the inflow and outflow gates, are:

- The discharges in and out of the lake are 2100 m^3 per day-night. On the gate-in Γ_1 the boundary values of $\bar{\mathbf{U}}_{in} = u.n_x + v.n_y$ are generated by the program with $u = 0.00174$ m/s and $v = -0.00164$ m/s.

- The content of BOD_5 in the gate Γ_1 into the lake is equal to the measurement value 24 mg/l.

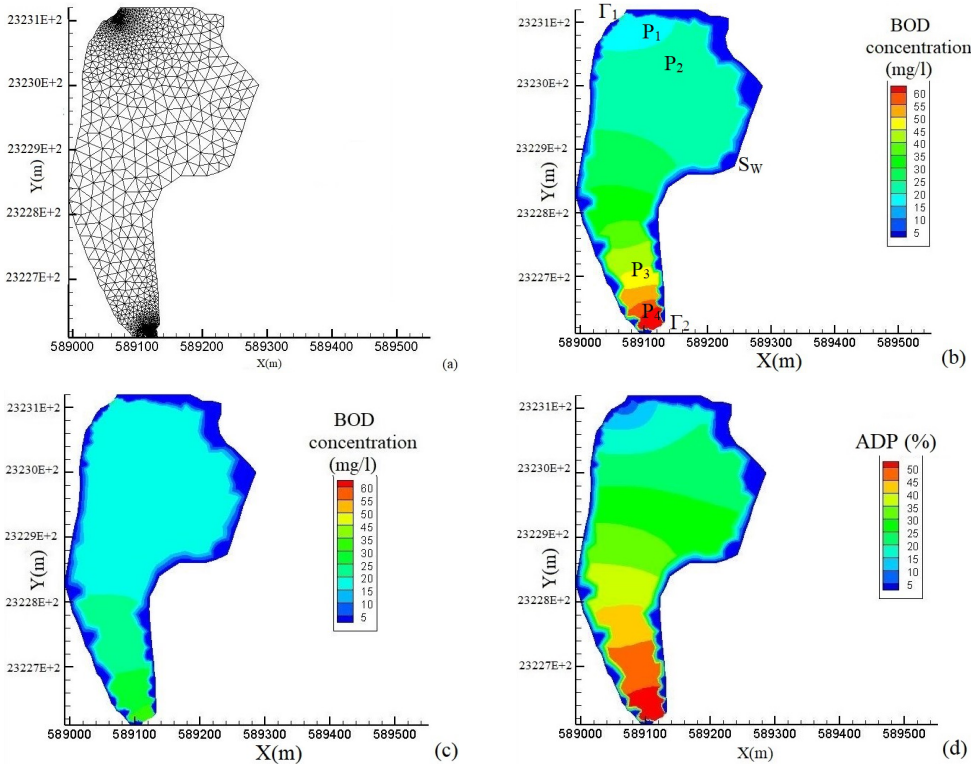

Fig. 1. (a) The unstructured grid of the lake; (b) The initial concentration $C_{ref}(0)$ of the reference model R_2; (c) The initial concentration $\bar{C}$ of 4 models $R_3^1, R_3^2, R_3^3, R_3^4$; (d) The absolute difference in percentage between the initial concentration $\bar{C}$ of 4 models $R_3^1, R_3^2, R_3^3, R_3^4$ and $C_{ref}(0)$ of the reference model R_2 with the scale from 5 to 50

On the boundary S_W and the gate-out, there are the conditions of concentration and velocity $\left.\dfrac{\partial C}{\partial n}\right|_{S_W \cup \Gamma_2} = 0$, $\left. U.\vec{n}\right|_{S_W} = 0$ where $\vec{n}$ is the unit normal vector to the outflow boundary. The locations of the gates Γ_1 and Γ_2 are shown in the Fig. 1(b).

The data for the Thanh Nhan lake model are given in Table 1.

Table 1. Data 1 of the model

K_x, K_y	η	K	Time step(s)	$h\big\|_{\Gamma_2}$ (m)
30.6	$1.7e^{-6}$	$-4.05E^{-6}$	0.2	4

In order to estimate the effects of ASEIKF let us introduce the Absolute Difference Percent (ADP) as a measure of the difference between two variables a and b

$$ADP(a,b) = \frac{|a-b| * 100}{|b|}. \tag{32}$$

6.2. Experiment setup

The experiment, reported here, is of the type "twin-experiment". A randomly chosen state is used to start the numerical model run based on Eqs. (5)–(6). First the sequence of the true states and of the observations are generated as follows:

6.2.1. *Simulation of true states and observations*

- The run R_1: During the first time period $P1 := [T(0), T(1)], T(0) = 0$ s, $T(1) =$ 300 s, the pollution sources, located at several points in the interior of the lake, are not activated. This run is denoted as R_1. The time-averaged concentration state, produced during from the time period $[T(0), T(1)]$, will be denoted as $\bar{C}$ and described by Fig. 1(c). $\bar{C}$ will serve as the estimate for the initial concentration in the SEIKF and ASEIKF.

- The run R_2 begins from $t = T(1)$. The period $P2$ is defined as $P2 := [T(1), T(2)]$, $T(2) = T(1) + 12$ s. This run R_2 is initialized by the state at the end of the run R_1. At the moment $T(1)$ the pollution source (with the concentration 5 mg/l) is activated continuously and the model is running over the period $P2$.

- Reference model denoted by R_2 and observations: The ensemble of states, generated during the run R_2, serves as a sequence of the true states (or *reference*). The state values of R_2 are denoted by Z_{ref}, C_{ref}. The values of $Z(T(1))$ and $C(T(1))$ in the run R_1 will be considered as the initial values in this reference model R_2 and denoted by $Z_{ref}(0), C_{ref}(0)$. The initial concentration field $C_{ref}(0)$ is described by Fig. 1(b).

For testing example, the measurements are at 0.2 s, 0.6 s, 1.0 s (one per 0.4 s) after the time moment $T(1)$. At these time instants and on some observation points located on the middle lake, the measurement values are extracted from the reference and served as observations Z_{obs}, C_{obs} to be used in the data assimilation experiment.

6.2.2. *Data assimilation experiment*

For this example, we will introduce the following 4 different models denoted by $R_3^1, R_3^2, R_3^3, R_3^4$. In these 4 models all data assimilation experiments start from $t = T(1)$ and are carried out over the period $P2$ (initialized by the values $\bar{Z} = Z(T(1)), \bar{C}$) and the pollution source (with the concentration 5 mg/l) is activated continuously. The Absolute Difference Percent (ADP) between the initial concentration value $\bar{C}$ (used in the 4 models $R_3^1, R_3^2, R_3^3, R_3^4$) and that $C_{ref}(0)$ of the reference is described by Fig. 1(d) with the scale from 5 to 50.

- The Model running without correction denoted by R_3^1: The numerical model is running from $T(1)$ to the end of the period $P2$ without using the observations. This DA experiment is denoted as $DA - Model$. This *no-assimilation* experiment produces the estimates denoted as (Z_{no-ass}, C_{no-ass});

- The model running with SEIKF denoted by R_3^2: This numerical model is running from $T(1)$ to the end of the period $P2$ using the observations in correction process by SEIKF. The SEIKF is running to assimilate the observations (this experiment is denoted as $DA - SEIKF$).

- The model running with ASEIKF(k) denoted by R_3^3: This model is running in the period P2 using the observations in correction process by adaptive version ASEIKF(k) of the SEIKF with the time-varying gain $\mathbf{G}_k$ of the SEIKF.

- The model running ASEIKF(0) denoted by R_3^4: This model is running in the period P2 using the observations in correction process by adaptive version ASEIKF(0) of the SEIKF with the constant gain $\mathbf{G}_0$ - the gain of the SEIKF at T_0.

6.3. Numerical results

We will test the simulation concentration differences of the models R_2, R_3^1, R_3^2, R_3^3, R_3^4 at 4 points $P_1 - P_4$, where their locations are shown on Fig. 1(b). The ADP between the initial concentration condition $\bar{C}$ used in the 4 filters R_3^1, R_3^2, R_3^3, R_3^4 and that $C_{ref}(0)$ used in the reference model R_2 makes the differences in their produced results (see Figs. 2, 6, 7).

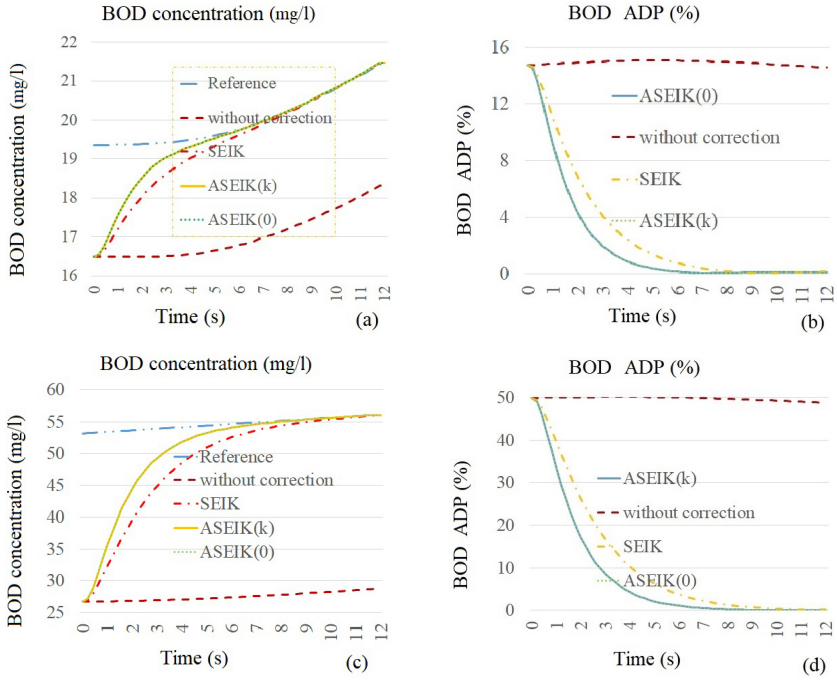

Fig. 2. Time period $[0, 12s]$ (starting from $T(1)$): BOD concentration at the points P_1 and P_4 (a, c); The ADP between 4 runs R_3^1, R_3^2, R_3^3, R_3^4 and R_2 at the points P_1 and P_4 (b, d)

Figs. 2(a) and 2(c) show the BOD concentration at the points P_1 and P_4. In these figures the simulation results of the reference model R_2 and those of the model without correction R_3^1 are shown by the curves with the blue dash two dots and dark red dash types, respectively. It is easy to see that the curves of the model R_3^1 are the most far from the reference curves, comparing with other curves of the models R_3^2, R_3^3, R_3^4 (i.e., by the SEIKF, ASEIKF(k), ASEIKF(0)). One sees that the curves of the model R_3^3, R_3^4 with the ASEIKF(0), ASEIKF(k) (see yellow and green color curves in Figs. 2(a, c)) are the most close to the reference curves. Figs. 2(b, d), 3(c), 3(d) show the ADP between 4 runs R_3^1, R_3^2, R_3^3, R_3^4 and R_2 at the points P_1, P_4, P_2 and P_3. One sees that for the time period $[T(1), T(2)]$, the assimilation allows to reduce significantly the estimation ADP (compare the ADP produced by the model without correction (see dark red dash curves in Figs. 2(b, d), 3(c, d)) with those generated by the other filters). The values of ADP are much less in the ASEIKF(0), ASEIKF(k) compared to those with the SEIKF. The ASEIKF(k) have produced the best estimates for the system states. It means that the adaptation has proved to be very an efficient tool for improving the performance of the non-adaptive filters.

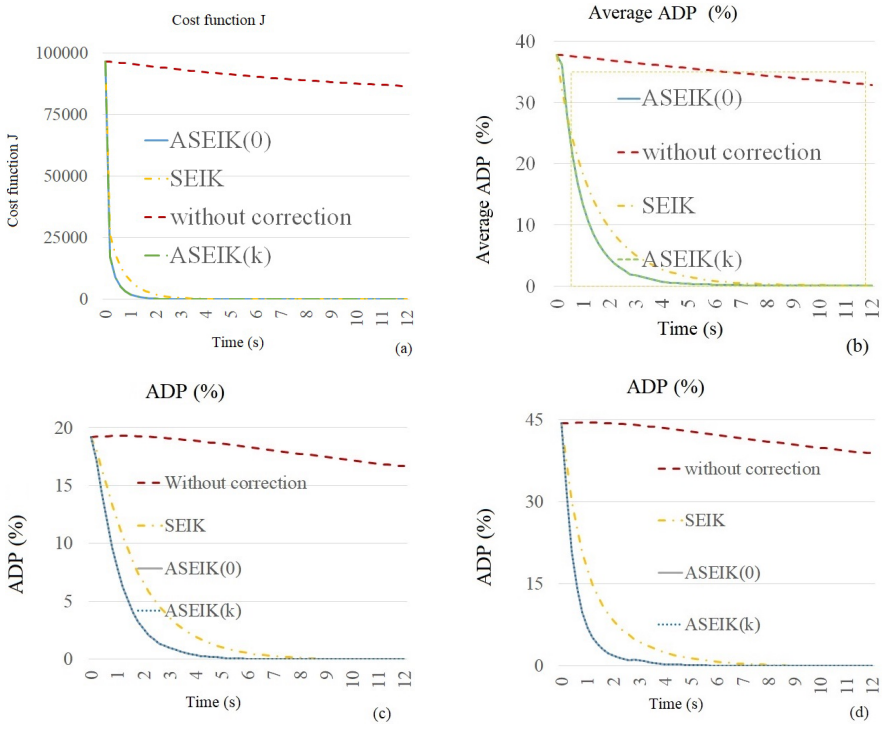

Fig. 3. Time period $[0, 12s]$ (starting from $T(1)$): Evolution of the cost function J (a); The Average ADP between 4 runs R_3^1, R_3^2, R_3^3, R_3^4 and R_2 (b); The ADP between 4 runs R_3^1, R_3^2, R_3^3, R_3^4 and R_2 at the points P_2 and P_3 (c, d)

During the assimilation period $[0, 12s]$ (starting from $T(1)$), the evolution of the time average of the sample cost functions $J = \Phi[\zeta(k)]$ (squares of the Euclidean norm of the

innovation) is depicted in Fig. 3(a). It is clear that the cost function is highest (see dark red dash curve) for the model without correction and the adaptation allows to better minimize the prediction error for the system outputs in the ASEIKF(0) and ASEIKF(k). The lowest prediction error is produced by the ASEIK(0) and ASEIKF(k). For this period time, the averaged ADP between 4 runs R_3^1, R_3^2, R_3^3, R_3^4 and R_2 are shown in Fig. 3(b). One sees that for the time period [T (1), T (2)], the assimilation allows to reduce significantly the estimation ADP (compare the ADP values produced by the model without correction (see dark red dash curves with the ADP values more than 35) with those generated by the other filters). It is also easy to see that the lowest ADP values are produced by the ASEIKF(0), ASEIKF(k).

Figs. 4, 5 show concentration ADP between the produced estimates of 4 runs R_3^2, R_3^3, R_3^4, R_3^1 and the reference model R_2 (with the percentage scales from 0.5 to 5.5 for the different filters ASEIKF(k), ASEIKF(0), SEIKF and the percentage scales 5 to 50 for the

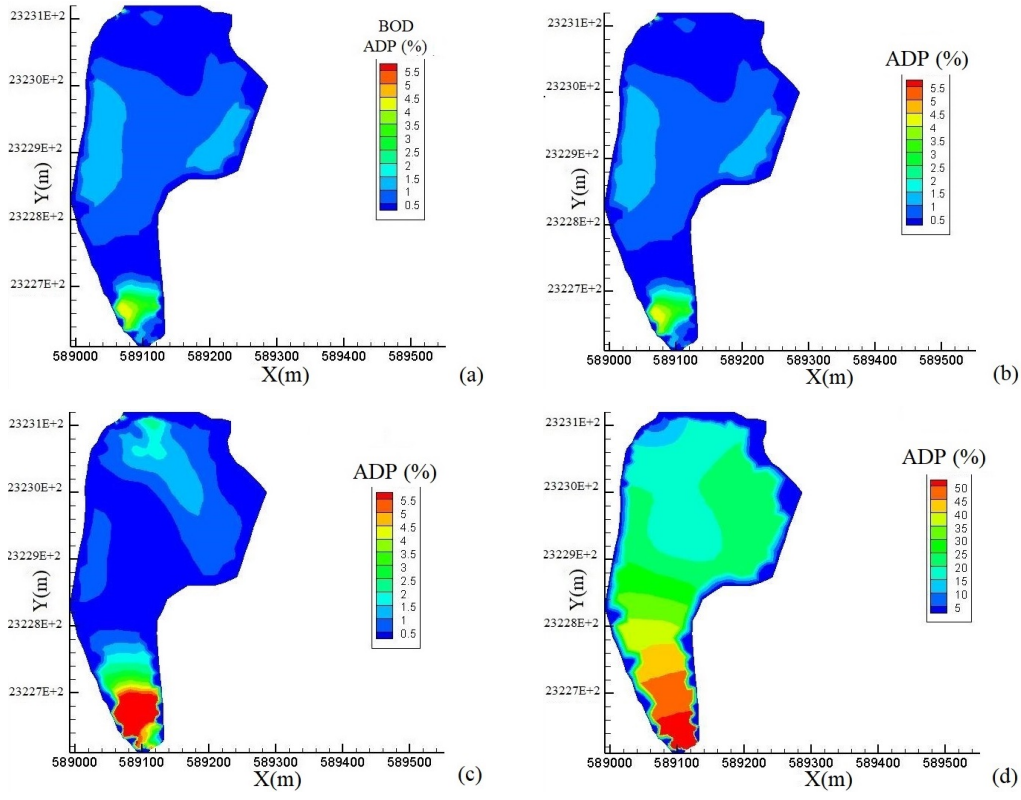

Fig. 4. Concentration estimation ADP between 4 runs R_3^2, R_3^3, R_3^4, R_3^1 and reference R_2 (with the scales from 0.5 to 5.5 for the filter models R_3^2, R_3^3, R_3^4 (a–c) and from 5 to 50 for the model without correction R_3^1 (d)) at the time instant $t = 2.4s$ after putting the source in the middle of the lake: Model R_3^3 with ASEIKF(k) (a); Model R_3^4 with ASEIKF(0) (b); Model R_3^2 with SEIKF (c); Model R_3^1 without correction (no assimilation) (d)

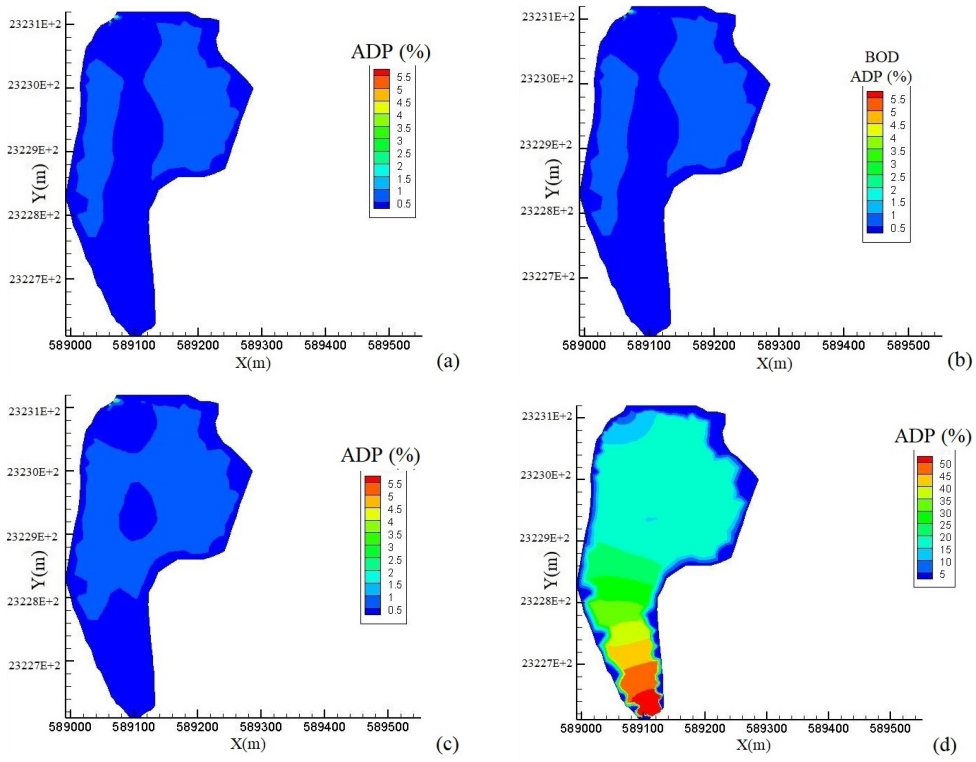

Fig. 5. Concentration estimation ADP between 4 runs R_3^2, R_3^3, R_3^4, R_3^1 and reference R_2 (with the scales from 0.5 to 5.5 for the filter models R_3^2, R_3^3, R_3^4 (a–c) and from 5 to 50 for the model without correction R_3^1 (d)) at the time instant $t = 8$ s after putting the source in the middle of the lake: Model R_3^3 with ASEIKF(k) (a); Model R_3^4 with ASEIKF(0) (b); Model R_3^2 with SEIKF (c); Model R_3^1 without correction (no assimilation) (d)

model without correction R_3^1) at $t = 2.4$ s and $t = 8$ s after the opening of the source in the middle of the lake. It is seen that the ADP levels are lowest in the ASEIKF(0) and ASEIKF(k) (see Figs. 4(a, b), 5(a, b)) and are biggest in the model without correction R_3^1 (see Figs. 4(d), 5(d)).

Looking at Figs. 6(a)–6(e) and 7(a)–7(e), corresponding to the time moments $t = 2.4$ s and $t = 8$ s, one concludes that the concentration fields, estimated by the model without correction R_3^1 are the most far different from the ones of reference model R_2. Beside that, the concentration fields, estimated by the ASEIKF(0) and ASEIKF(k), are closest to the references ones, compared to those produced by the other filters. It means that among all the employed filters, the ASEIKF(0) and ASEIKF(k) behave better in recovering the unknown true concentration.

In the popular PC (Intel Core i5 3.1 GHz), for our correction problem it takes about 45 min to run all 3 SEIKF, ASEIKF(0), ASEIKF(k) filter algorithms.

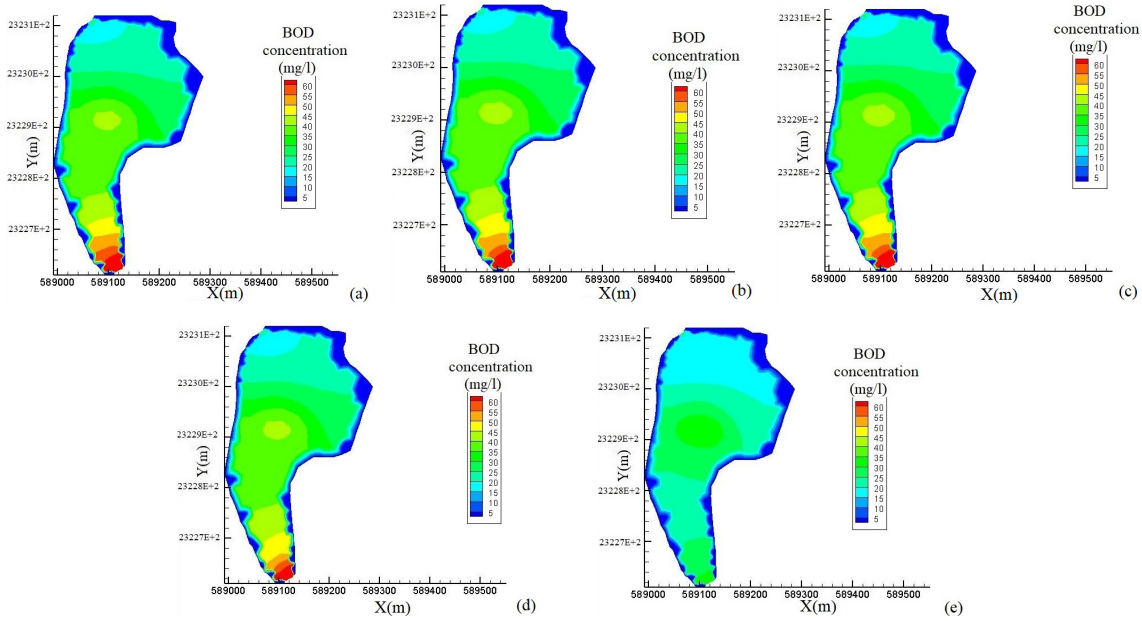

Fig. 6. Concentration estimate produced by different filters at the time $t = 2.4$ s after opening the source in the middle of the lake: the reference model R_2 (a); Model R_3^3 with ASEIKF(k) (b); Model R_3^4 with ASEIKF(0) (c); Model R_3^2 with SEIKF (d); Model R_3^1 (without correction) (e)

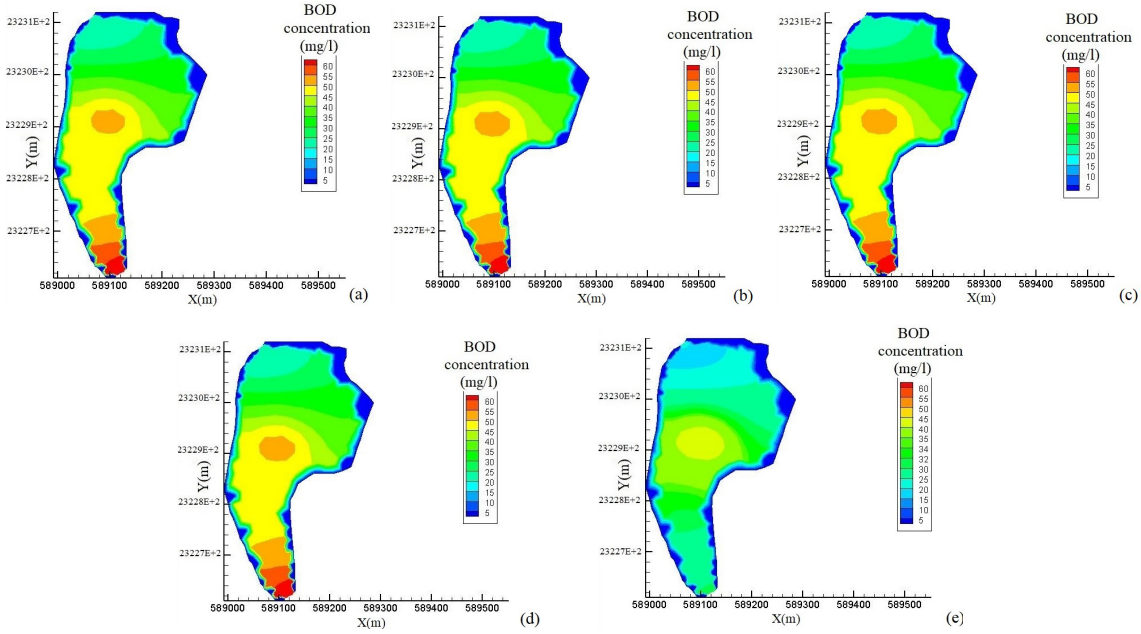

Fig. 7. Concentration estimate produced by the different filters at the time $t = 8$ s after opening the source in the middle of the lake: the reference model R_2 (a); Model R_3^3 with ASEIKF(k) (b); Model R_3^4 with ASEIKF(0) (c); Model R_3^2 with SEIKF (d); Model R_3^1 (without correction) (e)

7. CONCLUSIONS

In this paper, the idea of the adaptive filtering approach, developed in [6], was followed to construct the ASEIKF whose non-adaptive version is the SEIKF [4,7]. It is shown how the gain in the SEIKF can be parameterized in an appropriate way which allows to achieve an excellent performance of the ASEIKF by tuning some parameters of the gain. The objective of the ASEIKF is to minimize the mean prediction error of the system output. The difficulties in solving the optimization problem, formulated in probabilistic space, are overcome by applying the SPSA algorithm which works perfectly in accordance with the time-recursive character of sequential filtering approach. By perturbing simultaneously (and stochastically) the vector of unknown parameters in the gain, at each assimilation instant, it requires to integrate only two or three times the numerical model, without the need to compute numerically the gradient vector or to construct the adjoint model (or adjoint code). By this way, the original SEIKF become adaptive. The solution of the optimization problem based on the SPSA algorithm is presented in detail. The algorithms, obtained in Sections 3-6, constitute a background for solving the 2D water pollution problem, formulated in Section 2.

The numerical experiment on estimation of pollution propagation has been carried out which shows a high accuracy of the estimates produced by the ASEIKF. Due to space limit of the paper we cannot presented some numerical results showing the better performance of the ASEIKF(k) over the ASEIKF(0) especially when the initial state estimate is more far away from the initial true state. This proves the usefulness of the ASEIKF, especially for time-varying systems. It has been demonstrated that the ASEIKF, either based on a constant or time-varying gain, is overperforming the SEIKF. Another very positive point concerns the performance of the ASEIKF(k): with best performance, the ASEIKF is proved to be a very promising tool in the future for solving data assimilation problems in non-stationary dynamical systems.

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