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CRACK IDENTIFICATION IN BEAM BY ANTIRESONANT FREQUENCIES

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Abstract. The present paper deals with the concept of antiresonance in multiple cracked beams and application for multi-crack identification. First, governing equations for antiresonant frequency are conducted and used for both computing antiresonant frequencies versus crack parameters and measuring-loading colocation and identifying cracks by measured antiresonant frequencies. Then, a procedure is proposed for crack identification in cantilever beam by antiresonant frequencies based on the so-called crack scanning method. Theoretical development is illustrated by numerical examples.

Keywords: cracked beam, antiresonant frequency, crack identification.

1. INTRODUCTION

Resonance is the well known concept for both researchers and engineers in the field of vibration theory and engineering. This is the phenomena happened when frequency of external force approaches to natural frequency of a system and it leads amplitude of forced vibration to reach its maximum (peak). The frequency at which the forced vibration amplitude attains its maximum is called resonant frequency. The resonant frequencies of undamped dynamic systems are identical to natural frequency and therefore, the resonant frequencies have been comprehensively studied in the literature on vibration of machines and structures. Conversely, antiresonance is the phenomena that occurs when external force frequency makes the forced vibration amplitude vanished or reached its minimum and location of the vibration amplitude's zero or minimum is termed by antiresonant frequency. Obviously, antiresonance may appears only in multi-degree of freedom systems or continuous-elastic structures and from mathematical point of view there should be found at least one antiresonant frequency between two neighboring resonant frequencies.The antiresonant frequencies have found some application in the vibration control problems [1–3] and model updating [4–7], but they have not adequately and systematically studied with the aim to use for structural health monitoring [8–11]. The present paper is devoted to establish basic equations for computing antiresonant frequencies of multiple cracked beam and simple diagnostic equations for crack identification in beam by measured antiresonant frequencies.

First attempt to systematically investigate antiresonant frequencies of cantilever beam was accomplished by Wang et al. [12] who have revealed that antiresonant frequency measured at node of vibration mode is identical to resonant frequency corresponding to the vibration mode. Bamnios et al. [13] have studied effect of either crack or driving (loading) point on mechanical impedance and extracted from that antiresonant frequencies of cantilever and clamped end beam. They concluded that driving point of mechanical impedance could provide additional information useful for crack detection. This impedance method has been applied recently for crack identification in simply supported Euler–Bernoulli [14] and Timoshenko [15] beams. Dilena and Morassi [16–19] demonstrated that antiresonant frequencies used mutually with resonant frequencies are capable to solve numerous problems of crack identification in rods and beams that could not be solved by using only natural frequencies. Moreover, Meruane and Heylen [20] obtained an attractive result that antiresonant frequencies used mutually with natural ones are more efficient than using mode shapes in combination with natural frequencies for structural damage assessment, while antiresonant frequencies are more easily and accurately measured than the mode shapes.

The authors of Refs. [21, 22] have studied resonant and antiresonant frequencies of multiple cracked bars and proposed a useful procedure for crack identification in bar based on both resonant and antiresonant frequency equations. The present study aims at expanding the obtained in [21, 22] results for beam structures. Namely, first explicit equations for antiresonant frequency are established for multiple cracked beams that allow one to examine antiresonant frequencies in dependence upon not only crack location and depth but also measuring and driving collocation. Then, the simplest form of the antiresonant frequency equations are employed for solving the multi-crack identification from first antiresonant frequencies measured at various collocations using the so-called crack scanning method [23, 24]. Theoretical development has been illustrated by numerical examples.

2. FREQUENCY RESPONSE FUNCTION OF MULTIPLE CRACKED BEAM

It was well known that frequency response function measured at x under point load applied at location x_0 of a beam is defined as solution of the equation

$$EI\partial^4 W\left(x,t\right)/\partial x^4 + \rho F \partial^2 W\left(x,t\right)/\partial x^2 = P(t)\delta\left(x - x_0\right), \quad x \in (0,\ell) \tag{1}$$

with boundary conditions generally represented by

$$W^{(p_0)}\left(0,t\right) = W^{(q_0)}\left(0,t\right) = W^{(p_1)}\left(\ell,t\right) = W^{(q_1)}\left(\ell,t\right) = 0, \tag{2}$$

where $W^{(r)}\left(x,t\right)$ is derivative of function $W(x,t)$ with respect to spatial variable x and derivative order $r = p_0, q_0, p_1, q_1$ could be equal to one of the values 0, 1, 2, 3.

Under Fourier transform the equations (1) and (2) become

$$\phi^{(IV)}(x) - \lambda^4 \phi(x) = \overline{P}(\omega)\,\delta(x - x_0), x \in (0,1), \tag{3}$$

$$\lambda^4 = \omega^2 \rho F \ell^4 / EI, \quad \overline{P}(\omega) = P(\omega)\,\ell^4 / EI,$$

$$\phi^{(p_0)}(0) = \phi^{(q_0)}(0) = \phi^{(p_1)}(1) = \phi^{(q_1)}(1) = 0, \tag{4}$$

where

$$\phi(x) = \int_{-\infty}^{\infty} W(x,t)\,e^{-i\omega t} dt, \quad P(\omega) = \int_{-\infty}^{\infty} P(t)\,e^{-i\omega t} dt.$$

Furthermore, the beam is assumed to be damaged to n cracks of depth $a_1, \ldots, a_n$ at the locations $e_1, \ldots, e_n$ $(0 \le e_1 < \ldots < e_n \le 1)$ so that solution of Eq. (3) must satisfy also the following conditions at the crack locations

$$\begin{aligned} &\phi(e_j + 0) = \phi(e_j - 0), \phi'(e_j + 0) - \phi'(e_j - 0) = \gamma_j \phi''(e_j), \\ &\phi''(e_j + 0) = \phi''(e_j - 0) = \phi''(e_j), \quad \phi'''(e_j + 0) = \phi'''(e_j - 0). \end{aligned} \tag{5}$$

The parameter γ_j, acknowledged as crack magnitude calculated from crack depth as

$$\gamma_j = EI/R = 6\pi\left(1 - \nu_0^2\right) h f_c(z), \quad z = a/h,$$

$$f_c(z) = z^2 \Big(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + 47.1063z^6 - 40.7556z^7 + 19.6z^8\Big).$$

As well-known also in theory of differential equations, general solution of Eq. (3) can be found in the form

$$\phi(x) = \phi_0(x) + \phi_Q(x), \tag{6}$$

where $\phi_0(x,\omega)$ is general solution of homogeneous equation

$$\phi^{(IV)}(x) - \lambda^4 \phi(x) = 0, \tag{7}$$

and $\phi_Q(x,\omega)$ is a particular solution

$$\phi_Q(x) = \overline{P}(\omega) \int_0^x H(x-s)\,\delta(s - x_0)\,ds = \overline{P}(\omega)\,H(x - x_0) \tag{8}$$

with

$$H(x) = \begin{cases} 0 & : x \le 0 \\ h(x) & : x > 0 \end{cases}, \quad h(x) = [\sinh \lambda x - \sin \lambda x] / 2\lambda^3. \tag{9}$$

On the other hand, general solution of Eq. (7) satisfying conditions (5) inside the beam span could be found in the form [23]

$$\phi_0(x,\omega) = C\phi_1(x,\omega) + D\phi_2(x,\omega), \tag{10}$$

where C, D are constants and

$$\phi_1(x,\omega) = L_1(x,\lambda) + \sum_{j=1}^{n} \mu_{1j} K(x - e_j, \lambda), \phi_2(x,\omega) = L_2(x,\lambda) + \sum_{j=1}^{n} \mu_{2j} K(x - e_j, \lambda), \tag{11}$$

with the damage index vectors $\boldsymbol{\mu}_1 = (\mu_{11}, \ldots, \mu_{1n})^T, \boldsymbol{\mu}_2 = (\mu_{21}, \ldots, \mu_{2n})^T$ determined subsequently by recurrent relationships

$$\mu_{1j} = \gamma_j \left[L_1'' (e_j, \lambda) + \sum_{k=1}^{j-1} \mu_{1j} S'' (e_j - e_k, \lambda) \right], \mu_{2j} = \gamma_j \left[L_2'' (e_j, \lambda) + \sum_{k=1}^{j-1} \mu_{2j} S'' (e_j - e_k, \lambda) \right],$$
$$K(x, \lambda) = \begin{cases} 0, & x \le 0 \\ S(x, \lambda), & x \succ 0 \end{cases}, \quad S(x, \lambda) = [\sinh \lambda x + \sin \lambda x] / 2\lambda. \tag{12}$$

Functions $L_1(x, \lambda)$, $L_2(x, \lambda)$ in Eq. (11) are continuous solutions of Eq. (7) satisfying boundary conditions at the left end, $x = 0$, that are given in Table 1 for different boundary conditions. It is easily to verify that functions defined in Eq. (11) also satisfy boundary conditions at the left end of beam likely the functions $L_1(x, \lambda)$, $L_2(x, \lambda)$. Hence, it remains to satisfy only boundary conditions at the right end of the beam as

$$\begin{cases} C\phi_1^{(p_1)}(1) + D\phi_2^{(p_1)}(1) = -\overline{P}(\omega) h^{(p_1)}(1 - x_0), \\ C\phi_1^{(q_1)}(1) + D\phi_2^{(q_1)}(1) = -\overline{P}(\omega) h^{(q_1)}(\ell - x_0), \end{cases} \tag{13}$$

where p_1, q_1 are also given in Table 1. Obviously, Eqs. (13) allow one to find two constants C, D as

$$C = \overline{P}(\omega) C_1(x_0) / D(\lambda), D = \overline{P}(\omega) D_1(x_0) / D(\lambda) \tag{14}$$

with $D(\lambda) = \phi_1^{(p_1)}(1)\phi_2^{(q_1)}(1) - \phi_1^{(q_1)}(1)\phi_2^{(p_1)}(1)$ and

$$C_1(x_0) = \left[h^{(q_1)}(1 - x_0)\phi_2^{(p_1)}(1) - h^{(p_1)}(1 - x_0)\phi_2^{(q_1)}(1) \right],$$
$$D_1(x_0) = \left[h^{(p_1)}(1 - x_0)\phi_1^{(q_1)}(1) - h^{(q_1)}(1 - x_0)\phi_1^{(p_1)}(1) \right].$$

Table 1. Boundary functions and derivative orders

Boundary conditions	$L_1(x, \lambda)$	$L_2(x, \lambda)$	p_1	q_1
Clamped ends	$\sinh \lambda x - \sin \lambda x$	$\cosh \lambda x - \cos \lambda x$	0	1
Free ends	$\sinh \lambda x + \sin \lambda x$	$\cosh \lambda x + \cos \lambda x$	2	3
Symply sopported ends	$\sinh \lambda x$	$\sin \lambda x$	0	2
Cantilever	$\sinh \lambda x - \sin \lambda x$	$\cosh \lambda x - \cos \lambda x$	2	3

Thus, solution of Eq. (3) satisfying conditions (4) and (5) is found in the form

$$\phi(x, x_0, \omega) = \left[\overline{P}(\omega) / D(\lambda)\right] [C_1(x_0)(x) + D_1(x_0)\phi_2(x) + D(\lambda) H(x - x_0)], \tag{15}$$

and frequency response function of the damaged beam is therefore obtained as

$$\begin{aligned} FRF(x, x_0, \omega) &= \phi(x, x_0, \omega) / \overline{P}(\omega) \\ &= [C_1(x_0)\phi_1(x) + D_1(x_0)\phi_2(x) + D(\lambda) H(x - x_0)] / D(\lambda). \end{aligned} \tag{16}$$

3. ANTIRESONANT FREQUENCY EQUATION FOR MULTIPLE CRACKED BEAM

Recalling resonant and antiresonant frequencies defined as poles and zeros respectively of frequency response function, they are determined from so-called resonant and antiresonant frequency equations

$$\begin{aligned} &D(\lambda) = \phi_1^{(p_1)}(1)\,\phi_2^{(q_1)}(1) - \phi_1^{(q_1)}(1)\,\phi_2^{(p_1)}(1) = 0, \\ &A(x, x_0, \lambda) \equiv C_1(x_0)\,\phi_1(x) + D_1(x_0)\,\phi_2(x) + D(\lambda)\,H(x - x_0) = 0. \end{aligned} \tag{17}$$

Note, both the resonant and antiresonant frequencies determined from the above equations are related to parameter $\lambda = \sqrt[4]{\omega^2 \rho F \ell^4 / EI}$, well known as frequency parameter. So, solution of Eq. (17) with respect to λ is called respectively resonant and antiresonant frequency parameter.

Obviously, resonant frequencies or natural frequencies could be found in dependence upon only boundary conditions, while antiresonant frequencies are strongly dependent on where the frequency response is measured (cosh x) and where the point load is applied (x_0). The response measured location and load applied location are acknowledged herein as measuring and loading locations respectively. It can be noted that antiresonant frequencies may not be found for arbitrarily chosen the measuring and loading locations. Therefore, for simplification, antiresonant frequency is sough from frequency response function detemined for identical measuring and loading locations, i.e. $x = x_0$. Such identical location is called driving-measuring collocation or shortly collocation, keeping its notation by x. In the case, equation for seeking antiresonant frequencies is

$$A(x, \lambda) \equiv C_1(x)\,\phi_1(x, \lambda) + D_1(x)\,\phi_2(x, \lambda) = 0,$$

or

$$A(x, \lambda) \equiv h^{(q_1)}(1 - x)\,A_p(x, \lambda) - h^{(p_1)}(1 - x)\,A_q(x, \lambda) = 0, \tag{18}$$

where

$$\begin{aligned} A_p(x, \lambda) &= \left[\phi_2^{(p_1)}(1)\,\phi_1(x) - \phi_1^{(p_1)}(1)\,\phi_2(x)\right], \\ A_q(x, \lambda) &= \left[\phi_2^{(q_1)}(1)\,\phi_1(x) - \phi_1^{(q_1)}(1)\,\phi_2(x)\right]. \end{aligned}$$

First, using expressions (11) functions $A_p\left(x, \overline{\lambda}\right), A_q\left(x, \overline{\lambda}\right)$ can be rewritten as

$$\begin{aligned} A_p(x, \lambda) &= A_0^p(x, \lambda) + \sum_{j=1}^{n}\left[A_2^p(x, \lambda, e_j)\,\mu_{1j} - A_1^p(x, \lambda, e_j)\,\mu_{2j}\right] + \sum_{j,k=1}^{n} SK_p(x, \lambda, e_j, e_k)\,\mu_{1j}\mu_{2k}, \\ A_q(x, \lambda) &= A_0^q(x, \lambda) + \sum_{j=1}^{n}\left[A_2^q(x, \lambda, e_j)\,\mu_{1j} - A_1^q(x, \lambda, e_j)\,\mu_{2j}\right] + \sum_{j,k=1}^{n} SK_q(x, \lambda, e_j, e_k)\,\mu_{1j}\mu_{2k}, \end{aligned} \tag{19}$$

where

$$
\begin{aligned}
A_0^{p,q}(x,\lambda) &= L_2^{(p,q)}(\lambda)\,L_1(x) - L_1^{(p,q)}(\lambda)\,L_2(x),\\
A_i^{p,q}(x,\lambda,e_j) &= L_i^{(p,q)}(\lambda)\,K(x-e_j) - L_i(x)\,S^{(p,q)}(1-e_j),\, i=1,2,\\
SK_{p,q}(x,\lambda,e_j,e_k) &= K(x-e_j)\,S^{(p,q)}(1-e_k) - K(x-e_k)\,S^{(p,q)}(1-e_j).
\end{aligned}
$$

Putting (19) into (18) leads the latter equation to

$$
A(x,\lambda) = A_0(x,\lambda) + \sum_{j=1}^{n}\left[A_2(x,\overline{\lambda},e_j)\,\mu_{1j} - A_1(x,\lambda,e_j)\,\mu_{2j}\right] + \sum_{j,k=1}^{n} SK(x,\lambda,e_j,e_k)\,\mu_{1j}\mu_{2k} = 0, \tag{20}
$$

where

$$
\begin{aligned}
A_0(x,\lambda) &= h^{(q_1)}(1-x)\,A_0^{p_1}(x,\lambda) - h^{(p_1)}(1-x)\,A_0^{q_1}(x,\lambda),\\
A_i(x,\lambda,e_j) &= h^{(q_1)}(1-x)\,A_i^{p_1}(x,\lambda,e_j) - h^{(p_1)}(1-x)\,A_i^{q_1}(x,\lambda,e_j),\, i=1,2,\\
SK(x,\lambda,e_j,e_k) &= h^{(q_1)}(1-x)\,SK_{p_1}(x,\lambda,e_j,e_k) - h^{(p_1)}(1-x)\,SK_{q_1}(x,\lambda,e_j,e_k).
\end{aligned}
$$

It is easily to verify that $SK(x,\lambda,e_j,e_j) = 0$ and antiresonant frequency equation can be represented in the form

$$
R_0(\lambda) + \sum_{j=1}^{n}\left[R_{2j}(\lambda)\,\mu_{1j} - R_{1j}(\lambda)\,\mu_{2j}\right] + \sum_{j,k=1}^{n} Q_{jk}(\lambda)\,\mu_{1j}\mu_{2k} = 0, \tag{21}
$$

where

$$
\begin{aligned}
&R_0(\lambda) = A_0(x,\lambda), \quad R_{1j}(\lambda) = A_1(x,\lambda,e_j),\\
&R_{2j}(\lambda) = A_2(x,\lambda,e_j), \quad Q_{jk}(\lambda) = SK(x,\lambda,e_j,e_k).
\end{aligned} \tag{22}
$$

Now, the damage indexes (12) are rewritten as

$$
\begin{aligned}
&\mu_{11} = \gamma_1 L_1''(e_1), \quad \mu_{21} = \gamma_1 L_2''(e_1),\\
&\mu_{12} = \gamma_2 L_1''(e_2) + \gamma_2\gamma_1 S''(e_2-e_1)\,L_1''(e_1), \mu_{22} = \gamma_2 L_2''(e_2) + \gamma_2\gamma_1 S''(e_2-e_1)\,L_2''(e_1),\\
&\mu_{i3} = \gamma_3 L_i''(e_3) + \gamma_3\gamma_2 S''(e_3-e_2)\,L_i''(e_2) + \gamma_3\gamma_1 S''(e_3-e_1)\,L_i''(e_1) +\\
&\qquad + \gamma_3\gamma_2\gamma_1 S''(e_3-e_2)\,S''(e_2-e_1)\,L_i''(e_1), i=1,2,\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mu_{ij} = \gamma_j\Big[L_i''(e_j) + \sum_{k=1}^{j-1}\gamma_k S''(e_j-e_k)\,L_i''(e_k) + \sum_{k_1=2}^{j-1}\sum_{k_2=1}^{k_1-1}\gamma_{k_1}\gamma_{k_2} S''(e_j-e_{k_1})\,S''(e_{k_1}-e_{k_2})\,L_i''(e_{k_2})\\
&\qquad + \sum_{k_1=3}^{j-1}\sum_{k_2=2}^{k_1-1}\sum_{k_3=1}^{k_2-1}\gamma_{k_1}\gamma_{k_2}\gamma_{k_3} S''(e_j-e_{k_1})\,S''(e_{k_1}-e_{k_2})\,S''(e_{k_2}-e_{k_3})\,L_i''(e_{k_3}) + \ldots\\
&\qquad + \gamma_{j-1}\gamma_{j-2}\ldots\gamma_1 S''(e_j-e_{j-1})S''(e_{j-1}-e_{j-2})\ldots S''(e_2-e_1)\,L_i''(e_1)\Big], i=1,2, j=1,2,\ldots,n.
\end{aligned} \tag{23}
$$

Substituting (23) into (21) one gets

$$R_0(\lambda) + \sum_{j=1}^{n} R_1(\lambda, e_j)\,\gamma_j + \sum_{j=2}^{n}\sum_{k=1}^{j-1} R_2(\lambda, e_j, e_k)\,\gamma_j\gamma_k + \sum_{3=1}^{n}\sum_{k=2}^{j-1}\sum_{r=1}^{k-1} \gamma_j\gamma_k\gamma_r R_3(\lambda, e_j, e_k, e_r)$$
$$+ \ldots + R_n(\lambda, e_n, e_{n-1}, \ldots, e_1)\,\gamma_n\gamma_{n-1}\ldots\gamma_1 = 0, \tag{24}$$

where

$$\begin{aligned}
R_1(\lambda, e_j) &= R_{2j}(\lambda)\,L_1''(e_j) - R_{1j}(\lambda)\,L_2''(e_j),\\
R_2(\lambda, e_j, e_k) &= S''(e_j - e_k)\left[R_{2k}(\lambda)\,L_1''(e_k) - R_{1k}(\lambda)\,L_2''(e_k)\right]\\
&\quad + Q_{jk}(\lambda)\left[L_1''(e_j)\,L_2''(e_k) - L_1''(e_k)\,L_2''(e_j)\right],\\
R_3(\lambda, e_j, e_k, e_r) &= S''(e_j - e_k)\left[R_{2r}(\lambda)\,L_1''(e_r) - R_{1r}(\lambda)\,L_2''(e_r)\right]\\
&\quad + Q_{jk}(\lambda)\begin{bmatrix} S''(e_j - e_r)\left[L_1''(e_r)\,L_2''(e_k) - L_1''(e_k)\,L_2''(e_r)\right] - \\ -S''(e_k - e_r)\left[L_2''(e_r)\,L_1''(e_j) - L_2''(e_j)\,L_1''(e_r)\right]\end{bmatrix},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$
$$R_n(\lambda, e_n, e_{n-1}, \ldots, e_1) = S''(e_n - e_{n-1})\,S''(e_n - e_{n-1})\ldots S''(e_2 - e_1)\left[R_{21}(\lambda)\,L_1''(e_1) - R_{11}(\lambda)\,L_2''(e_1)\right]. \tag{25}$$

Thus, in case of single, double and triple cracks the exact characteristic equation gets to be

$$R_0(\lambda) + \gamma R_1(\lambda, e) = 0, \tag{26}$$
$$R_0(\lambda) + \gamma_1 R_1(\lambda, e_1) + \gamma_2 R_1(\lambda, e_2) + \gamma_2\gamma_1 R_2(\lambda, e_2, e_1) = 0, \tag{27}$$
$$R_0(\lambda) + \gamma_1 R_1(\lambda, e_1) + \gamma_2 R_1(\lambda, e_2) + \gamma_3 R_1(\lambda, e_3) + \gamma_3\gamma_2 R_2(\lambda, e_3, e_2)$$
$$+ \gamma_2\gamma_1 R_2(\lambda, e_2, e_1) + \gamma_3\gamma_1 R_2(\lambda, e_3, e_1) + \gamma_3\gamma_2\gamma_1 R_3(\lambda, e_3, e_2, e_1) = 0, \tag{28}$$

with coefficients $R_1(\lambda, e_1), R_2(\lambda, e_2, e_1), R_3(\lambda, e_3, e_2, e_1)$ are defined in (25).

In case of multiple cracked beam with small magnitude of cracks, first and second asymptotic approximations of the characteristic equation are respectively

$$R_0(x, \lambda) + \sum_{j=1}^{n} R_1(x, \lambda, e_j)\,\gamma_j = 0, \tag{29}$$

$$R_0(x, \lambda) + \sum_{j=1}^{n} R_1(x, \lambda, e_j)\,\gamma_j + \sum_{j=2}^{n}\sum_{k=1}^{j-1} R_2(x, \lambda, e_j, e_k)\,\gamma_j\gamma_k = 0. \tag{30}$$

The latter equations are first obtained herein and they will be employed below for developing a crack identification procedure in beam by antiresonant frequencies. For this purpose, the coefficients $R_0(x, \lambda)$, $R_1(x, \lambda, e_j)$, $R_2(x, \lambda, e_j, e_k)$ are recalled for antiresonant frequencies as follow:

$$
\begin{aligned}
R_0(x,\lambda) &= h^{(q_1)}(1-x)\left[L_2^{(p_1)}(\lambda)L_1(x) - L_1^{(p_1)}(\lambda)L_2(x)\right] \\
&\quad + h^{(p_1)}(1-x)\left[L_1^{(q_1)}(\lambda)L_2(x) - L_2^{(q_1)}(\lambda)L_1(x)\right], \\
R_1(x,\lambda,e_j) &= h^{(q_1)}(1-x)\left\{\begin{array}{c} K(x-e_j)\left[L_2^{(p_1)}(\lambda)L_1''(e_j) - L_1^{(p_1)}(\lambda)L_2''(e_j)\right] + \\ +S^{(p_1)}(1-e_j)\left[L_2''(e_j)L_1(x) - L_1''(e_j)L_2(x)\right] \end{array}\right\} \\
&\quad + h^{(p_1)}(1-x)\left\{\begin{array}{c} K(x-e_j)\left[L_1^{(q_1)}(\lambda)L_2''(e_j) - L_2^{(q_1)}(\lambda)L_1''(e_j)\right] + \\ +S^{(q_1)}(1-e_j)\left[L_1''(e_j)L_2(x) - L_2''(e_j)L_1(x)\right] \end{array}\right\}, \\
R_2(x,\lambda,e_j,e_k) &= S''(e_j-e_k)\left[M_1(x,\lambda,e_k)K(x-e_k) + M_2(x,\lambda,e_k)S^{(p_1)}(1-e_k)\right] \\
&\quad + \left[N(x,\lambda,e_j)K(x-e_k) + N(x,\lambda,e_k)K(x-e_j)\right]\left[L_1''(e_j)L_2''(e_k) - L_2''(e_j)L_1''(e_k)\right]
\end{aligned} \tag{31}
$$

with

$$
\begin{aligned}
M_1(x,\lambda,e_k) &= h^{(q_1)}(1-x)\left[L_2^{(p_1)}(\lambda)L_1''(e_k) - L_1^{(p_1)}(\lambda)L_2''(e_k)\right] \\
&\quad + h^{(p_1)}(1-x)\left[L_1^{(q_1)}(\lambda)L_2''(e_k) - L_2^{(q_1)}(\lambda)L_1''(e_k)\right], \\
M_2(x,\lambda,e_k) &= h^{(q_1)}(1-x)\left[L_1(x)L_2''(e_k) - L_2(x)L_1''(e_k)\right] \\
&\quad + h^{(p_1)}(1-x)\left[L_2(x)L_1''(e_k) - L_1(x)L_2''(e_k)\right], \\
N(x,\lambda,e) &= h^{(q_1)}(1-x)S^{(p_1)}(1-e) - h^{(p_1)}(1-x)S^{(q_1)}(1-e).
\end{aligned} \tag{32}
$$

The latter equations would be used below for developing a procedure for multi-crack identification by antiresonant frequencies.

$$
\begin{aligned}
A(\overline{x},\lambda) \equiv h^{(q_1)}(1-\overline{x})&\left[\phi_2^{(p_1)}(1)\phi_1(\overline{x}) - \phi_1^{(p_1)}(1)\phi_2(\overline{x})\right] \\
- h^{(p_1)}(1-\overline{x})&\left[\phi_2^{(q_1)}(1)\phi_1(\overline{x}) - \phi_1^{(q_1)}(1)\phi_2(\overline{x})\right] = 0.
\end{aligned} \tag{33}
$$

While the resonant frequency equation, simply acknoledged as frequency equation in case of ignored damping was thoroughly studied in the literature for multiple cracked beam, the antiresonant frequency equations are first obtained herein and that will be involved below in subsequent section to propose a procedure for multi-crack identification from given antiresonant frequencies.

4. A PROCEDURE FOR CRACK IDENTIFICATION BY ANTIRESONANT FREQUENCIES

Suppose that m antiresonant frequencies $\overline{\omega}_1,\ldots,\overline{\omega}_m$ of a beam have been determined at locations $x_1,\ldots,x_m$, it is required to identify amount of cracks possibly occurred in the beam and their location and depth. Evidently, for given antiresonant frequencies

$\overline{\omega}_1, \ldots, \overline{\omega}_m$ the antiresonant frequency parameter can be easily calculated as

$$\overline{\lambda}_k = \sqrt[4]{\overline{\omega}_k^2 \rho F \ell^4 / EI}, \quad k = 1, \ldots, m \tag{34}$$

Based on the fact that an assumed crack at location e is confirmed to actually exist if its depth a could be predicted definitely greater than zero, a procedure acknowledged as crack scanning method was proposed [24] for crack identification in beam structures. The crack scanning method applied for antiresonant frequency-based crack identification in beam can be briefly presented as follows [23,24]:

First, a mesh of cracks assumed to occur at selected locations $(e_1, \ldots, e_n)$ and to have unknown depth $(a_1, \ldots, a_n)$ is introduced for conducting a model of multiple cracked beam that enables to establish a relationship between the crack parameters and antiresonant frequencies of the beam measured at a given grid (the characteristic equation for antiresonant frequencies).

Second, using the established above relationship, crack depth vector $\boldsymbol{a} = (a_1, \ldots, a_n)^T$ is predicted accordingly to the selected mesh of crack locations and given antiresonant frequencies and, as result, a new mesh of crack locations $\left(e'_1, \ldots, e'_{n'}\right), n' < n$, corresponding to the positively predicted crack depths could be generated.

Third, the newly generated crack mesh allows a new model to be reconstructed so that the crack depth vector could be reproduced and this iteration in estimating the vector of crack depths would be stopped until no new crack mesh could be obtained. The lastly obtained crack location mesh and predicted crack depth vector provide desired crack locations and depths. The identified crack locations and depths give also the number of cracks obtained and thus the problem of crack identification based on antiresonant frequencies is thus solved. Note, the most important task in the provided procedure is to estimate crack depth from given antiresonant frequencies, especially, when the scanning crack location mesh should be very large compared to number of given antiresonant frequencies being usually limited ($m < n$).

Let's consider Eq. (30) for antiresonant frequency that is rewritten as

$$[\mathbf{A}(\gamma)]\{\gamma\} = \{\boldsymbol{b}\}, \tag{35}$$

where $[\mathbf{A}]$ is $m \times n$-matrix with elements

$$a_{kj} = R_1\left(x_k, \overline{\lambda}_k, e_j\right) + \sum_{r=1}^{j-1} R_2\left(x_k, \overline{\lambda}_k, e_j, e_r\right)\gamma_r, \quad k = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{36}$$

and vectors

$$\{\gamma\} = (\gamma_1, \ldots, \gamma_n)^T, \ \ \{\boldsymbol{b}\} = (b_1, \ldots, b_m)^T, \ \ b_k = -R_0\left(x_k, \overline{\lambda}_k\right), \ \ k = 1, \ldots, m. \tag{37}$$

In general, solution of Eq. (34) can be sought by iteration procedure

$$[\mathbf{A}_{i-1}]\left\{\gamma^{(i)}\right\} = \{\boldsymbol{b}\}, \tag{38}$$

with $\mathbf{A}_{i-1} = \mathbf{A}\left(\gamma^{(\boldsymbol{i}-\mathbf{1})}\right); i = 1, 2, 3, \ldots, \gamma^{(\mathbf{0})} = \mathbf{0}$ and this procedure would be stopped when

$$\left|\left|\gamma^{(\boldsymbol{i})} - \gamma^{(\boldsymbol{i}-\mathbf{1})}\right|\right| \leq \text{ tolerance.} \tag{39}$$

Since the crack scanning mesh should be large, the system of equations (38) is usually underdetermined, so that it should be solved by the regularization method. Regularized solution of Eq. (38) can be found in the form

$$\gamma^{(i)} = \sum_{r=1}^{NR} \left(\frac{\sigma_r \boldsymbol{u}_r^T \boldsymbol{b}}{\delta + \sigma_r^2} \right) \boldsymbol{v}_r, \tag{40}$$

where $NR, \sigma_r, \boldsymbol{u}_r, \boldsymbol{v}_r, r = 1, 2, \ldots, NR$ are respectively the rank, singular values and left and right singular vectors of matrix $\mathbf{A}_{i-1}$, δ is regularization factor determined from equation

$$\theta(\delta) = \sum_{r=1}^{NR} \left(\frac{\delta \boldsymbol{u}_r^T \boldsymbol{b}}{\delta + \sigma_r^2} \right)^2 - \sum_{r=NR+1}^{n} \left(\boldsymbol{u}_r^T \boldsymbol{b} \right)^2 = \varepsilon, \tag{41}$$

with ε being noise level in the right hand side of Eq. (35).

After the crack magnitude estimated, depth of the identified cracks is calculated from the equation

$$\begin{aligned} &F(a) = 6\pi \left(1 - \nu_0^2\right) h f_c(a/h) = \overline{\gamma}_j, \\ &f_c(z) = z^2 \Big(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 \\ &\qquad + 47.1063z^6 - 40.7556z^7 + 19.6z^8\Big). \end{aligned} \tag{42}$$

5. NUMERICAL EXAMPLES AND DISCUSSION

5.1. Antiresonant frequencies of multiple cracked beam

First, fundamental antiresonant frequency parameter (λ) of beam with conventional boundary conditions has been computed using the antiresonant frequency equations (26)–(28) for various collocations in different cases of multi-crack scenarios (single, double and triple cracks at different locations and with various depth including also the case of uncracked beam). Computation results given in Tables 2–4 show that the antiresonant frequency of the beam with symmetric boundary conditions is unchanged due to a crack appeared at the beam middle. However, symmetric cracks produce different changes in antiresonant frequency measured at any location. Moreover, the second approximate equation (30) gives a solution almost identical to that of the exact one even for crack depth reached to 40% beam thickness. This allows one to confidently use the second approximate antiresonant frequency equation for multi-crack identification of beam by measured antiresonant frequencies.

Table 2. First antiresonant frequency parameter of multiple cracked simply supported beam

Crack location	Driving-measuring collocation, x									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	Single crack of $a/h = 0.4$									
0.25	4.0459	4.3697	4.9132	5.5918	5.8422	5.2865	4.7410	4.3180	3.9912	-
0.5	5.6063	5.1937	5.3612	5.8069	6.2832	5.7800	5.0608	4.4792	4.0545	-
0.75	4.2638	4.5419	4.8497	5.1363	5.4813	5.3950	4.8152	4.4572	4.1906	-

Crack location	Driving-measuring collocation, x								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Double cracks of $a/h = 0.4$										
0.25-0.50	4.5533	4.8006	4.9016	4.9028	4.9428	4.9683	4.9043	4.6353	4.2153	-
0.25-0.75	3.8681	4.0130	4.1103	4.1830	4.2701	4.3731	4.4881	4.4304	4.1046	-
0.50-0.75	4.2292	4.4742	4.7430	5.0035	5.3045	5.6294	5.0165	4.7342	5.4074	-
Triple cracks at locations (0.25-0.50-0.75)										
$a/h = 0.5$	3.5958	3.7013	3.7630	3.8168	3.8934	3.9398	3.9698	3.8504	3.6092	
0.4 EX	3.8545	3.9964	4.0938	4.1711	4.2636	4.3326	4.3884	4.2657	3.9511	-
A2	3.8538	3.9955	4.0931	4.1704	4.2624	4.3303	4.3845	4.2627	3.9500	
A1	4.2879	4.5368	4.7290	4.8378	4.9965	5.2234	4.6298	4.2329	3.9275	
0.3 EX	4.1154	4.3144	4.4612	4.5629	4.6650	4.7625	4.8756	4.7855	4.3716	-
A2	4.1150	4.3138	4.4607	4.5625	4.6643	4.7610	4.8729	4.7832	4.3709	
A1	4.5010	4.8578	5.1713	5.2830	5.4565	5.3820	4.7800	4.3550	4.0307	
0.2 EX	4.4387	4.7547	4.9777	5.0751	5.1575	5.3357	5.1894	4.6385	4.4313	-
A2	4.4385	4.7544	4.9774	5.0750	5.1572	5.3350	5.1902	4.6386	4.4314	
A1	4.7528	5.3700	5.7306	6.0375	6.1116	5.5563	4.9386	4.4773	4.1259	
0.1 EX	4.9526	4.8838	5.3378	6.0485	5.7991	5.7825	5.0899	4.5879	4.2153	
A2	4.9527	4.8838	5.3378	6.0486	5.7991	5.7826	5.0899	4.5879	4.2153	
A1	4.7470	4.8332	5.2663	5.8508	6.2282	5.7117	5.0730	4.5763	4.1974	
0.0	4.2264	4.6183	5.1318	5.7826	6.2832	5.7826	5.1318	4.6183	4.2264	
Notation: EX – exact equation; A2 – second approximation; A1 – first approximation										

Table 3. First antiresonant frequency parameter of multiple cracked clamped end beam

Crack location	Driving-measuring collocation, x								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Single crack of $a/h = 0.4$										
0.3	5.2058	5.4726	5.9770	6.9441	7.4069	6.6894	6.0080	5.4708	5.0371	-
0.5	7.7053	6.9407	6.7904	7.2324	7.8532	7.1305	6.1353	5.3930	4.8767	-
0.7	5.4897	5.7393	6.1137	6.5252	7.0038	6.7182	5.9770	5.5969	5.1262	-
Double cracks of $a/h = 0.4$										
0.3–0.5	5.4386	5.6143	5.8218	5.9382	6.0333	5.9959	5.8122	5.4422	4.5699	-
0.3–0.7	5.0999	5.1811	5.2869	5.3647	5.4445	5.4582	5.4057	5.2335	5.0602	
0.5–0.7	5.3669	5.5630	5.8354	6.1544	6.5452	7.1475	6.3643	6.8946	5.8087	-
Triple cracks at locations (0.3–0.5–0.7)										
$a/h = 0.5$	4.9504	5.0119	5.0881	5.1323	5.1871	5.1899	5.1429	5.0040	4.8733	
0.4 EX	5.0999	5.1811	5.2869	5.3647	5.4445	5.4582	5.4057	5.2335	5.0602	-
A2	5.0994	5.1803	5.2858	5.3637	5.4432	5.4565	5.4039	5.2324	5.0597	
A1	5.5890	5.8824	6.3009	6.5906	6.9944	6.5889	5.8239	5.3504	4.8880	

Crack location		Driving-measuring collocation, x									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.3	EX	5.2564	5.3750	5.5332	5.6609	5.7714	5.8014	5.7468	5.5278	5.2652	-
	A2	5.2562	5.3746	5.5327	5.6605	5.7708	5.8005	5.7458	5.5272	5.2650	
	A1	5.6816	6.0364	6.5651	6.8828	7.5523	6.7368	5.9527	5.4402	4.9687	
0.2	EX	5.4461	5.6591	5.9243	6.1192	6.2523	6.3228	6.3199	6.0536	5.6077	-
	A2	5.4459	5.6590	5.9241	6.1190	6.2521	6.3224	6.3194	6.0533	5.6076	
	A1	5.8004	6.3104	7.4263	7.5617	7.4865	6.9097	6.1012	5.5322	5.0458	
0.1	EX	5.8050	6.4030	7.0549	7.1095	7.0413	7.4390	6.2920	5.6362	5.1311	
	A2	5.8050	6.4029	7.0549	7.1095	7.0413	7.4391	6.2920	5.6362	5.1311	
	A1	6.0605	6.0129	6.5015	7.2592	7.7938	7.0726	6.2382	5.6076	5.1052	
0.0		5.1296	5.6399	6.3015	7.1494	7.8532	7.1494	6.3015	5.6399	5.1296	
Notation: EX – exact equation; A2 – second approximation; A1 – first approximation											

Table 4. First antiresonant frequency parameter of multiple cracked cantilever beam

Crack location		Driving-measuring collocation, x									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Single crack of $a/h = 0.4$											
0.3		2.8176	1.8623	2.1441	2.5948	3.0666	3.6402	4.3296	4.6185	4.3154	3.9190
0.5		2.6316	1.8443	2.1045	2.3950	2.7547	3.3905	4.0964	4.3626	4.0846	3.7437
0.8		2.3773	2.0489	2.2926	2.5805	2.9479	3.4435	4.1211	4.6202	4.2714	3.7811
Double cracks of $a/h = 0.4$											
0.3-0.5		2.6318	2.3695	1.9411	2.3542	2.7930	3.4969	4.2356	3.9214	4.0506	3.6756
0.3-0.8		2.5796	2.4101	1.8897	2.2592	2.6605	3.2596	3.9400	3.7127	3.9065	3.6748
0.5-0.8		2.5849	2.3488	2.0292	2.3093	2.6520	3.2511	3.9931	3.8732	4.0459	3.6482
Triple cracks at locations (0.3 – 0.5 – 0.8)											
$a/h = 0.5$		3.7526	2.5149	1.6986	2.1759	2.7315	3.2611	3.2333	3.2889	3.4460	3.4234
0.4	EX	2.6776	2.4611	1.8911	2.3044	2.7614	3.7039	3.5696	3.5878	3.8007	3.6067
	A2	2.6782	2.4615	1.8911	2.3044	2.7614	3.7045	3.5697	3.5881	3.8011	3.6065
	A1	2.5796	2.4101	1.8897	2.2592	2.6605	3.2596	3.9400	3.7127	3.9065	3.6748
0.3	EX	2.6772	1.8102	2.0777	2.4490	2.8685	3.5271	3.9972	3.9167	4.2079	3.7459
	A2	2.6773	1.8102	2.0777	2.4490	2.8685	3.5272	3.9971	3.9167	4.2080	3.7459
	A1	2.6263	1.8098	2.0767	2.4307	2.8303	3.4151	4.1057	4.0303	4.1660	3.7696
0.2	EX	2.5997	1.9905	2.2513	2.5888	2.9921	3.5747	4.3255	4.2655	4.2686	3.8443
	A2	2.5997	1.9905	2.2513	2.5888	2.9921	3.5747	4.3255	4.2655	4.2686	3.8443
	A1	2.5822	1.9904	2.2509	2.5839	2.9822	3.5492	4.2438	4.3602	4.2582	3.8495
0.1	EX	1.9188	2.1453	2.3892	2.6994	3.0964	3.6472	4.3439	4.6594	4.3228	3.9047
	A2	1.9188	2.1453	2.3892	2.6994	3.0964	3.6472	4.3439	4.6594	4.3228	3.9047
	A1	1.9188	2.1453	2.3891	2.6989	3.0955	3.6452	4.3379	4.6527	4.3219	3.9051
0.0		2.0291	2.2160	2.4484	2.7462	3.1416	3.6830	4.3737	4.6826	4.3465	3.9266
Notation: EX – exact equation; A2 – second approximation; A1 – first approximation											

5.2. Results of crack identification by antiresonant frequencies for cantilever beam

In this subsection the proposed above crack detection procedure is examined for cantilever beam with single and triple cracks at different locations and depths and various number of measured antiresonant frequencies. Results of the crack detection are shown in Figs. 1–4, where identified crack magnitude versus crack locations assumed in the scanning mesh. Inputs used for the crack detection are antiresenant frequencies computed by exact antiresonant frequency equations that implies the frequencies measured without noise.

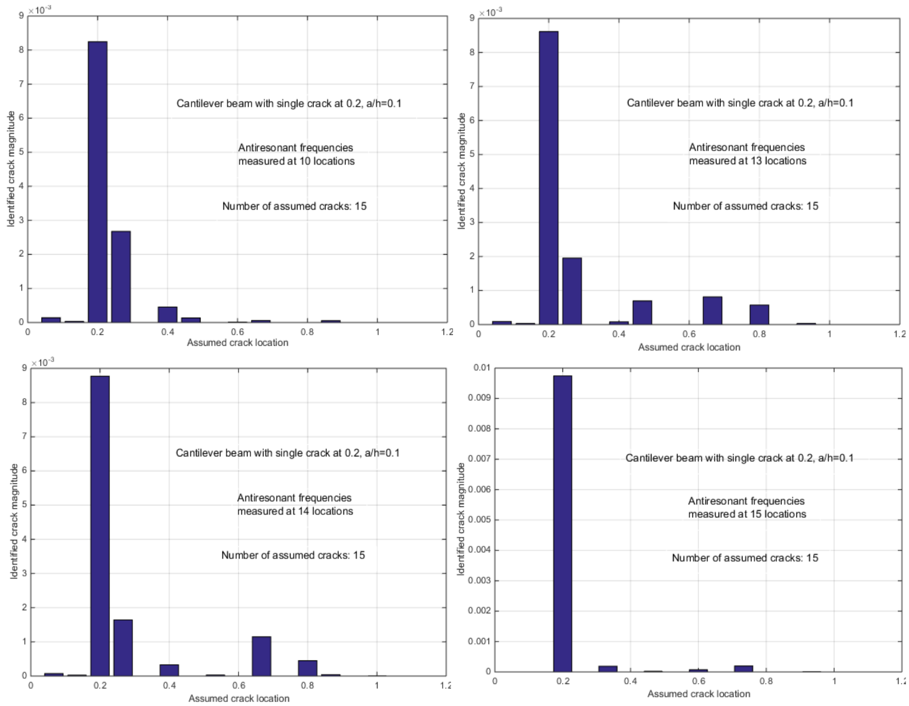
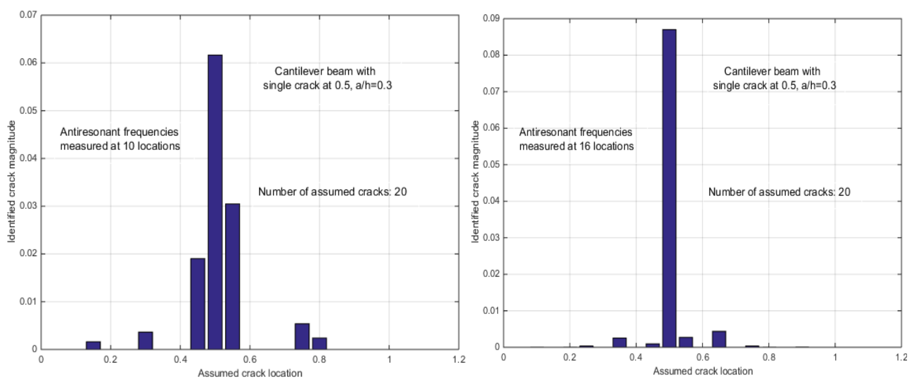


Fig. 1. Results of crack detection by antiresonant frequencies measured at 10, 13, 14, 15 positions on cantilever beam with single crack at location $e/L = 0.2; a/h = 10\%$, using model with 15 crack mesh



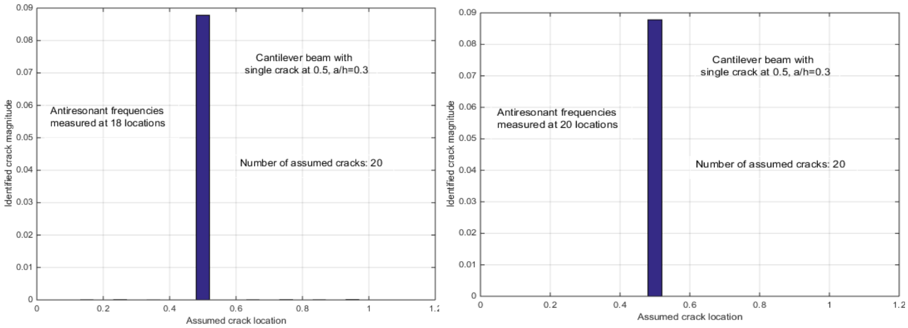


Fig. 2. Results of crack detection by antiresonant frequencies measured at 10, 16, 18, 20 positions on cantilever beam with single crack at location $e/L = 0.5$; $a/h = 30\%$, using model with 20 crack mesh

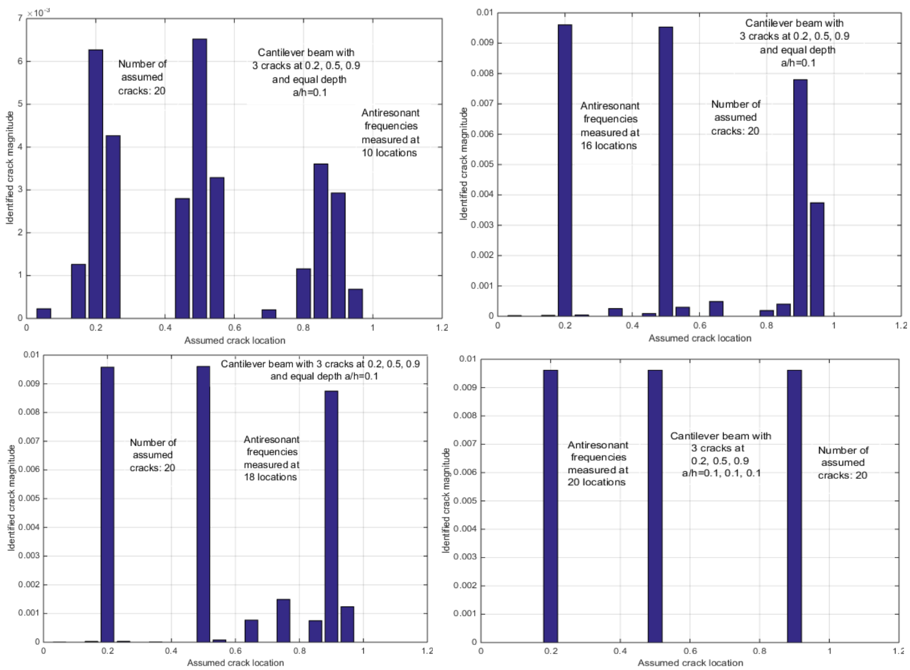


Fig. 3. Results of crack detection by antiresonant frequencies measured at 10, 16, 18, 20 positions on cantilever beam with three cracks at location $e/L = 0.2$; 0.5; 0.9 of equal depth $a/h = 10\%$, using model with 20 crack mesh

It can be seen that single crack of 10% depth could be reliably detected with antiresonant frequencies measured at 10 locations, but it would be uniquely identified by amount of antiresonant frequencies measured at all 15 locations of scanning mesh (Fig. 1). Larger crack of 30% beam thickness could be detected uniquely by 18 measured antiresonant frequencies using 20 location mesh (Fig. 2). Detecting triple cracks at positions 0.2-0.5-0.9 is accomplished in two cases of crack depth scenario: in the first case, three cracks have equal 10% depth (Fig. 3) and in the other one they have different depth of 30%, 20% and 50% (Fig. 4). Obviously, though triple cracks might be also consistently localized from 10 antiresonant frequencies, they could be exactly with equal depth only by the antiresonant frequencies measured at 20 locations of scanning mesh. In case of triple cracks with different depth, both location and magnitude could be accurately identified with 16 antiresonant frequencies.

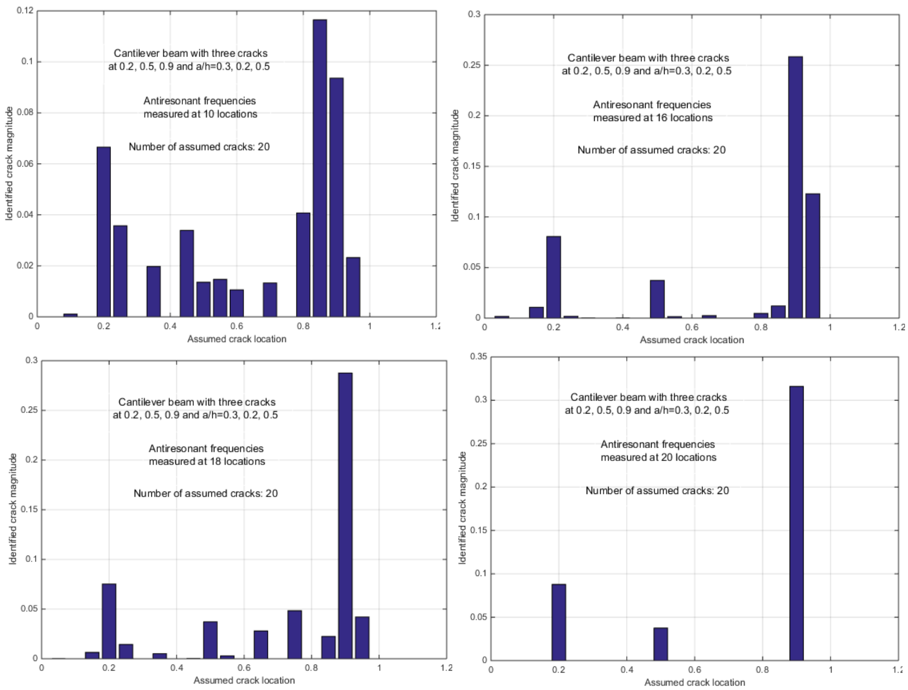


Fig. 4. Results of crack detection by antiresonant frequencies measured at 10, 16, 18, 20 positions on cantilever beam with three cracks of depth $a/h = 30\%$; 20%; 50% at locations $e/L = 0.2$; 0.5; 0.9 using model with 20 crack mesh

6. CONCLUDING REMARKS

Thus, in the present paper, concept of antiresonant frequency defined as zero point of frequency response function has been generally addressed for multiple cracked beam. First, governing equation for computing anti- resonant frequencies called antiresonant frequency equation has been established explicitly in term of crack magnitudes. From the governing equation, first and second asymptotic approximate antirequency equations in case of number of small cracks greater than 3 can be derived and used for analysis of antiresonant frequencies versus crack parameters and position on beam where the antiresonant frequencies have been measured. Finally, a procedure has been proposed to detect multi-cracks by given antiresonant frequencies based on the crack scanning method. Numerical examples show that the second approximate antiresonant frequency equation is consistent for computing antiresonant frequencies and antiresonant frequencies in combination with positions, where the antiresonant frequencies have been measured, provide more efficient indicator for crack identification compared with natural frequencies or even with mode shapes.

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