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AXIAL VIBRATION OF DOUBLE-WALLED CARBON NANOTUBES USING DOUBLE-NANOROD MODEL WITH VAN DER WAALS FORCE UNDER PASTERNAK MEDIUM AND MAGNETIC EFFECTS

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Abstract. The present study investigates the axial vibration of double-walled nanotubes. Using the nanorod continuum model with the van der Waals effect, the vibrational frequencies are studied. Aydogdu (Journal of Vibration and Control, Vol. 21, Issue 16, (2015), 3132-3154) proposed a reliable model for the study of axial vibration in a double-walled nanotube. This model provided a detailed investigation of axial vibration using van der Waals effects. But sometimes, the wrong equation might lead to erroneous scientific results. The incorrect term for axial vibration in the double-walled nanotube model is taken care of in the present study for the correct scientific inferences. Effectively, the axial vibrational frequencies appear without decoupling the continuum model as for primary and secondary nanotubes. The semi-analytical method estimates the axial vibrational frequencies of the double-walled nanotube as a coupled model. Two different boundary conditions like clamped-clamped and clamped-free support, are considered in this computation. The Pasternak medium support and magnetic effects influence the vibrational frequencies of the first and second nanotube for the first time. The Pasternak constant and magnetic parameters don't vary with the length of the nanotube for axial vibration. It means that still more understanding requires in modeling the Pasternak medium and magnetic force for the double-nanotube to model axial vibration.

Keywords: critical buckling of double-walled nanotube, differential transform method, Pasternak medium support, magentic force effect.

1. INTRODUCTION

The discovery of carbon nanotube by Iijima [1] influenced the researchers to understand its properties and behaviour. Endo et al. [2] discussed the carbon nanotube's properties in detail. Dresselhaus et al. [3] elaborated on the physical properties of carbon nanotube. Elishakoff et al. [4] investigated the mechanical properties and nanotube's structural behavior for one-dimensional nanostructures. Hierold et al. [5] examined the modelling and application of carbon nanotube devices. Aydogdu [6] proposed the nonlocal

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model for the axial vibration of a single nanorod and obtained the analytical solutions for two different boundary conditions. Kiani [7] used the perturbation technique to compute the vibrational frequencies of tapered nanowire/nanorods. Narendar and Gopalakrishnan [8] studied the wave propagation analysis for carbon nanorod using nonlocal elasticity theory. Aydogdu [9] used a cylindrical nanorod model to study the wave propagation and calibrated the nonlocal parameter compared to lattice dynamics. Islam et al. [10] used the analytical solutions to investigate the nonlocal effects in torsional nanorods. Aydogdu and Gul [11] investigated the vibrational behaviour of double nanorod systems using the doublet mechanics theory. Li et al. [12] used analytical solutions for specified boundary conditions and developed finite element solutions for clamped-strained ends using nonlocal strain gradient theory. Kiani [13] used an integro-differential model to investigate the vibrational frequencies of elastically connected nanorods. Zhu and Li [14] analysed the integral form of stress, strain, stress-strain based nonlocal theories for vibrational analysis. Nazemnezhad and Kamali [15] investigated the vibrational studies of the nonlocal nanorod model using analytical solutions. Farajpour et al. [16] compiled the review on the mechanics of nanostructures. It includes the research works about nanorods, nanorings, nanobeams, nanoplates, and nanoshells. Numanoğlu et al. [17] studied the nonlocal effects over vibrational frequencies of nanorod for various boundary conditions. Babaei and Yang [18] estimated the vibrational frequencies for rotating rods with nonlocal effects using the Galerkin method. Civalek and Numanoğlu [19] solved the nonlocal nanorod model for dynamic analysis using analytical and finite element methods. Babaei [20] studied the effect of harmonic excitation over frequencies of nanorod using nonlocal strain gradient theory. From this literature, it is understood that the vibrational analysis of nanorods is of greater interest among researchers.

Erol and Gürgöze [21] proposed the mathematical model of a double-rod system along with springs and dampers. Though two different cross-sectional area properties exist in their model, they simplified the material properties as $E_1A_1 = E_2A_2 = e$. Further, the relative displacement function assumes in terms of the primary and secondary nanorod in their study. Using this assumption, the coupled model of double-rod became two separate models as primary and secondary nanorod. Finally, Erol and Gürgöze [21] obtained the closed-form solution of double-rod for longitudinal vibrational frequencies along with springs and dampers effects. Murmu and Adhikari [22] analyzed the nonlocal effects in the vibrational frequencies of double-nanorod connected with springs using analytical solutions. Further, the decoupled model evaluated the frequencies using relative displacements, and it interlinks the primary and secondary nanorod. Narendar and Gopalakrishnan [23] elaborated the nonlocal effects in coupled nanorod systems using wave propagation studies. Karličić et al. [24] investigated the viscoelastic and nonlocal effects over the frequencies of double-nanorod. Xu et al. [25] determined the longitudinal vibrational frequencies of double and triple-nanorods using the Fourier series with the Rayleigh-Ritz approach. Using doublet mechanics theory, Aydogdu and Gul [11] studied the longitudinal vibrational frequencies of double-nanorod systems. Zhou [26], proposed the differential transform method for semi-analytical solutions. Chai and Wang [27], used the differential transformation method to compute the critical load for heavy columns.

Using the differential transform method, Senthilkumar [28] computed the Euler's critical buckling loads of carbon nanotube with nonlocal effects. Further, Senthilkumar et al. [29] determined Timoshenko's critical buckling load of single-walled carbon nanotube by implementing the differential transform method. Senthilkumar [30] evaluated the frequencies of double-nanorod and double-walled carbon nanotube by applying the differential transform method. Aydogdu [31] proposed the governing equations for axial vibration of double-nanotube using nonlocal elasticity. This model includes van der Waals force with suitable transformation. Using the interaction coefficient, the van der Waals coefficient of double-walled carbon nanotube yielded the axial effect for double-nanorod in Aydogdu's model. Kiani and Żur [32] analysed the vibrational frequencies of double-nanorod systems with defects using nonlocal-integral-surface-energy models. In Aydogdu's [31] model, an erroneous mistake occurred as ρ instead of ρA_1 for the first nanotube. Also, for the second nanotube, instead of ρA_2, the same ρ took place. Further, Aydogdu reported the dimensionless axial frequency term as $f^2 = \dfrac{\rho\omega^2 L^2}{EA_2}$ for double-nanotube model. It is worth noting that the dimensionless axial frequency of double-nanotube doesn't depend on the cross-sectional area of the nanorod. Due to this, the second carbon nanorod's axial frequencies resulted in the wrong values. So this has to be corrected as $f^2 = \dfrac{\rho\omega^2 L^2}{E}$. So the present objective of this work is to estimate the double-nanotubes axial frequencies using the corrected model. The Differential Transform Method estimates the axial frequencies of primary and secondary nanotubes effectively in a reliable manner by using the updated model. The first research work about modelling of Pasternak medium support while investigating the frequency analysis of nanorod is reported by Mohammadimehr et al. [33]. Later, Lv et al. [34] investigated the wave propagation of nanorods using the Pasternak medium with uncertainty in material properties using the nonlocal model. Murmu et al. [35] proposed the magnetic force term for the axial frequency analysis of nanorods. Also, the Pasternak effect and magnetic effect in the axial vibrational frequencies of double-nanotube don't exist in the literature as per the author's knowledge. So this work addresses the impact of the Pasternak medium and magnetic effect over axial frequencies of double-nanotube.

2. MATHEMATICAL MODEL OF DOUBLE-WALLED CARBON NANOTUBE

Aydogdu [31] proposed the mathematical model for double-walled carbon nanotube with axial vibration effect as,

$$c'(u_2 - u_1) = -EA_1\frac{\partial^2 u_1}{\partial x^2} + \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{1}$$

$$-c'(u_2 - u_1) = -EA_2\frac{\partial^2 u_2}{\partial x^2} + \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{2}$$

where u_1 and u_2 are the axial deformation of the first and second nanotube. Also, the cross-section areas of the first and second nanotube are A_1 and A_2. c' is the van der Waals interaction coefficient for axial effect with the first-order approximation [31] of van der Waals force. In the nonlinear analysis, the higher-order coefficients of van der Walls force involve with c'_3, c'_7 and c'_9 etc along with the linear term coefficient c'. Since the present study considers linear analysis [31], the coefficient c' takes care of the first-order approximation effect by ignoring the higher-order nonlinear effects of van der Waals coefficients values.

Two erroneous mistakes appeared in Eq. (1) and Eq. (2). The inertial force effect term for the first nanotube ρA_1 incorrectly appeared as ρ in Eq. (1). Similarly, the second nanotube's inertial force inaccurately appeared as ρ instead of ρA_2 in Eq. (2). The corrected model is,

$$c'(u_2 - u_1) = -EA_1 \frac{\partial^2 u_1}{\partial x^2} + \rho A_1 \frac{\partial^2 u_1}{\partial t^2}, \tag{3}$$

$$-c'(u_2 - u_1) = -EA_2 \frac{\partial^2 u_2}{\partial x^2} + \rho A_2 \frac{\partial^2 u_2}{\partial t^2}. \tag{4}$$

The present model extends the investigation by including the effect of the Pasternak medium and magnetic environment. By ignoring Winkler and damping moduli, c_u=0 and k_1=0 (Mohammadimehr et al. [33] and Lv et al. [34]), the axially distributed force per unit length f_u with magnetic effect by neglecting terms of nonlocal (Murmu et al. [35]) assumes as,

$$f_u = k_{up} \frac{\partial^2 u}{\partial x^2} + \eta_m A(H_y^2 + H_z^2) \frac{\partial^2 u}{\partial x^2}. \tag{5}$$

Here k_{up} and η_m represent the Pasternak medium (Mohammadimehr et al. [33] and Lv et al. [34]) and magnetic effects (Murmu et al. [35]) for nanorod. Aydogdu [31] defined the axial force in the following form as $f_{ij} = c'(u_j - u_i)$ for linear analysis. For the consistent notation purpose, assuming $p^u_{(N)(N+1)} = f_{ij}$ and $c^u_{(N)(N+1)} = c'$. The forces $p^u_{(N)(N+1)}$ [36] and $c^u_{(N)(N+1)}$ [37] take form as,

$$p^u_{(N)(N+1)} = c^u_{(N)(N+1)}(u_{N+1} - u_N), \tag{6}$$

$$c_{(N)(N+1)} = \frac{320\times(2R_N)\ \mathrm{erg/cm^2}}{0.16a^2_{C-C}}, \quad a_{C-C} = 0.142\ \mathrm{nm} \quad N = 1, 2, \ldots n. \tag{7}$$

By defining a ratio between the van der Waals interaction coefficient and linear van der Waals interaction coefficient as c_{Ratio},

$$c_{Ratio} = \frac{c^u_{(N)(N+1)}}{c_{(N)(N+1)}}, \tag{8}$$

$$c = c_{12} = \frac{320\times(2R_1)\ \mathrm{erg/cm^2}}{0.16a^2_{C-C}}, \quad a_{C-C} = 0.142\ \mathrm{nm}, \tag{9}$$

$$c_{Ratio} = \frac{c^u_{12}}{c_{12}} = \frac{c'}{c}, \tag{10}$$

$$c' = c_{Ratio}\times c, \tag{11}$$

$$c^u_{12} = c_{Ratio}\times c_{12}. \tag{12}$$

As suggested by Aydogdu [31], c_{Ratio} assumed in the range of 0.01–0.1.

Using Aydogdu [38] expression for the equation of motion and adding the force term p^u, the equation becomes,

$$\frac{\partial N}{\partial x} = \rho A \frac{\partial^2 u(x,t)}{\partial t^2} - f_u - p^u. \tag{13}$$

Also, the well-known relation takes the following form for nanorod,

$$\frac{\partial N}{\partial x} = EA \frac{\partial^2 u}{\partial x^2}. \tag{14}$$

Substituting Eq. (5) and Eq. (6) in Eq. (13) , the equation of motion for nanorod in terms of displacement using Eq. (14) is as follows,

$$\rho A \frac{\partial^2 u}{\partial t^2} - k_{up} \frac{\partial^2 u}{\partial x^2} - \eta_m A (H_y^2 + H_z^2) \frac{\partial^2 u}{\partial x^2} - p^u = EA \frac{\partial^2 u}{\partial x^2}. \tag{15}$$

Using Eq. (6) in Eq. (15) , the axial vibration model for double-walled nanotube attains as

$$EA_1 \frac{\partial^2 u_1}{\partial x^2} - \rho A_1 \frac{\partial^2 u_1}{\partial t^2} + k_{up} \frac{\partial^2 u_1}{\partial x^2} + \eta_m A_1 (H_y^2 + H_z^2) \frac{\partial^2 u_1}{\partial x^2} + c_{12}^u (u_2 - u_1) = 0, \tag{16}$$

$$EA_2 \frac{\partial^2 u}{\partial x^2} - \rho A_2 \frac{\partial^2 u_2}{\partial t^2} + k_{up} \frac{\partial^2 u_2}{\partial x^2} + \eta_m A_2 (H_y^2 + H_z^2) \frac{\partial^2 u_2}{\partial x^2} - c_{12}^u (u_2 - u_1) = 0. \tag{17}$$

By substituting the values as $k_{up} = 0$ and $\eta_m = 0$, in Eq. (16) and Eq. (17) yield the corrected governing equations for double-walled carbon nanotube for axial vibration.

3. DIFFERENTIAL TRANSFORM METHOD

Assuming the axial vibration is in the form of harmonic vibration and transforming Eq. (16) and Eq. (17) in dimensionless form as,

$$\frac{d^2 \overline{U}_1}{d\overline{X}^2} = \frac{-\Omega_{dr}^2 \overline{U}_1 - \alpha_{dr} (\overline{U}_2 - \overline{U}_1)}{\left[1 + \overline{K}_{updr} + \psi_m (1 + \delta_m^2)\right]}, \tag{18}$$

$$\frac{d^2 \overline{U}_2}{d\overline{X}^2} = \frac{-\Omega_{dr}^2 \overline{U}_2 + \left(\frac{\alpha_{dr}}{A_{dr}}\right)(\overline{U}_2 - \overline{U}_1)}{\left[1 + \left(\frac{\overline{K}_{updr}}{A_{dr}}\right) + \left(\frac{\psi_m (1+\delta_m^2)}{A_{dr}}\right)\right]}, \tag{19}$$

here the following dimensionless relations occur,

$$u_1(x,t) = U_1(x)\, e^{i\omega_{dr} t}, \quad u_2(x,t) = U_2(x)\, e^{i\omega_{dr} t}, \tag{20}$$

$$\overline{U}_1 = \frac{U_1}{L}, \quad \overline{U}_2 = \frac{U_2}{L}, \quad \overline{X} = \frac{x}{L}, \tag{21}$$

$$\Omega_{dr}^2 = \frac{\rho \omega_{dr}^2 L^2}{E}, \quad \alpha_{dr} = \frac{c_{12}^u L^2}{EA_1}, \quad \overline{K}_{updr} = \frac{k_{up}}{A_1 E}, \quad A_{dr} = \frac{A_2}{A_1}, \tag{22}$$

$$\psi_m = \frac{\eta_m H_y^2}{E}, \quad \delta_m = \frac{H_y}{H_z}, \quad \psi_m (1 + \delta_m^2) = \frac{\eta_m (H_y^2 + H_z^2)}{E}. \tag{23}$$

Applying Differential Transform to Eq. (18) and Eq. (19) by referring to Table 1 as

$$\overline{U}_1(k+2) = \frac{-\Omega_{dr}^2 \overline{U}_1(k) - \alpha_{dr}\left[\,\overline{U}_2(k) - \overline{U}_1(k)\,\right]}{(k+1)(k+2)\left[1 + \overline{K}_{updr} + \psi_m(1+\delta_m^2)\right]}, \tag{24}$$

$$\overline{U}_2(k+2) = \frac{-\Omega_{dr}^2 \overline{U}_2(k) + \left(\frac{\alpha_{dr}}{A_{dr}}\right)\left[\,\overline{U}_2(k) - \overline{U}_1(k)\,\right]}{(k+1)(k+2)\left[1 + \left(\frac{\overline{K}_{updr}}{A_{dr}}\right) + \left(\frac{\psi_m(1+\delta_m^2)}{A_{dr}}\right)\right]}. \tag{25}$$

Table 1. DTM Transformation

Original Function	Transformed Function
Original B.C	Transformed B.C
$y(x) = \lambda u(x)$	$Y(k) = \lambda \mathrm{U(k)}^a$
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)^a$
$y(x) = \dfrac{d^n u(x)}{dx^n}$	$Y(k) = (k+1)(k+2)...(k+n)U(k+n)^a$

[a]Zhou [26].

3.1. Boundary Conditions

The present investigation of double-walled nanotube for axial vibration effect deals with two different boundary conditions: clamped-clamped and clamped-free.

3.1.1. *Clamped-Clamped Support*

The boundary conditions for double-walled with clamped-clamped support for axial vibration effect satisfy the following requirements,

$$u_1\Big|_{x=0} = 0, \quad u_1\Big|_{x=L} = 0, \quad u_2\Big|_{x=0} = 0, \quad u_2\Big|_{x=L} = 0, \tag{26}$$

using Eq. (21) , the dimensionless form of Eq. (26) appears as,

$$\overline{U}_1\Big|_{\overline{X}=0} = 0, \quad \overline{U}_1\Big|_{\overline{X}=1} = 0, \quad \overline{U}_2\Big|_{\overline{X}=0} = 0, \quad \overline{U}_2\Big|_{\overline{X}=1} = 0. \tag{27}$$

The Differential Transform converts Eq. (27) with the assumption of unknown initial conditions as c_1 and c_2 and boundary conditions for clamped-clamped support becomes,

$$\overline{U}_1(0) = 0, \quad \overline{U}_1(1) = c_1, \quad \overline{U}_2(0) = 0, \quad \overline{U}_2(1) = c_2, \tag{28}$$

$$\sum_{k=0}^{\infty} \overline{U}_1(k) = 0, \qquad \sum_{k=0}^{\infty} \overline{U}_2(k) = 0. \tag{29}$$

3.1.2. *Clamped-Free Support*

The double-walled nanotube with axial vibration effect satisfies the following boundary conditions for clamped-free support

$$u_1\Big|_{x=0} = 0, \quad N_1\Big|_{x=L} = 0, \quad u_2\Big|_{x=0} = 0, \quad N_2\Big|_{x=L} = 0. \tag{30}$$

Eq. (30) converts to dimensionless form using Eq. (21) as follows,

$$\overline{U}_1\Big|_{\overline{X}=0} = 0, \quad \overline{N}_1\Big|_{\overline{X}=1} = 0, \quad \overline{U}_2\Big|_{\overline{X}=0} = 0, \quad \overline{N}_2\Big|_{\overline{X}=1} = 0, \tag{31}$$

where

$$\overline{N}_1\Big|_{\overline{X}=1} = \frac{d\overline{U}_1}{d\overline{X}}, \qquad \overline{N}_2\Big|_{\overline{X}=1} = \frac{d\overline{U}_2}{d\overline{X}}. \tag{32}$$

By using suitable unknown values as c_1 and c_2, the boundary conditions for clamped-free support takes form with the assistance of the Differential Transform approach,

$$\overline{U}_1(0) = c_1, \quad \overline{U}_1(1) = 0, \quad \overline{U}_2(0) = c_2, \quad \overline{U}_2(1) = 0. \tag{33}$$

$$\sum_{k=0}^{\infty} \overline{U}_1(k) = 0, \qquad \sum_{k=0}^{\infty} \overline{U}_2(k) = 0. \tag{34}$$

4. COMPUTATION OF DIFFERENTIAL TRANSFORMATION METHOD

The computation of double-walled carbon nanotube's axial vibrational frequencies for clamped-clamped support involves by writing Eq. (28) and Eq. (29) in matrix form as

$$\begin{bmatrix} A_{11}^n(\Omega_{dr}^2) & A_{12}^n(\Omega_{dr}^2) \\ A_{21}^n(\Omega_{dr}^2) & A_{22}^n(\Omega_{dr}^2) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{35}$$

$$\begin{vmatrix} A_{11}^n(\Omega_{dr}^2) & A_{12}^n(\Omega_{dr}^2) \\ A_{21}^n(\Omega_{dr}^2) & A_{22}^n(\Omega_{dr}^2) \end{vmatrix} = 0, \tag{36}$$

$$[\Omega_{dr}]^2 = \left[\Omega_{drj}^{(n)}\right]^2, \quad j = 1, 2, 3, \ldots, n \tag{37}$$

$$\left| \left[\Omega_{drj}^{(n)}\right]^2 - \left[\Omega_{drj}^{(n-1)}\right]^2 \right| \leq \varepsilon. \tag{38}$$

The tolerance parameter ε value as 0.0001 in Eq. (38) determines the axial vibration computation with four decimal accuracy for clamped-clamped supported double-walled nanotube. Similarly, for the clamped-free boundary conditions, by using Eq. (33) and Eq. (34) in matrix form yields the axial vibrational frequencies for double-walled nanotube following the steps from Eq. (35) to Eq. (38).

5. DISCUSSIONS

The double-walled nanotube's properties [4] used in this analysis are Young's modulus $E = 1.0$ TPa, van der Waals coefficient $c = c_{12} = 0.06943$ TPa, inner mean radius of nanotube $R_1 = 0.35$ nm, outer mean radius of nanotube $R_2 = 0.70$ nm, cross sectional area of first nanotube $A_1 = 7.476990\times10^{-19}$ m^2 and cross sectional area of second nanotube $A_2 = 1.495398\times10^{-18}$ m^2, $A_{dr} = A_2/A_1 = 2$, inner mean diameter $d_1 = 2\times R_1 = 0.7$ nm, outer mean diameter $D_o = 2\times R_2 = 1.4$ nm and nanotube thickness $t = 0.34$ nm. The present study considers the value of the van der Waals coefficient as $c^u_{12} = c_{Ratio}\times\dfrac{320\times(2R_1)\ \mathrm{erg/cm}^2}{0.16a^2_{C-C}}$ or $c' = c_{Ratio}\times0.06943$ TPa. The values of c_{Ratio} [31] varies from 0.01 to 0.1. The differential transform method estimates the dimensionless axial frequencies. Fig. 1 shows, the convergence of the first six dimensionless axial frequencies happens at $n = 40$ for double-walled nanotube with clamped-clamped boundary conditions. The dimensionless axial frequencies of the first nanotube appear from the first, third, and fifth modes. Also, the second, fourth, and sixth modes correspond to dimensionless axial frequencies of the second nanotube. The first and second modes for clamped-clamped support appear at n=16 terms. Emergences of third and fourth mode happen at $n = 26$ terms. Finally, the fifth and sixth modes evolve at $n = 36$ terms. However, Fig. 1 represents that the difference between the first and second modes is negligible. But it is seen from Table 2 that the first frequency is different from the second frequency. So the difference between first and second frequencies is not ignorable. Similar observations happen from third to sixth modes. The axial frequencies of the double-walled nanotube with clamped-free support show identical phenomena compared to clamped-clamped results of the convergency study using Fig. 2.

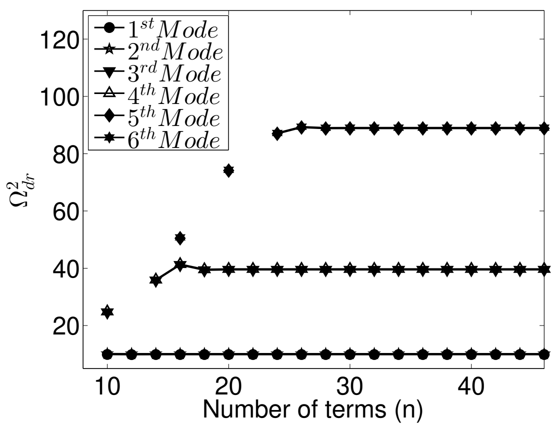

Fig. 1. Convergence of dimensionless frequency (Ω^2) for Clamped-Clamped support of Double-walled Carbon Nanorod with $c^u_{12} = 0.01\times c_{12}$ and $L/D_o = 10$

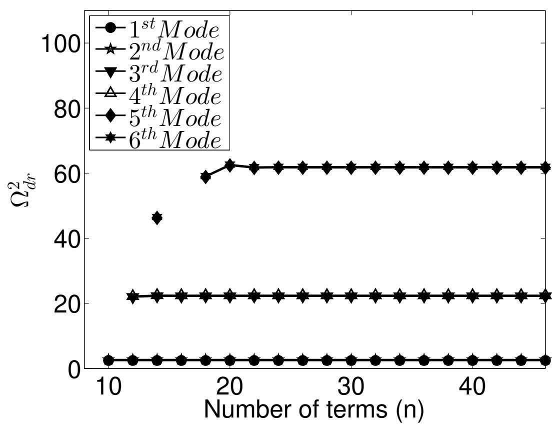

Fig. 2. Convergence of dimensionless frequency (Ω^2) for Clamped-Free support of Double-walled Carbon Nanorod with $c^u_{12} = 0.01\times c_{12}$ and L/D_o=10

Table 2. Critical Buckling Loads of DWCNTs for various boundary conditions in nN, $c_{12} = 71.11$ GPa, $L/d_2 = 10$

No	L/D_o	DTM Ω_{dr}	First Nanotube Ω_{dr}	Second Nanotube Ω_{dr}	Single Nanorod Ω_{sr} [38]
1	5	3.1416	$3.1416^{(1)}$	-	3.141
2		3.1524	-	$3.1524^{(1)}$	-
3		6.2832	$6.2832^{(2)}$	-	6.284
4		6.2886	-	$6.2886^{(2)}$	-
5		9.4248	$9.4248^{(3)}$	-	9.425
6		9.4284	-	$9.4284^{(3)}$	-
1	25	3.1416	$3.1416^{(1)}$	-	3.141
2		3.4023	-	$3.4023^{(1)}$	-
3		6.2832	$6.2832^{(2)}$	-	6.284
4		6.4175	-	$6.4175^{(2)}$	-
5		9.4248	$9.4248^{(3)}$	-	9.425
6		9.5149	-	$9.5149^{(3)}$	-
1	50	3.1416	$3.1416^{(1)}$	-	3.141
2		4.0859	-	$4.0859^{(1)}$	-
3		6.2832	$6.2832^{(2)}$	-	6.282
4		6.8047	-	$6.8047^{(2)}$	-
5		9.4248	$9.4248^{(3)}$	-	9.425
6		9.7802	-	$9.7802^{(3)}$	-

$^{(1)}$First Vibrational Mode; $^{(2)}$Second Vibrational Mode; $^{(3)}$Third Vibrational Mode.

Table 2 compares the first six dimensionless axial frequencies of the double-walled nanotube using a nanorod model with a single nanorod. It is evident from Table 2 that the dimensionless axial frequency parameter increases with various L/D_o ratios. It implies that the length of the nanotube plays a crucial role in the increase in dimensionless frequency compared to short-length nanotubes. This phenomenon happens in the second nanotube alone. For the first nanotube, the dimensionless axial frequency parameter is insensitive to L/D_o ratios. Also, the dimensionless frequency of the first nanotube and the single nanorod are the same. Similar observations happen in the dimensionless axial frequencies of the double-walled nanotube with clamped-free support by referring to Table 3 and Fig. 2. Using the relation $\Omega_{dr}^2 = \frac{\rho\omega_{dr}^2 L^2}{E}$, the computation of angular frequency happens easily. Fig. 3 demonstrates, the L/D_o ratios influence the dimensionless frequency parameter of the second nanotube significantly for the clamped-clamped support. Notably, the first mode of the second nanotube increases with the length of the nanotube compared to the second and third modes. A similar trend continues in the dimensionless axial frequencies of the double-walled nanotube with clamped-free support, as shown in Fig. 4.

Table 3. First six dimensionless axial frequency parameter (Ω_{dr}) of double-walled nanotube with clamped-free support ($c_{12}^{u} = 0.01 \times c_{12}$)

No	L/D_o	DTM Ω_{dr}	First Nanotube Ω_{dr}	Second Nanotube Ω_{dr}	Single Nanorod Ω_{sr} [38]
1.	5	1.5708	1.5708$^{(1)}$	-	1.570
2.		1.5924	-	1.5924$^{(1)}$	-
3.		4.7124	4.7124$^{(2)}$	-	4.712
4.		4.7196	-	4.7196$^{(2)}$	-
5.		7.8540	7.8540$^{(3)}$	-	7.853
6.		7.8583	-	7.8583$^{(3)}$	-
1.	25	1.5708	1.5708$^{(1)}$	-	1.570
2.		2.0430	-	2.0430$^{(1)}$	-
3.		4.7124	4.7124$^{(2)}$	-	4.712
4.		4.8901	-	4.8901$^{(2)}$	-
5.		7.8540	7.8540$^{(3)}$	-	7.853
6.		7.9619	-	7.9619$^{(3)}$	-
1.	50	1.5708	1.5708$^{(1)}$	-	1.570
2.		3.0484	-	3.0484$^{(1)}$	-
3.		4.7124	4.7124$^{(2)}$	-	4.712
4.		5.3881	-	5.3881$^{(2)}$	-
5.		7.8540	7.8540$^{(3)}$	-	7.853
6.		8.2771	-	8.2771$^{(3)}$	-

$^{(1)}$First Vibrational Mode; $^{(2)}$Second Vibrational Mode; $^{(3)}$Third Vibrational Mode.

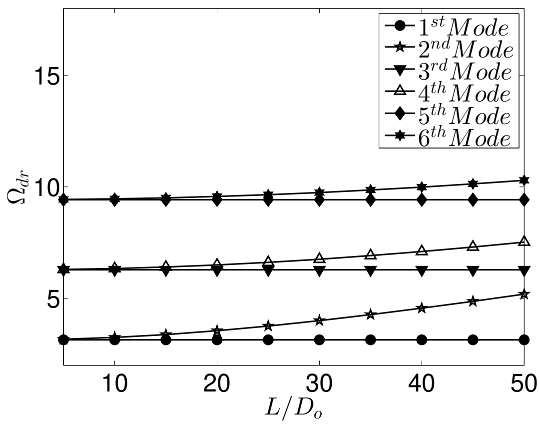

Fig. 3. Effect of L/D_o in dimensionless frequency (Ω) for Clamped-Clamped support of Double-walled Carbon Nanorod with $c_{12}^{u} = 0.025 \times c_{12}$

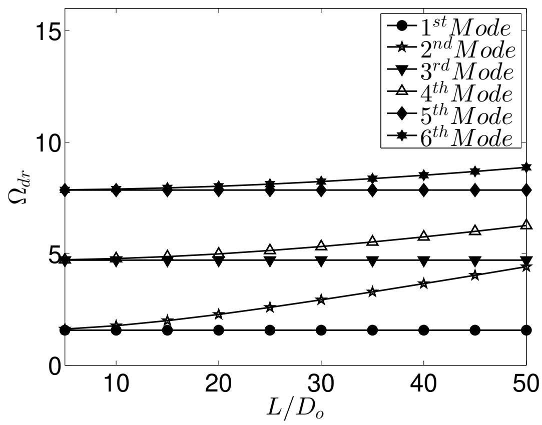

Fig. 4. Effect of L/D_o in dimensionless frequency (Ω) for Clamped-Free support of Double-walled Carbon Nanorod with $c_{12}^{u} = 0.025 \times c_{12}$

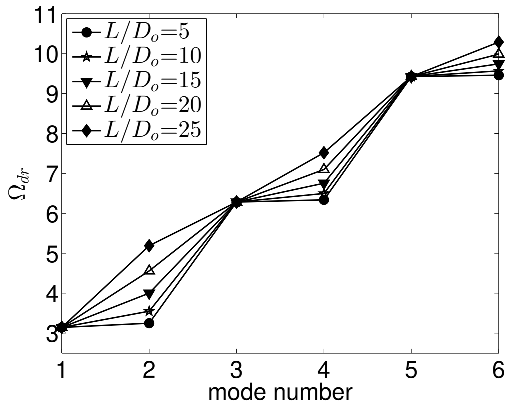

Fig. 5. First six modes effect of L/D_o in dimensionless frequency (Ω) for Clamped-Clamped support of Double-walled Carbon Nanorod with $c_{12}^u = 0.1 \times c_{12}$

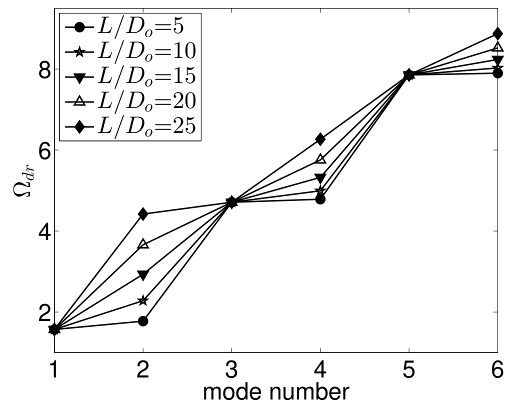

Fig. 6. First six modes effect of L/D_o in dimensionless frequency (Ω) for Clamped-Free support of Double-walled Carbon Nanorod with $c_{12}^u = 0.1 \times c_{12}$

From Fig. 5, the dimensionless axial frequencies increase for higher modes for clamped-clamped support of double-walled nanotube. Interestingly, the first, third, and fifth modes are insensitive with the various value of L/D_o ratio. These are the dimensionless axial frequency of the first nanotube, which doesn't change with the length of the nanotube. But for the second nanotube, the dimensionless axial frequency parameter changes. The reason for this is due to the contribution of the van der Waals force. Fig. 6 shows similar phenomena for clamped-free support. Fig. 7 shows the influence of van der Waals force over dimensionless axial frequency for clamped-clamped support. Again the second nanotube's dimensionless axial frequency increase as the van der Waals effect

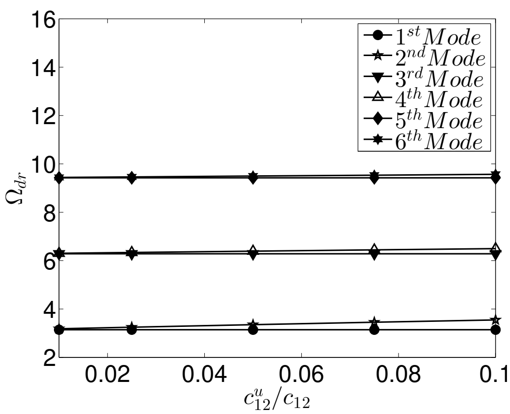

Fig. 7. Effect of van der Waals in dimensionless frequency (Ω) for Clamped-Clamped support of Double-walled Carbon Nanorod with $c_{12}^u = 0.1 \times c_{12}$ and $L/D_o = 10$

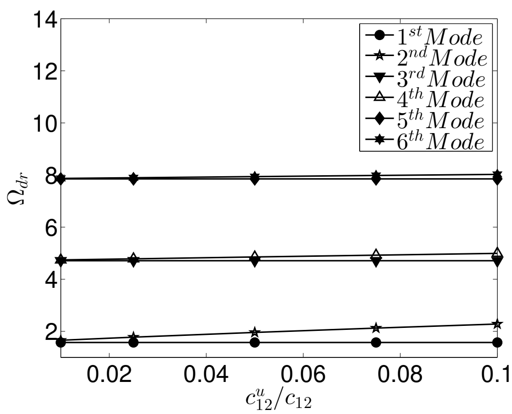

Fig. 8. Effect of van der Waals in dimensionless frequency (Ω) for Clamped-Free support of Double-walled Carbon Nanorod with $c_{12}^u = 0.1 \times c_{12}$ and $L/D_o = 10$

expands. In Fig. 8, though similar trends happen in clamped-free support, the dimensionless axial frequencies of the second nanotube significantly increase compared with the clamped-clamped boundary condition.

6. EFFECT OF PASTERNAK MEDIUM AND MAGNETIC EFFECT

The Pasternak medium for axial vibration (Mohammadimehr et al. [33]) of nanorod in the literature discusses viscous effects. As per Mohammadimehr et al. [33], the Pasternak medium [34] impacts the non-dimensional frequencies of micro-rod. In the present analysis, the double-walled nanotube includes the Pasternak medium using the nanorod models as shown in Eqs. (16) and (17). The dimensionless Pasternak constant takes form using Eq. (3.18) as $\overline{K}_{updr} = \dfrac{k_{up}}{A_1 E}$. The present investigation considers three different values of k_{up} as 1.0e-7 N, 2.0e-7 N and 3.0e-7 N [39]. The corresponding dimensionless Pasternak constant values of $\overline{K}_{updr}$ are 0.13, 0.27 and 0.40. These dimensional values are not dependent on the length of the nanotube, and it is constant for each Pasternak constant k_{up}. So the dimensionless Pasternak constant without depending on the nanotube's length term is not acceptable because it must be in the function of nanotube length. So the Pasternak medium support effect for axial vibration of nanorod is not understood properly in the literature, and more understanding is required. Since the dimensional Pasternak constant is not depending on the nanotube's length, the Pasternak effect won't yield any meaningful impact over the dimensional axial frequencies of the double-walled nanotube for clamped-clamped and clamped-free support.

Further, this study focuses on the impact of magnetic forces (Murmu et al. [35]) on axial vibrational frequencies of double-walled nanotubes using the nanorod model. According to Murmu et al. [35], the magnetic effect significantly affects the axial vibration frequencies of a single nanorod. The magnetic forces modify the mathematical model to predict the dimensionless axial vibrational frequencies as shown in Eq. (16) and Eq. (17). The dimensionless magnetic parameter converts as $\psi_m(1+\delta_m^2) = \dfrac{\eta_m(H_y^2 + H_z^2)}{E}$. It is interesting to observe that even this dimensionless magnetic parameter does not vary with the length of the nanotube. So the dimensionless axial vibrational frequencies can not be influenced by magnetic effects. Again, further work is required to understand magnetic force effects for axial vibrational modelling for double-walled nanotube for clamped-clamped and clamped-free support.

7. CONCLUSIONS

In the present study, the erroneous terms ρ in the inertial force are taken care of as ρA_1 and ρA_2 to predict the dimensionless axial vibrational frequencies of double-walled nanotube correctly. Unlike solving the coupled model by assuming a relative displacement function, the present approach estimated the dimensionless axial frequencies as a coupled model using the differential transform method. The dimensionless axial frequencies don't depend on the area of the first or second nanotube. Though the double-walled nanotube model predicts the dimensionless axial frequencies, the first nanotube's frequencies are the same as the single nanotube's frequencies. But the second nanotube's dimensionless axial frequencies are different from the first nanotube predicted by the present model. The dimensionless axial frequencies increase with the shorter length of the nanotube to the longer length of the nanotube. Further, an increase in the linear van der Waals coefficient increases the dimensionless frequencies. So the van der walls force plays a crucial role in the second nanotube's frequency. Also, for the higher modes, the dimensionless axial frequencies of the second nanotube change with the various L/D_o values. Again, this happens for the second nanotube's dimensionless axial frequencies alone. Since the dimensionless Pasternak constant doesn't depend on the length of the nanotube, it remains a constant value, and it is not acceptable. So the Pasternak medium effects for axial vibrational frequency analysis remain a challenge, and still more to be done in understanding this effect. Similarly, the dimensional magnetic parameter is insensitive to the length of the nanotube. So the magnetic effect on axial vibration of the double-walled nanotube is incorrectly done, and it needs further understanding for correct modelling.

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