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STATIC REPAIR OF MULTIPLE CRACKED BEAM USING PIEZOELECTRIC PATCHES

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Abstract. This paper addresses the problem of repairing multiple cracked beams subjected to static load using piezoelectric patches. First, the problem is formulated and solved analytically for the case of two cracks that results in ratio of restoring moments produced by employed piezoelectric patches. Since the ratio is dependent only on crack positions but not their depth, the result obtained for case of two cracks has been extended for the case of multiple cracks. This proposition is then validated by finite element simulation where repairing piezoelectric patches are replaced by mechanical moment load equivalent to the restoring bending moments produced by the piezoelectric patches. The excellent agreement between analytical solution and numerical simulation results in case of single and double cracks allows making a conclusion that a piezoelectric patch could productively repair a cracked beam by producing a restoring moment due to its piezoelectricity. Thus, the problem of repairing multiple cracked beam using piezoelectric patches is solved.

Keywords: multiple cracked beam, piezoelectric patches, static repair.

1. INTRODUCTION

Damages or cracks appearing in a structure will inevitably reduce its serviceability and might lead to serious accidence if the deteriorations would not be early detected and repaired. Therefore, there are a lot of studies devoted to developing efficient techniques for structural damage detection and major results obtained in the last decade were reviewed in [1]. Recently, smart material such as piezoelectric one has found widespread application in structural health monitoring and repair [2]. Wang and his coworkers [3–10] have solved numerous problems of repairing cracked structures using piezoelectric patches. The advantage of the piezoelectric material in repairing cracked structures consists of that effectiveness of the repair can be controlled when the output voltage of the piezoelectric patch used as a sensor is applied to the repaired structure through a collocated piezoelectric actuator. As a result, the repaired structure gets from the actuator an action of a local bending moment that could restore the slope increased due to the crack. Obviously, the applied bending moment is dependent on external load applied to the structure, crack parameters and on the design parameters of the piezoelectric patch. All the mentioned above parameters can be chosen to disregard the slope discontinuity caused by the crack that is acknowledged as the principle for repairing cracked structures. Some other problems were studied in Ref. [11, 12], however, there are absent studies on the repair of multiple cracked structures.

Thus, the present study addresses the problem of repairing multiple cracked beams subjected to static load by using piezoelectric patches as shown in Fig. 1. First, the problem of repairing beams with two and three cracks is analytically solved to establish relationship between coefficients of the so-called restoring moments defined for repairing the cracks. After finding that ratio of the restoring moment coefficients is dependent only on crack positions, the restoring moment for every crack can be determined from the first one. This hypothesis is further approved numerically by using the finite element method that proposes to replace the repairing patches by applying mechanical bending moments equal to the restoring moments so that an equivalent repair is achieved.

2. THEORETICAL DEVELOPMENT

Let's consider a cantilever beam of length L (m), elastic modulus E (N/m^2), mass density ρ (kg/m^3), cross section area $D \times H$, subjected to a static load F at free end of the beam, i.e. at the position L. Suppose furthermore that the beam is cracked at positions $L_1, L_2, L_3, \ldots$ and the cracks are repaired by bonding piezoelectric patches of thickness $\delta_1, \delta_2, \delta_3, \ldots$ and length $p_1 + p_2, p_3 + p_4, p_5 + p_6, \ldots$ respectively to the beam at the crack positions as shown in Fig. 1.

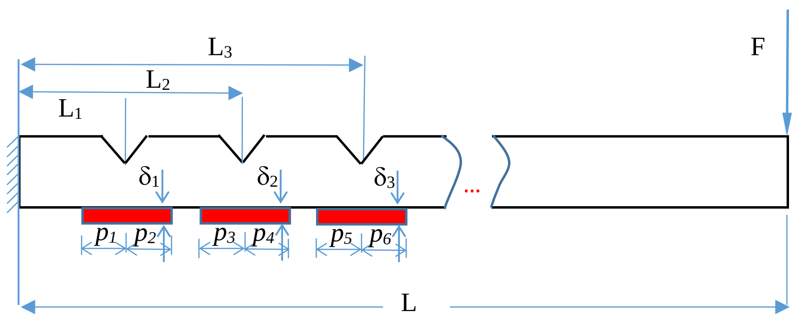


Fig. 1. Model of multiple cracked beam repaired by piezoelectric patches

2.1. Crack modelling

The open edged cracks are represented by the well-known equivalent spring model with the spring stiffness defined and calculated as [13, 14]

$$K_r = \frac{EI}{L\Theta}, \quad \Theta = \frac{5.346H}{L} f(z), \tag{1}$$

where I is the moment of inertia and

$f(z) = 1.8624z^2 - 3.95z^3 + 16.375z^4 - 37.226z^5 + 78.81z^6 - 126.9z^7 + 172z^8 - 143.97z^9 + 66.56z^{10}$.

2.2. Effect of piezoelectric patches on beam

Assuming that deflection curve of the beam under the load F is $y(x)$ and considering the piezoelectric patch as sensor, electric charge induced in the patch is calculated as [15]

$$Q = -e_{31} \int_0^{L_p} D\left(\frac{H+\delta}{2}\right) y'' dx, \tag{2}$$

where e_{31} is piezoelectric constant and δ, L_p are the patch thickness and length, respectively. Therefore, output voltage of the sensor is given by

$$V_s = \frac{Q}{C_v} = -\frac{e_{31} D\,(H+\delta)}{2C_v} \int_0^{L_p} y'' dx, \tag{3}$$

with C_v is electric capacitance of the sensor.

In case if the piezoelectric patch is used as collocated sensor and actuator, the voltage applied to the patch is

$$V_a = gV_s = -g\frac{e_{31} D\,(H+\delta)}{2C_v} \int_0^{L_p} y'' dx = -s\frac{e_{31} D\,(H+\delta)}{2} \int_0^{L_p} y'' dx, \tag{4}$$

where g is so-called gain factor and $s = g/C_v$. Under the voltage, axial stress induced along the piezoelectric patch can be expressed as

$$\sigma_x = e_{31}\frac{V_a}{\delta} = -s\frac{e_{31}^2 D\,(H+\delta)}{2\delta} \int_0^{L_p} y'' dx, \tag{5}$$

and, in consequence, bending moment applied to the beam will be

$$M_e = \sigma_x \delta D \frac{H+\delta}{2} = -s\frac{e_{31}^2 D^2 (H+\delta)^2}{4} \int_0^{L_p} y'' dx = -G \int_0^{L_p} y'' dx = -G\, y'\big|_0^{L_p}, \tag{6}$$

where G, defined as coefficient of restoring moment, is given by

$$G = \frac{s e_{31}^2 D^2 (H+\delta)^2}{4}. \tag{7}$$

2.3. Repair of beam with two cracks by piezoelectric patches

Based on the theoretical development and the beam model given in Fig. 1, equations for deflection in the beam segments divided by the patches and cracks can be written as

$$y_1''(x) = \frac{F}{EI}(L-x), \quad 0 \le x \le L_1 - p_1,$$

$$EIy_2''(x) = F(L-x) + G_1\left(y'_4\big|_{L_1+p_2} - y'_1\big|_{L_1-p_1}\right), \quad L_1 - p_1 \le x \le L_1,$$

$$EIy_3'' = F(L-x) + G_1\left(y'_4\big|_{L_1+p_2} - y'_1\big|_{L_1-p_1}\right), \quad L_1 \le x \le L_1 + p_2,$$

$$\begin{aligned} y_4'' &= \frac{F}{EI}(L-x), \quad L_1+p_2 \le x \le L_2-p_3, \\ EIy_5'' &= F(L-x) + G_2\left(y'_7\big|_{L_2+p_4} - y'_4\big|_{L_2-p_3}\right), \quad L_2-p_3 \le x \le L_2, \\ EIy_6'' &= F(L-x) + G_2\left(y'_7\big|_{L_2+p_4} - y'_4\big|_{L_2-p_3}\right), \quad L_2 \le x \le L_2+p_4, \\ EIy_7'' &= F(L-x), \quad L_2+p_4 \le x \le L. \end{aligned} \tag{8}$$

Solving the differential equations (8) gives

$$\begin{aligned} y_1(x) &= \frac{F}{6EI}(L-x)^3 + b_1x + b_2, \\ y_2(x) &= \frac{F}{6EI}(L-x)^3 + \frac{G_1}{2EI}\left(-\frac{F}{2EI}(L-L_1-p_2)^2 + b_3 + \frac{F}{2EI}(L-L_1+p_1)^2 - b_1\right)x^2 + b_7x + b_8, \\ y_3 &= \frac{F}{6EI}(L-x)^3 + \frac{G_1}{2EI}\left(-\frac{F}{2EI}(L-L_1-p_2)^2 + b_3 + \frac{F}{2EI}(L-L_1+p_1)^2 - b_1\right)x^2 + b_9x + b_{10}, \\ y_4 &= \frac{F}{6EI}(L-x)^3 + b_3x + b_4, \\ y_5 &= \frac{F}{6EI}(L-x)^3 + \frac{G_2}{2EI}\left(-\frac{F}{2EI}(L-L_2-p_4)^2 + b_5 + \frac{F}{2EI}(L-L_2+p_3)^2 - b_3\right)x^2 + b_{11}x + b_{12}, \\ y_6 &= \frac{F}{6EI}(L-x)^3 + \frac{G_2}{2EI}\left(-\frac{F}{2EI}(L-L_2-p_4)^2 + b_5 + \frac{F}{2EI}(L-L_2+p_3)^2 - b_3\right)x^2 + b_{13}x + b_{14}, \\ y_7 &= \frac{F}{6EI}(L-x)^3 + b_5x + b_6, \end{aligned} \tag{9}$$

The constants $b_i (i = 1, 2, 3, \ldots, 14)$ would be determined from conditions at crack sections, ends of piezoelectric patches and at the beam boundaries. Namely, the conditions are

$$\begin{gathered} y_1(L_1-p_1) = y_2(L_1-p_1), \quad y_1'(L_1-p_1) = y_2'(L_1-p_1), \quad y_3(L_1+p_2) = y_4(L_1+p_2), \\ y_3'(L_1+p_2) = y_4'(L_1+p_2), \quad y_4(L_2-p_3) = y_5(L_2-p_3), \quad y_4'(L_2-p_3) = y_5'(L_2-p_3), \\ y_6(L_2+p_4) = y_7(L_2+p_4), \quad y_6'(L_2+p_4) = y_7'(L_2+p_4), \\ y_2(L_1) = y_3(L_1), \quad y_3'(L_1) - y_2'(L_1) = \Theta_1 y_3''(L_1), \quad y_5(L_2) = y_6(L_2), \\ y_6'(L_2) - y_5'(L_2) = \Theta_2 y_6''(L_2), \quad y_1(0) = 0, \quad y_1'(0) = 0. \end{gathered} \tag{10}$$

Substituting solutions (9) into conditions (10) leads to system of equations

$$[\mathbf{A}]\{\mathbf{b}\} = \{\mathbf{C}\}, \tag{11}$$

where matrix $\mathbf{A}$ is given in Appendix A, vectors $\{\mathbf{b}\} = \{b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}\}^T$, $b_1 = \frac{FL^2}{2EI}$, $b_2 = -\frac{FL^3}{6EI}$ and $\{\mathbf{C}\} = \{C_1, C_2, \ldots, C_{12}\}$ with

$$C_1 = -\frac{FG_1(L_1-p_1)^2}{4(EI)^2}\left((L-L_1-p_2)^2 - (L-L_1+p_1)^2 + L^2\right) - \frac{FL^2}{2EI}(L_1-p_1) + \frac{FL^3}{6EI},$$

$$C_2 = -\frac{FG_1}{4(EI)^2}\left((L-L_1-p_2)^2-(L-L_1+p_1)^2+L^2\right)(L_1+p_2)^2,$$
$$C_3 = -\frac{FG_1}{2(EI)^2}\left((L-L_1-p_2)^2-(L-L_1+p_1)^2+L^2\right)(L_1-p_1)-\frac{FL^2}{2EI},\quad C_4=0,$$
$$C_5 = -\frac{FG_1}{2(EI)^2}\left((L-L_1-p_2)^2-(L-L_1+p_1)^2+L^2\right)(L_1+p_2),$$
$$C_6 = \frac{F\Theta_1}{EI}(L-L_1)-\frac{FG_1\Theta_1}{2(EI)^2}\left((L-L_1-p_2)^2-(L-L_1+p_1)^2+L^2\right),$$
$$C_7 = -\frac{FG_2}{4(EI)^2}\left((L-L_2-p_4)^2-(L-L_2+p_3)^2\right)(L_2-p_3)^2,$$
$$C_8 = -\frac{FG_2}{4(EI)^2}\left((L-L_2-p_4)^2-(L-L_2+p_3)^2\right)(L_2+p_4)^2,$$
$$C_9 = -\frac{FG_2}{2(EI)^2}\left((L-L_2-p_4)^2-(L-L_2+p_3)^2\right)(L_2+p_4),\quad C_{10}=0,$$
$$C_{11} = -\frac{FG_2}{2(EI)^2}\left((L-L_2-p_4)^2-(L-L_2+p_3)^2\right)(L_2-p_3),$$
$$C_{12} = \frac{F\Theta_2}{EI}(L-L_2)-\frac{FG_2\Theta_2}{2(EI)^2}\left((L-L_2-p_4)^2-(L-L_2+p_3)^2\right).$$

The cracked beam would be considered repaired if its slope at the cracks is continuous, i.e.

$$y_3'(L_1)-y_2'(L_1)=\Theta_1 y_3''(L_1)=0,\quad y_6'(L_2)-y_5'(L_2)=\Theta_2 y_6''(L_2)=0. \tag{12}$$

The latter conditions yield $b_9-b_7=0$ and $b_{13}-b_{11}=0$ that in consequence allow one to calculate restoring moment coefficients as

$$\begin{aligned} G_1 &= -\frac{2EI\,(L-L_1)}{\left(p_1^2-p_2^2\right)} = \frac{2EI\,(L-L_1)}{(p_2-p_1)\,(p_2-p_1)} \neq \frac{EI}{p_1+p_2+\Theta_1},\\ G_2 &= -\frac{2EI\,(L-L_2)}{\left(p_3^2-p_4^2\right)} = \frac{2EI\,(L-L_2)}{(p_4-p_3)\,(p_4+p_3)} \neq \frac{EI}{p_3+p_4+\Theta_2}, \end{aligned} \tag{13}$$

or

$$G_2/G_1 = \frac{(p_2+p_1)\,(p_2-p_1)\,(L-L_2)}{(p_4+p_3)\,(p_4-p_3)\,(L-L_1)}. \tag{14}$$

So that restoring moments and voltages of the piezoelectric patches are calculated as

$$M_1=-F\,(L-L_1)\,,\ M_2=-F\,(L-L_2)\,,\ V_1=-\frac{2F\,(L-L_1)}{e_{31}\,(H+\delta_1)},\ V_2=-\frac{2F\,(L-L_2)}{e_{31}\,(H+\delta_2)}. \tag{15}$$

It can be seen from Eq. (15) that $M_2/M_1=(L-L_2)\,/\,(L-L_1)$ and in case if the piezoelectric patches have the same design, we obtain also

$$G_2/G_1=(L-L_2)\,/\,(L-L_1)\,,\quad V_2/V_1=(L-L_2)\,/\,(L-L_1)\,. \tag{16}$$

Since the ratios obtained above are dependent only on crack positions but not crack depths, it can be proposed that for any subsequent crack at position L_n one obtains

$$G_k = G_1 \frac{(L - L_n)}{(L - L_1)}, \quad k = 2, 3, \ldots, n \tag{17}$$

and voltages and restoring moments can be calculated as

$$\begin{aligned} V_1 &= -\frac{2F(L - L_1)}{e_{31}(H + \delta_1)}, V_2 = -\frac{2F(L - L_2)}{e_{31}(H + \delta_2)}, \ldots, V_n = -\frac{2F(L - L_n)}{e_{31}(H + \delta_n)}, \\ M_1 &= -F(L - L_1), M_2 = -F(L - L_2), \ldots, M_n = -F(L - L_n). \end{aligned} \tag{18}$$

This fact will be approved by using finite element simulation performed in subsequent section.

2.4. Repairing cracked beam by applying restoring moments – the finite element simulation

This subsection is devoted to study static response of the cracked beam subjected to static force F and bending moments (18) by the well-known finite element method (FEM) [16–18]. The aim of this study is to verify the fact that multiple cracked beam could be repaired by applying bending moments (18) instead of using piezoelectric patches. So, the finite element model of cracked beam can be established as following: the beam is divided to N_e elements of the same length L_e and stiffness matrix [19]

$$\mathbf{K}_c^e = \mathbf{T}\tilde{\mathbf{C}}^{e^{-1}}\mathbf{T}^T, \tag{19}$$

where matrices $\mathbf{T} = \begin{bmatrix} -1 & -L_e & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^T$ and

$$\tilde{\mathbf{C}}^{e^{-1}} = \begin{bmatrix} \dfrac{12EI}{L_e^3 + 24mR_2EI} & -\dfrac{6EI}{L_e^3 + 24mR_2EI} \\ -\dfrac{6EI}{L_e^3 + 24mR_2EI} & -\dfrac{2\left(2L_e^3 + 3nL_e^2R_1EI + 12mR_3EI\right)EI}{\left(L_e^3 + 24mR_2EI\right)\left(L_e + 2nR_1EI\right)} \end{bmatrix},$$

$$n = \frac{36\pi}{EbH^4}, \quad R_1 = \int_0^a aF_I^2(s)\,da, \quad m = \frac{\pi}{EbH^2}, \quad R_2 = \int_0^a aF_{II}^2(s)\,da.$$

with

$$\begin{aligned} F_I\left(z = \frac{a}{H}\right) &= \sqrt{\frac{2}{\pi z}\tan\left(\frac{\pi z}{2}\right)} \frac{0.923 + 0.199[1 - \sin(\pi z/2)]^4}{\cos(\pi z/2)}, \\ F_{II}\left(z = \frac{a}{H}\right) &= \left(3z - 2z^2\right) \frac{1.122 - 0.561z + 0.085z^2 + 0.18z^3}{\sqrt{1 - z}}. \end{aligned}$$

The nodal load vector for an element is calculated as [19]

$$\mathbf{P}^e = \int_{Le} \mathbf{N}^T q(x)dx + \sum_{i=1}^{n_Q} \mathbf{N}^T(x_{Q_i})Q_i + \sum_{i=1}^{n_M} \frac{d}{dx}\mathbf{N}^T(x_{M_i})M_i, \tag{20}$$

where $q(x)$ is distributed load density; Q_i is concentrated load at position x_{Q_i}, M_i is concentrated moment at section x_{M_i}, n_Q and n_M are the numbers of concentrated loads and moments. Shape function vector

$$\mathbf{N}^T(x) = \left\{1 - 3\frac{x^2}{L_e^2} + 2\frac{x^3}{L_e^3}, x - 2\frac{x^2}{L_e} + \frac{x^3}{L_e^2}, 3\frac{x^2}{L_e^2} - 2\frac{x^3}{L_e^3}, -\frac{x^2}{L_e} + \frac{x^3}{L_e^2}\right\}^T.$$

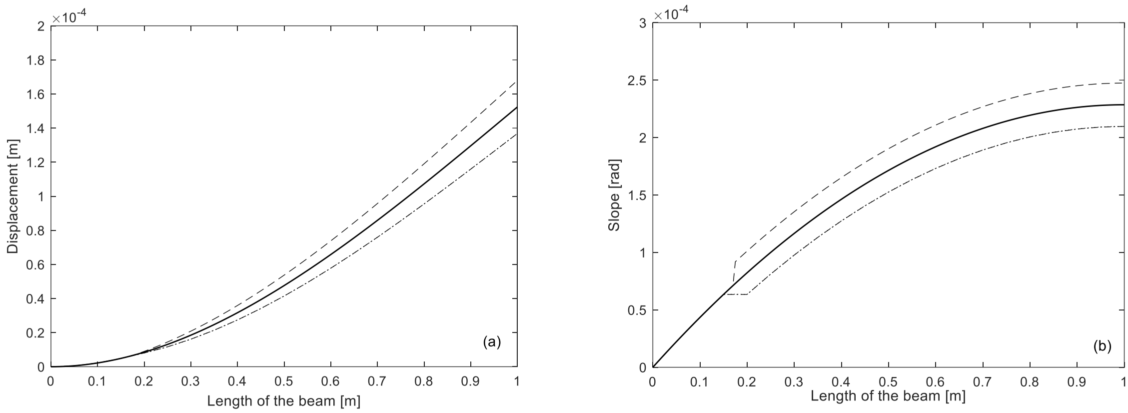
Assembling element load vectors and stiffness matrices one obtains equation

$$[\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{P}\}, \tag{21}$$

that can be solved using the CAFEM toolbox [19] and results in nodal displacement vector $\{\mathbf{q}\}$ including both deflection and slope at the nodes.

3. NUMERICAL RESULTS AND DISCUSSION

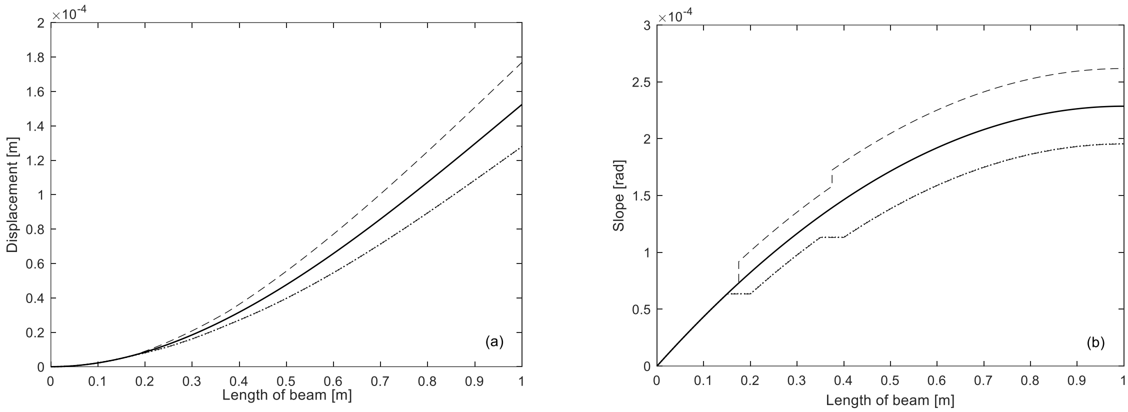
Let's consider cantilevered beam with $E = 210$ GPa, $L = 1.0$ m, rectangular cross section of high $H = 0.05$ m and wide $D = 0.1$ m. Concentrated load $F = 100$ N applied to free end of the beam $L = 1.0$ m and piezoelectric patches, made of PZT-4 with $e_{31} = -9.29$, have thickness $\delta = 0.15H$ and $p_1 = 0.0249$ mm, $p_2 = 0.025$ mm [4]. Deflection and slope diagrams in case of single, two, three and four cracks obtained by both the analytical solution and FEM are depicted in Figs. 2–5. In Fig. 6 there is given dependence of voltage needed to repair single crack on crack position along the beam length.



uncracked , – – – – – cracked beam without patch, ... cracked beam with patch,
– · – · – · cracked beam with restoring moments (FEM with 20 elements)

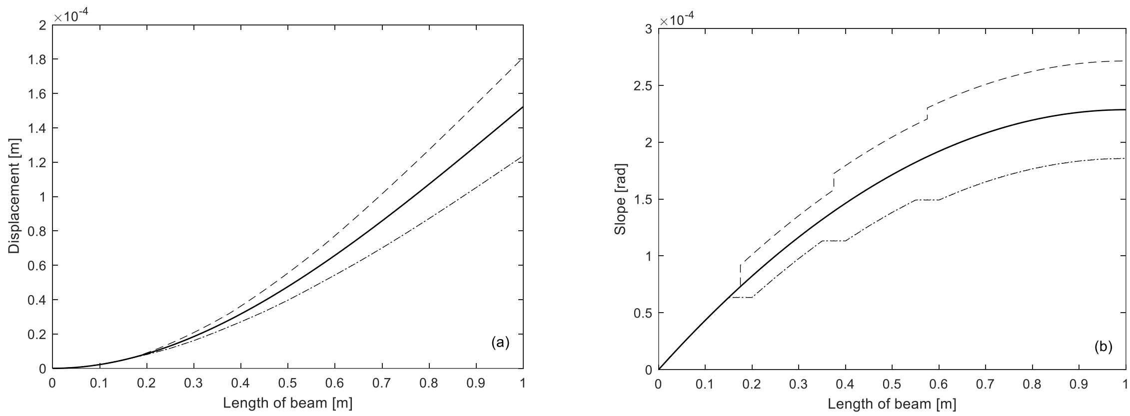
Fig. 2. Deflection (a) and slope (b) of beam with single crack of $L_1 = 0.175$m, $\Theta = 0.05$

Observing the graphics given in Figs. 2–5 allows one to make following remarks: (1) both deflection and slope curves calculated for beam with piezoelectric patches (dot lines) and those computed (by FEM) for beam subjected to restoring moments (dash-pot lines) are overlapped. This implies equivalence of piezoelectric repair and action of mechanical moments; (2) deflection of beam repaired by piezoelectric patches is really decreased in comparison with not repaired beam and even with uncracked beam that demonstrates



uncracked , - - - - - cracked beam without patch, ... cracked beam with patch,
- · - · - · cracked beam with restoring moments (FEM with 20 elements)

Fig. 3. Deflection (a) and slope (b) of beam with two cracks of $L_1 = 0.175$ m, $L_2 = 0.375$ m, $\Theta_1 = \Theta_2 = 0.05$



uncracked , - - - - - cracked beam without patch, ... cracked beam with patch,
- · - · - · cracked beam with restoring moments (FEM with 20 elements)

Fig. 4. Deflection (a) and slope (b) of beam with three cracks at positions $L_1 = 0.175$ m, $L_2 = 0.375$ m, $L_3 = 0.575$ m and $\Theta_1 = \Theta_2 = \Theta_3 = 0.05$.

productiveness of the repair; (3) the slope diagrams show clearly that discontinuity of slope at cracked section has disappeared after repairing and the aim of the repair is thus achieved. Moreover, Fig. 6 shows that voltage needed for repairing crack decreases as the crack moves to free end of beam.

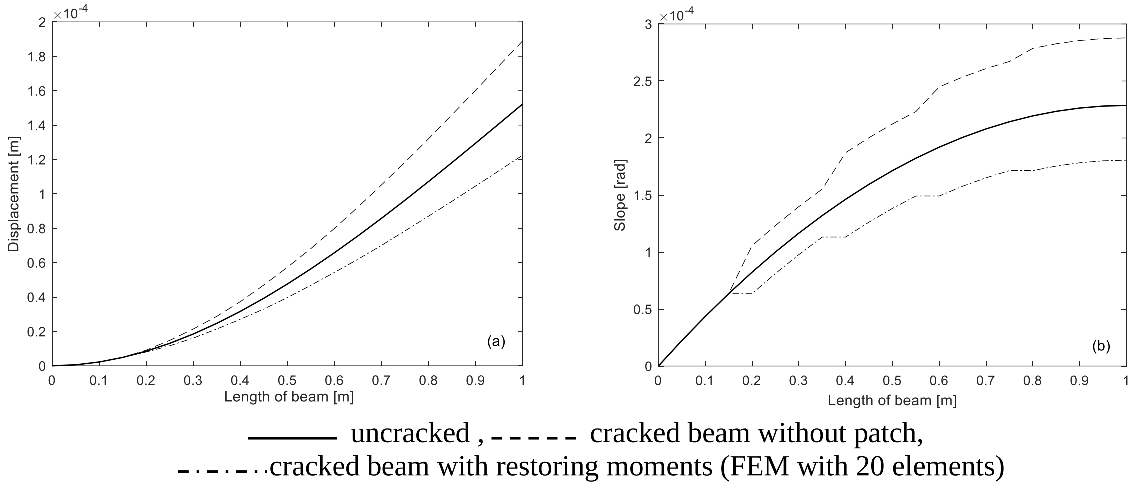


Fig. 5. Deflection (a) and slope (b) of beam with four cracks at positions $L_1 = 0.175$ m, $L_2 = 0.375$ m, $L_3 = 0.575\ m$, $L_4 = 0.775$ m and $\Theta_1 = \Theta_2 = \Theta_3 = \Theta_4 = 0.05$

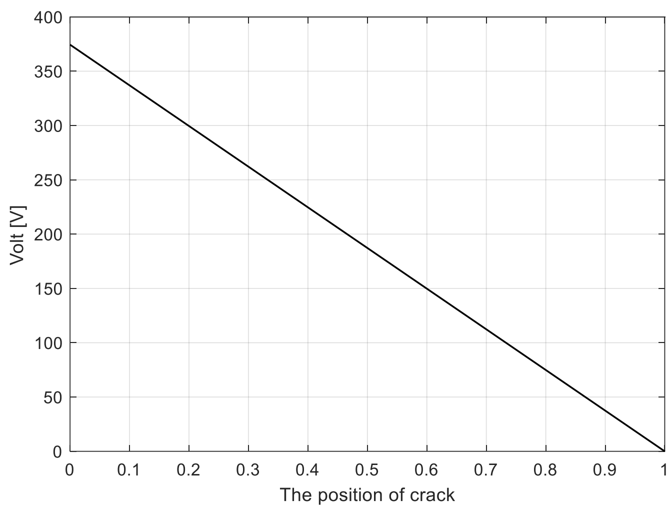


Fig. 6. Restoring voltage in dependence on the crack position $(L = 1.0\ \text{m}, H = 0.05\ \text{m}, e_{31} = -9.29, \delta = 0.15H, F = 100\ \text{N})$

4. CONCLUSION

The obtained in this study results demonstrated that beam with arbitrary number of open transverse cracks under static concentrated load can be productively repaired by using piezoelectric patches bonded to the beam segments surrounding cracks. Moreover, it was approved in the study that repair of multiple cracked beam by piezoelectric patches is equivalent to applying mechanical bending moments equal to so-called restoring moments calculated from the piezoelectric patches. In the context, the equivalent finite element method-based technique was proposed for static repair of multiple cracked beam.

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APPENDIX A

$$\mathbf{A}=\begin{bmatrix}
L_1-p_1+\frac{G_1(L_1-p_1)^2}{2EI} & -\frac{G_1(L_1-p_1)^2}{2EI} & 0 & 0 & 0 & p_1-L_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1+\frac{G_1(L_1-p_1)}{EI} & -\frac{G_1(L_1-p_1)}{EI} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{G_1(L_1+p_2)^2}{2EI} & -\frac{G_1(L_1+p_2)^2}{2EI}+L_1+p_2 & 1 & 0 & 0 & 0 & 0 & -L_1-p_2 & -1 & 0 & 0 & 0 & 0 & 0\\
\frac{G_1(L_1+p_2)}{EI} & 1-\frac{G_1(L_1+p_2)}{EI} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & L_2-p_3+\frac{G_2(L_2-p_3)^2}{2EI} & 1 & -\frac{G_2(L_2-p_3)^2}{2EI} & 0 & 0 & 0 & 0 & 0 & p_3-L_2 & -1 & 0 & 0 & 0\\
0 & 1+\frac{G_2(L_2-p_3)}{EI} & 0 & -\frac{G_2(L_2-p_3)}{EI} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & \frac{G_2(L_2+p_4)^2}{2EI} & 0 & -\frac{G_2(L_2+p_4)^2}{2EI}+L_2+p_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -L_2-p_4 & -1 & 0\\
0 & \frac{G_2(L_2+p_4)}{EI} & 0 & 1-\frac{G_2(L_2+p_4)}{EI} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & -L_3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_3-L\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & L_1 & 1 & -L_1 & -1 & 0 & 0 & 0 & 0 & 0\\
\frac{G_1\Theta_1}{EI} & -\frac{G_1\Theta_1}{EI} & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_2 & 1 & -L_2 & -1 & 0\\
0 & \frac{G_2\Theta_2}{EI} & 0 & -\frac{G_2\Theta_2}{EI} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0
\end{bmatrix}.$$