Vietnam Journal of Mechanics, VAST, Vol. 44, No. 1 (2022), pp. 59 – 68
DOI: https://doi.org/10.15625/0866-7136/16726

BUCKLING OF CHIRAL ELASTIC RINGS SPANNED BY FLUID FILMS

Hoang Minh Tuan[1,2,*], **Nguyen Thi Lan**[2], **Tran Minh Tuyen**[2]
[1]*Graduate University of Science and Technology, VAST, 18 Hoang Quoc Viet, Hanoi, Vietnam*
[2]*Institute of Mechanics, VAST, 18 Hoang Quoc Viet, Hanoi, Vietnam*
[*]E-mail: tmhoang@imech.vast.vn

Received: 11 November 2021 / Published online: 30 March 2022

Abstract. We use variational method to examine buckling of chiral elastic rings spanned by fluid films. We find that the critical surface tension of the fluid film at which buckling occurs depends on the degree of chirality, regardless whether the ring possesses left-handed chirality or right-handed chirality. Additionally, the chirality always has a destabilizing effect on buckling, yielding to buckle at a critical surface tension smaller than that of the achiral elastic rings. The destabilizing effect of chirality, however, can be reduced by increasing the twisting rigidity to bending rigidity of filaments (twist-to-bend ratio).

Keywords: buckling, elastic ring, chirality, cosseratsserat rod, fluid film.

1. INTRODUCTION

Buckling of elastic rings is an old problem dated back to the original paper of Levy [1] whose buckling of a cylindrical shell under pressure is examined. When the shell is infinitely long, the problem reduces to buckling of an elastic ring under pressure. The buckling of elastic rings plays an important role in many phenomena, including designing nanorings in MEM and NEM devices [2], designing slender structures such as tubes, pressure tankers [3], and others.

Buckling of elastic rings has recently attracted researchers as it appears in physical and biological systems whose the rings spanned by fluid films and the shape of the rings results from competition between the energies of boundaries and surfaces. A typical illustration is the dorsal mesenteric membrane spanning vertebrate gut tube; the surface tension of the membrane accounts for the chirality and knot of tube [4]. Another illustration is the lipid bilayers enclosing the helical protein belts which have both planar and eight-figured shapes [5]. Buckling of twisted elastic rings spanned by fluid films originated from the groundbreaking work of Plateau [6] on minimum surfaces spanning a closed, simple, and given boundary. A generalization of the Plateau's work was first considered Bernatzki and Ye [7] who found the existence condition for solutions of the generalized problem in which the boundary is elastic and not extensible. Giomi and Mahadevan [8] then experimentally and numerically identified the forms of the boundary using fishing strands and soapy fluid. Fried and co-workers [9–13] subsequently investigated theoretically the stability and bifurcation of the planar configuration.

Chirality is present in many structures at various scales, ranging from DNA to plant vines, and to animal horns. Chirality results in many useful facts such as the coupling between twisting, bending, and stretching of the filaments. Whereas the twist-stretch coupling is well understoood [14–16], the twist-bend coupling has not been so although the former was suggested after the latter [17, 18]. Motivated by recent work [19, 20] in which the coupling of twisting and bending is taken to trigger the out-of-plane buckling, we aim in this study to explore how this coupling affects buckling of elastic rings spanned by a fluid film. We show that an instability of the chiral elastic rings arises at a critical surface tension smaller than that of the achiral ones.

The paper is divided as follows. The kinematics and energetics of the chiral filaments and spanning surface are presented in Section **2**. While first variational criteria are provided in Section **3**, buckling results of chiral elastic rings appear in Section **4**. Our salient results are summarized in Section **5**.

2. KINEMATICS AND ENERGETICS

We model chiral elastic rings as unshearable and not extensible filaments of circular cross sections with midline C. Moreover, the ring encloses a fluid film modeled as a surface S of a constant surface tension σ, depicted schematically in Fig. 1. Since the boundary ∂S of S and the midline C are coincident, we have $C = \partial S$.

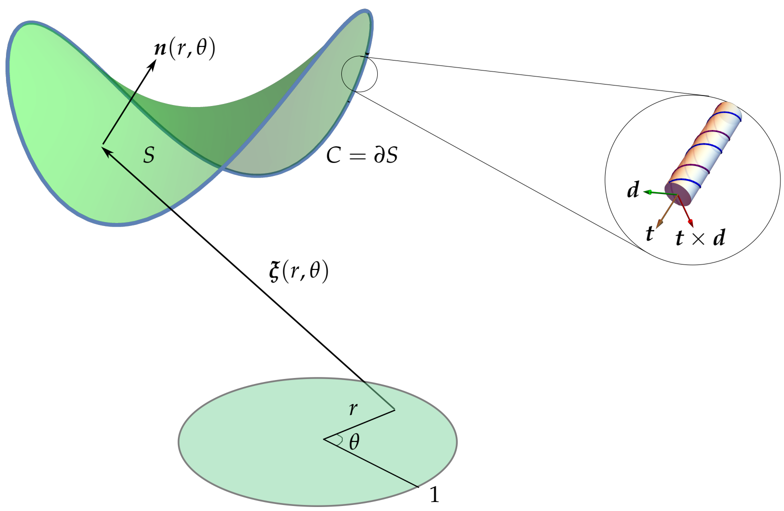

Fig. 1. An orientable surface $S = \{\boldsymbol{\xi} \in \mathbb{R}^3, \boldsymbol{\xi} = \boldsymbol{\xi}(r,\theta), 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ models a fluid film. The midline of chiral elastic ring is coincident with the boundary ∂S of S and given as $C = \{\boldsymbol{\xi} \in \mathbb{R}^3, \boldsymbol{\xi} = \boldsymbol{\xi}(1,\theta), 0 \leq \theta \leq 2\pi\}$. The surface S has unit normal $\boldsymbol{n}$ and the space curve C is provided with a material frame $\{\boldsymbol{t} \times \boldsymbol{d}, \boldsymbol{t}, \boldsymbol{d}\}$. A closed-up image of the filament shows its chirality, demonstrated as right-handedness

Throughout the paper, lengths are measured relative to R which is the radius of a circle with perimeter equal to the length $2\pi R$ of C. The surface S, thus, admits a parametrization

$$S = \{\boldsymbol{\xi} \in \mathbb{R}^3, \boldsymbol{\xi} = \boldsymbol{\xi}(r,\theta),\, 0 \le r \le 1,\, 0 \le \theta \le 2\pi\}, \tag{1}$$

where 2π is the dimensionless length of C, the vector $\boldsymbol{\xi}$ is continuously differentiable and injective up to the fourth order, and r, θ are polar coordinates on unit disc, as illustrated in Fig. 1. Denoting differentiation be indicated by subscripts, then periodicity requires that $\boldsymbol{\xi}$ and its higher derivatives are periodic with period of 2π. The midline C, then, admits a parametrization

$$C = \{\boldsymbol{\xi} \in \mathbb{R}^3, \boldsymbol{\xi} = \boldsymbol{\xi}(1,\theta),\, 0 \le \theta \le 2\pi\}, \tag{2}$$

with parameter θ being its dimensionless arclength. Denoting $\{\boldsymbol{t} \times \boldsymbol{d}, \boldsymbol{t}, \boldsymbol{d}\}$ the material frame of the filament [21], the tangent $\boldsymbol{t}$, the bending density $\boldsymbol{\kappa}$, and the twisting density ω of C are defined by [21]

$$\boldsymbol{t} = \boldsymbol{\xi}_\theta, \qquad \boldsymbol{\kappa} = \boldsymbol{t}_\theta, \qquad \omega = (\boldsymbol{d} \times \boldsymbol{d}_\theta) \cdot \boldsymbol{t}, \tag{3}$$

and the inextensibility condition of C along with the other constraints are [19]

$$|\boldsymbol{t}| = 1, \qquad |\boldsymbol{d}| = 1, \qquad \boldsymbol{t} \cdot \boldsymbol{d} = 0. \tag{4}$$

The net free-energy is [9,19]

$$E = \int_0^{2\pi} \frac{1}{2}\Big(|\boldsymbol{\kappa}|^2 + \alpha\omega^2 + 2\eta\omega\boldsymbol{\kappa}\cdot\boldsymbol{d}\Big)\,\mathrm{d}\theta + \int_0^{2\pi}\!\!\int_0^1 \nu|\boldsymbol{\xi}_r \times \boldsymbol{\xi}_\theta|\,\mathrm{d}r\mathrm{d}\theta. \tag{5}$$

where nondimensional quantities are introduced [19] as

$$\alpha = \frac{c}{a} > 0, \qquad -\sqrt{\alpha} \le \eta = \frac{e}{a} \le \sqrt{\alpha}, \qquad \nu = \frac{R^3\sigma}{a} \ge 0. \tag{6}$$

where $a, c > 0, e \neq 0$ are bending, twisting, and twisting-bending rigidities of the filament, respectively. To incorporate the constraints (4), we augment the potential energy as follows

$$\Phi = E + \int_0^{2\pi} \frac{1}{2}\Big(\lambda_1(|\boldsymbol{t}|^2 - 1) + \lambda_2(|\boldsymbol{d}|^2 - 1) + 2\lambda_3(\boldsymbol{t}\cdot\boldsymbol{d})\Big)\,\mathrm{d}\theta. \tag{7}$$

where λ_1, λ_2, and λ_3 are Lagrange multipliers.

3. EQUILIBRIUM CRITERIA

Let $\boldsymbol{v} = \delta\boldsymbol{\xi}$ and $\boldsymbol{\zeta} = \delta\boldsymbol{d}$ be smooth variations of $\boldsymbol{\xi}$ and $\boldsymbol{d}$. Equilibrium criteria are sought by setting to zero the first variation of (7). Following [9,19], we obtain

$$\begin{aligned}\delta\Phi = \int_0^{2\pi} \Big\{ &\big([(\boldsymbol{\kappa} + \eta\omega\boldsymbol{d})_\theta - (\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{d}\times\boldsymbol{d}_\theta)]_\theta + \nu(\boldsymbol{\xi}_\theta \times \boldsymbol{n}) \\ &- (\lambda_1\boldsymbol{t} + \lambda_3\boldsymbol{d})_\theta\big)\cdot\boldsymbol{v} + \big((\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{d}_\theta \times \boldsymbol{t}) - [(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{t}\times\boldsymbol{d})]_\theta \\ &+ \eta\omega\boldsymbol{\kappa} + \lambda_2\boldsymbol{d} + \lambda_3\boldsymbol{t}\big)\cdot\boldsymbol{\zeta}\Big\}\,\mathrm{d}\theta - \int_0^{2\pi}\!\!\int_0^1 \nu(\boldsymbol{\xi}_\theta \times \boldsymbol{n}_r + \boldsymbol{n}_\theta \times \boldsymbol{\xi}_r)\cdot\boldsymbol{v}\,\mathrm{d}r\mathrm{d}\theta,\end{aligned} \tag{8}$$

where $\boldsymbol{n}$ is a unit normal vector of S. Vanishing the first variation (8) leads to the areal equilibrium condition

$$\boldsymbol{\xi}_\theta \times \boldsymbol{n}_r + \boldsymbol{n}_\theta \times \boldsymbol{\xi}_r = \boldsymbol{0}, \tag{9}$$

and the two lineal equilibrium equations

$$[(\boldsymbol{\kappa} + \eta\omega\boldsymbol{d})_\theta - (\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{d}\times\boldsymbol{d}_\theta) - \lambda_1\boldsymbol{t} - \lambda_3\boldsymbol{d}]_\theta + \nu(\boldsymbol{\xi}_\theta \times \boldsymbol{n}) = \boldsymbol{0}, \tag{10}$$

and

$$(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{d}_\theta \times \boldsymbol{t}) - [(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{t}\times\boldsymbol{d})]_\theta + \eta\omega\boldsymbol{\kappa} + \lambda_2\boldsymbol{d} + \lambda_3\boldsymbol{t} = \boldsymbol{0}. \tag{11}$$

Whereas the areal equilibrium condition (9) expresses force equilibrium in the normal direction of S, the two lineal equilibrium conditions (10),(11) express, respectively force balance and moment balance on C. An alternative derivation of (10) and (11) is presented in A.1.

Multiplying both sides of equation (11) with $\boldsymbol{t}\times\boldsymbol{d}$, $\boldsymbol{t}$, and $\boldsymbol{d}$, we obtain,

$$\left.\begin{aligned} \lambda_3 &= -(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{t}\times\boldsymbol{d})\cdot\boldsymbol{\kappa}, \\ \lambda_2 &= -2(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{t}\times\boldsymbol{d})\cdot\boldsymbol{d}_\theta - \eta\omega(\boldsymbol{\kappa}\cdot\boldsymbol{d}), \end{aligned}\right\} \tag{12}$$

and

$$(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))_\theta = \eta\omega\boldsymbol{\kappa}\cdot(\boldsymbol{t}\times\boldsymbol{d}). \tag{13}$$

Substituting λ_3 in $(12)_1$ into (10) yields

$$[(\boldsymbol{\kappa} + \eta\omega\boldsymbol{d})_\theta - (\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d}))(\boldsymbol{t}\times\boldsymbol{\kappa}) - \lambda\boldsymbol{t}]_\theta + \nu(\boldsymbol{\xi}_\theta\times\boldsymbol{n}) = \boldsymbol{0}, \tag{14}$$

where a new Lagrange multiplier λ is introduced as

$$\lambda = \lambda_1 + \omega(\alpha\omega + \eta(\boldsymbol{\kappa}\cdot\boldsymbol{d})). \tag{15}$$

4. BUCKLING OF CHIRAL ELASTIC RINGS MADE FROM CHIRAL FILAMENTS AND SPANNED BY FLUID FILMS

Having determined the equilibrium conditions, we now study buckling of a ring which has a (dimensionless) constant curvature of 1, zero twist density, and is spanned by a planar surface S of the fluid film. We next introduce the Frenet frame $\{\boldsymbol{t},\boldsymbol{p},\boldsymbol{b}\}$ of the unit circle satisfying

$$\boldsymbol{t}_\theta = \boldsymbol{p}, \qquad \boldsymbol{p}_\theta = -\boldsymbol{t}, \qquad \boldsymbol{b}_\theta = \boldsymbol{0}. \tag{16}$$

The planar surface S has a parametrization

$$\boldsymbol{\xi}(r,\theta) = -r\boldsymbol{p}(\theta), \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi. \tag{17}$$

With (17), the equilibrium criteria (9), (13) are met trivially while the remaining equilibrium condition (14) is satisfied if

$$\lambda = -(1+\nu). \tag{18}$$

Now consider perturbation $\boldsymbol{v}(r,\theta)$ of the planar surface S, perturbation $\boldsymbol{\zeta}(\theta)$ of the director $\boldsymbol{d}$, and perturbation $\epsilon(\theta)$ of the Lagrange multiplier λ with $|\boldsymbol{v}| \ll 1$, $|\boldsymbol{\zeta}| \ll 1$, and $|\epsilon| \ll 1$,

$$\left.\begin{aligned} \boldsymbol{v}(r,\theta) &= u(r,\theta)\boldsymbol{t} + v(r,\theta)\boldsymbol{p} + w(r,\theta)\boldsymbol{b}, \\ \boldsymbol{\zeta}(\theta) &= h(\theta)\boldsymbol{t} + \ell(\theta)\boldsymbol{p} + k(\theta)\boldsymbol{b}, \end{aligned}\right\} \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi. \tag{19}$$

Since linearization of the constraints (4) yields

$$\boldsymbol{t} \cdot \boldsymbol{v}_\theta = 0, \qquad \boldsymbol{d} \cdot \boldsymbol{\zeta} = 0, \qquad \boldsymbol{v}_\theta \cdot \boldsymbol{d} + \boldsymbol{t} \cdot \boldsymbol{\zeta} = 0, \tag{20}$$

the components of perturbation fields $\boldsymbol{v}$ and $\boldsymbol{\zeta}$ must satisfy

$$v = u_\theta, \qquad \ell = 0, \qquad h = -(u + v_\theta)\cos\psi. \tag{21}$$

Following [9], linearization of the equilibrium condition (9) leads to the governing equation for the transverse perturbation $w(r,\theta)$.

$$w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = 0, \tag{22}$$

Similarly, following [19], linearization of the equilibrium condition (13) and (14) yields to

$$\alpha(w_\theta + k_\theta) + \eta(v_{\theta\theta} + v) = 0. \tag{23}$$

and

$$\left.\begin{aligned} &3(v_{\theta\theta} + v) + 2\eta(w_\theta + k_\theta) + \epsilon = 0, \\ &v_{\theta\theta\theta\theta} + v_{\theta\theta} + (\nu - 2)(v_{\theta\theta} + v) + \eta(w_{\theta\theta\theta} + k_{\theta\theta\theta} - k_\theta) - \epsilon = 0, \\ &w_{\theta\theta\theta\theta} + (1+\nu)w_{\theta\theta} + \nu w_r - \eta(v_\theta + v_{\theta\theta\theta}) = 0. \end{aligned}\right\} \tag{24}$$

Since the scalar Laplace equation (22) admits separable solutions of the form

$$w(r,\theta) = a_n r^n \cos n\theta, \qquad n = 0,1,2,\ldots, \tag{25}$$

we assume that the restriction of $v(1,\theta)$ to the boundary of the fluid film or the ring, $k(\theta)$, and $\epsilon(\theta)$ are of the form

$$\left.\begin{aligned} v(1,\theta) &= b_n \sin n\theta, \\ k(\theta) &= c_n \cos n\theta, \\ \epsilon(\theta) &= d_n \sin n\theta, \end{aligned}\right\} \qquad n = 0,1,2,\ldots \tag{26}$$

Substituting (25) and (26) into (23) and (24), we obtain a system of four homogeneous equations for a_n, b_n, c_n, d_n. We then eliminate c_n and d_n to obtain a system of two homogeneous equations for a_n, b_n

$$\begin{bmatrix} n\eta & \left(1 - \frac{\eta^2}{\alpha}\right)(n^2-1) - \nu \\ n(n+1) - \nu & (n+1)\eta \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27}$$

Nontrivial solutions of (27) exist if its determinant vanishes

$$f(\nu) \equiv \left[(n^2-1)\left(1 - \frac{\eta^2}{\alpha}\right) - \nu\right]\left[n(n+1) - \nu\right] - n(n+1)\eta^2 = 0. \tag{28}$$

Since $n = 0$ and $n = 1$ correspond, respectively, to rigid body translations and rotations so we consider $n \geq 2$.

Also, since

$$f(\nu) = -n(n+1)\eta^2 \leq 0, \quad \text{at} \quad \nu = (n^2-1)\left(1 - \frac{\eta^2}{\alpha}\right), \tag{29}$$

the quadratic equation (28) has two roots $\nu_{1,2}$ satisfying

$$\nu_1 \leq (n^2-1)\left(1-\frac{\eta^2}{\alpha}\right) \leq \nu_2, \qquad n \geq 2. \tag{30}$$

As we are interested in chiral elastic rings spanned by fluid films which buckle when the surface tension reaches a smallest critical value first, we, thus, consider the smaller $\nu = \nu_1$ of the two critical values $\nu = \nu_{1,2}$ of (28). From (30), we have

$$\nu_1 \leq (n^2-1)\left(1-\frac{\eta^2}{\alpha}\right) \leq n^2-1, \qquad \forall\, \alpha, \eta \quad \text{and} \quad n \geq 2. \tag{31}$$

Since $\nu = n^2 - 1$ is the critical value of surface tension at which elastic rings made from achiral filaments and spanned by fluid films buckle [8,9], we, therefore, conclude that a fluid film spanning a chiral elastic ring is less stable than that spanning an elastic ring made from achiral filaments.

The critical values ν_1 of ν as α and η vary is shown in Fig. 2. We see from this figure that the surface with $n = 2$ always stays inside surfaces with $n \geq 3$ for $\nu \geq 0$. Thus, the critical surface tension ν_1 of ν at which chiral elastic rings spanned by fluid films first buckle occurs when $n = 2$, in accordance with previous results [9,19].

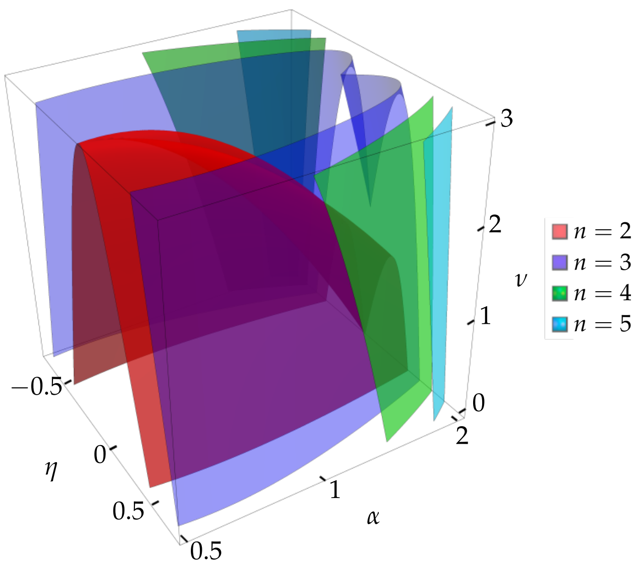

Fig. 2. A diagram showing the critical surface tension satisfying (28) at which buckling occurs for various chirality and twist-to-bend ratio for the modes $n = 2$, $n = 3$, $n = 4$, and $n = 5$

For $n = 2$ and with reference to (28), the influence of chirality on the critical surface tension ν_1 of ν for various values of α are determined and illustrated as in Fig. 3. All the curves in these figures are symmetric about the axis $\eta = 0$ implying that the critical surface tension of ν at which rings first buckle are independent of whether chirality is positive or negative. Additionally, we observe from Fig. 3 that $\nu_1 \leq 3$ and the equality

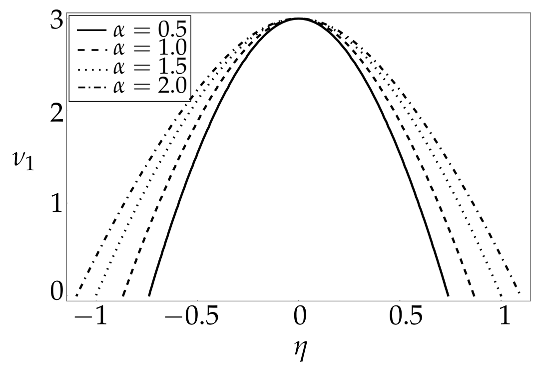

Fig. 3. A Plot showing effect of the chirality on the critical surface tension ν_1 of ν for $n = 2$ and different twist-to-bend ratio

occurs only if $\eta = 0$, regardless of values α. This results reflects the destabilizing influence of chirality of filaments on buckling of rings made from these filaments and spanned by fluid films. The destabilizing effect of chirality, however, can be reduced by increasing the twisting rigidity to the bending rigidity α, as intuitively expected, since doing so will suppress the buckling mode via which the elastic ring buckles out of its plane. This is demonstrated in Fig. 3 in which for a given value of η, a curve with a smaller value of α stays below the other curves with larger values of α.

5. SUMMARY

A variational approach is used to study buckling of chiral elastic rings spanned by fluid film. We considered rings of circular cross sections and constant material properties. Contrary to previous studies, the chirality leads to coupling of twisting and bending of the rings. Our model involves four dimensional constant: the bending rigidity $a > 0$, twisting rigidity $c > 0$, and twisting-bending coupling rigidity e of the filament, and the surface tension σ of the fluid film. Using parametrization, we reduce to three dimensionless parameters: the twist-to-bend ratio $\alpha = c/a > 0$, the chirality $\eta = e/a$, and the surface tension $\nu = \sigma R^3/a \geq 0$. We find that the critical surface tension of the fluid film at which buckling occurs depends only on the degree of chirality, no matter it is left-handed or right-handed chirality. Moreover, the chirality always has a negative influence on buckling behavior, yielding to critical values of surface tension less than those for achiral elastic rings spanned by fluid films ($\nu(\eta) < \nu(\eta = 0) = 3$). The destabilizing effect of chirality, however, can be reduced by increasing twisting rigidity to bending rigidity.

ACKNOWLEDGEMENTS

This work was supported by Graduate University of Science and Technology under Grant No. GUST.STS.ĐT2019-CH01 and Vietnam National Foundation for Science and Technology Development under Grant No. 107.02-2020.11.

REFERENCES

[1] M. M. Lévy. Mémoire sur un nouveau cas intégrable du probléme de l'élastique et lune de ses applications. *Journal of Pure and Applied Mathematics*, **10**, (1884), pp. 5–42.

[2] R. Hashemi and R. Avazmohammadi. Surface effects on stability of nanorings under uniform pressure. *Journal of Applied Physics*, **114**, (2013). https://doi.org/10.1063/1.4824818.

[3] D. F. Windenburg and C. Trilling. Collapse by instability of thin cylindrical shells under external pressure. *Transactions of the American Society of Mechanical Engineers*, **56**, (11), (1934), pp. 819–825.

[4] T. Savin, N. A. Kurpios, A. E. Shyer, P. Florescu, H. Liang, L. Mahadevan, and C. J. Tabin. On the growth and form of the gut. *Nature*, **476**, (7358), (2011), pp. 57–62. https://doi.org/10.1038/nature10277.

[5] A. Catte, J. C. Patterson, M. K. Jones, W. G. Jerome, D. Bashtovyy, Z. Su, F. Gu, J. Chen, M. P. Aliste, S. C. Harvey, L. Li, G. Weinstein, and J. P. Segrest. Novel Changes in Discoidal High Density Lipoprotein Morphology: A Molecular Dynamics Study. *Biophysical Journal*, **90**, (12), (2006), pp. 4345 –4360. https://doi.org/10.1529/biophysj.105.071456.

[6] J. Plateau. Récherches expérimentales et théoriques sur les figures d'équilibre d'une masse liquide sans pesanteur, Septième série. *Mémoires de l'Académie royale des sciences, des lettres et des beaux-arts de Belgique, Classe des sciences*, **XXXVI**, (1866), pp. 1–66.

[7] F. Bernatzki and R. Ye. Minimal surfaces with an elastic boundary. *Annals of Global Analysis and Geometry*, **19**, (2001), pp. 1–9.

[8] L. Giomi and L. Mahadevan. Minimal surfaces bounded by elastic lines. *Proceedings of the Royal Society of London Series A, Mathematical, Physical and Engineering Sciences*, **468**, (2012), pp. 1851–1864. https://doi.org/10.1098/rspa.2011.0627.

[9] Y.-C. Chen and E. Fried. Stability and bifurcation of a soap film spanning a flexible loop. *Journal of Elasticity*, **116**, (1), (2014), pp. 75–100.

[10] A. Biria and E. Fried. Buckling of a soap film spanning a flexible loop resistant to bending and twisting. *Proceedings of the Royal Society of London Series A, Mathematical, Physical and Engineering Sciences*, **470**, (2014), pp. 1–17. https://doi.org/10.1098/rspa.2014.0368.

[11] A. Biria and E. Fried. Theoretical and experimental study of the stability of a soap film spanning a flexible loop. *International Journal of Engineering Science*, **94**, (2015), pp. 86–102. https://doi.org/10.1016/j.ijengsci.2015.05.002.

[12] G. Giusteri, P. Franceschini, and E. Fried. Instability paths in the Kirchhoff–Plateau problem. *Journal of Nonlinear Science*, **26**, (4), (2016), pp. 1097–1132. https://doi.org/10.1007/s00332-016-9299-4.

[13] T. M. Hoang and E. Fried. Influence of a spanning liquid film on the stability and buckling of a circular loop with intrinsic curvature or intrinsic twist density. *Mathematics and Mechanics of Solids*, **23**, (1), (2018), pp. 43–66.

[14] P. N. R. D. Kamien, T. C. Lubensky and C. S. O'Hern. Direct determination of DNA twist-stretch coupling. *Europhys. Lett.*, **38**, (3), (1997), pp. 237–242. https://doi.org/10.1209/epl/i1997-00231-y.

[15] J. D. Moroz and P. Nelson. Torsional directed walks, entropic elasticity, and DNA twist stiffness. *Proc. Natl. Acad. Sci. USA*, **94**, (1997), pp. 14418–14422. https://doi.org/10.1073/pnas.94.26.14418.

[16] B. Đuričković, A. Goriely, and J. H. Maddocks. Twist and stretch of helices explained via the Kirchhoff-Love rod model of elastic filaments. *Physical Review Letters*, **111**, (2013). https://doi.org/10.1103/physrevlett.111.108103.

[17] J. F. Marko and E. D. Sigga. Bending and twisting elasticity of DNA. *Macromolecules*, **27**, (1994), pp. 981–988. https://doi.org/10.1021/ma00082a015.
[18] J. F. Marko. Stretching must twist DNA. *Europhys. Lett.*, **38**, (3), (1997), pp. 183–188. https://doi.org/10.1209/epl/i1997-00223-5.
[19] T. M. Hoang. Influence of chirality on buckling and initial postbuckling of inextensible rings subject to central loadings. *International Journal of Solids and Structures*, **172-173**, (2019), pp. 97–109. https://doi.org/10.1016/j.ijsolstr.2019.03.029.
[20] T. M. Hoang. Postbuckling of chiral elastic rings with intrinsic twist. *International Journal of Solids and Structures*, **225**, (2021). https://doi.org/10.1016/j.ijsolstr.2021.03.020.
[21] S. S. Antman. *Nonlinear Problems of Elasticity*. Springer, (1995).
[22] A. Goriely. Twisted elastic rings and the rediscoveries of Michell's instability. *Journal of Elasticity*, **84**, (3), (2006), pp. 281–299. https://doi.org/10.1007/s10659-006-9055-3.

APPENDIX A.

A.1. Another derivation of equilibrium conditions

Alternative derivation of the equilibrium criteria (10) and (11) can be done using balance of force and moment on C [21]:

$$\begin{cases} \boldsymbol{n}' + \boldsymbol{f} = \boldsymbol{0}, \\ \boldsymbol{m}' + \boldsymbol{t} \times \boldsymbol{n} = \boldsymbol{0}, \end{cases} \tag{A.1}$$

where $\boldsymbol{f} = -\nu(\boldsymbol{\xi}_\theta \times \boldsymbol{n})$ is nondimensional surface tension of the fluid. The internal moment is expressed in terms of strains for an unshearable, inextensible, and chiral ring [19] as follows

$$\boldsymbol{m} = (\alpha\omega + \eta\Omega_2)\boldsymbol{t} + \Omega_1\boldsymbol{d} + (\Omega_2 + \eta\omega)\boldsymbol{t} \times \boldsymbol{d}, \tag{A.2}$$

where

$$\Omega_1 = -\boldsymbol{\kappa} \cdot \boldsymbol{t} \times \boldsymbol{d}, \qquad \Omega_2 = \boldsymbol{\kappa} \cdot \boldsymbol{d}. \tag{A.3}$$

Repeating [22], we calculate

$$\begin{aligned} \boldsymbol{m}' = (\alpha\omega' + \eta(\omega\Omega_1 + \Omega_2'))\boldsymbol{t} + (\Omega_1' + (\alpha-1)\omega\Omega_2 + \eta(\Omega_2^2 - \omega^2))\boldsymbol{d} \\ + (\Omega_2' + \eta(\omega' - \Omega_1\Omega_2) - (\alpha-1)\omega\Omega_1)\boldsymbol{t} \times \boldsymbol{d}, \end{aligned} \tag{A.4}$$

and then plug it into $(A.1)_2$ to get

$$\alpha\omega' + \eta(\omega\Omega_1 + \Omega_2') = 0 \tag{A.5}$$

and

$$\boldsymbol{n} = \mu\boldsymbol{t} - (\Omega_2' + \eta(\omega' - \Omega_1\Omega_2) - (\alpha-1)\omega\Omega_1)\boldsymbol{d} + (\Omega_1' + (\alpha-1)\omega\Omega_2 + \eta(\Omega_2^2 - \omega^2))\boldsymbol{t} \times \boldsymbol{d}, \tag{A.6}$$

where μ is an unknown constant. Evidently, the equilibrium criterion (A.5) is identical with (13) describing moment balance in the tangential direction on C. Therefore, to prove that the force balance condition (14) coincides with $(A.1)_1$, it is sufficient to prove that

$$\boldsymbol{n} = -(\boldsymbol{\kappa} + \eta\omega\boldsymbol{d})_\theta + (\alpha\omega + \eta(\boldsymbol{\kappa} \cdot \boldsymbol{d}))(\boldsymbol{t} \times \boldsymbol{\kappa}) + \lambda\boldsymbol{t}. \tag{A.7}$$

By directly substituting (A.3) into (A.6) and requiring that the Lagrange multipliers λ, μ are related as

$$\mu = \lambda + |\boldsymbol{\kappa}|^2 + \alpha\omega^2 + 2\eta\omega(\boldsymbol{\kappa} \cdot \boldsymbol{d}), \tag{A.8}$$

we confirm that (A.6) coincides with (A.7).