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ON TWO IMPROVED NUMERICAL ALGORITHMS FOR VIBRATION ANALYSIS OF SYSTEMS INVOLVING FRACTIONAL DERIVATIVES

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Abstract. Zhang and Shimizu (1998) proposed a numerical algorithm based on Newmark method to calculate the dynamic response of mechanical systems involving fractional derivatives. On the basis of Runge–Kutta–Nyström method and Newmark method, the present study proposes two new numerical algorithms, namely, the improved Newmark algorithm using the second order derivative and the improved Runge–Kutta–Nyström algorithm using the second order derivative to solve the fractional differential equations of vibration systems. The accuracy of new algorithms is investigated in detail by numerical simulation. The simulation result demonstrated that the Runge–Kutta–Nyström algorithm using the second order derivative for the vibration analysis of systems involving fractional derivatives is more effective than the Newmark algorithm of Zhang and Shimizu.

Keywords: vibration, fractional differential equation, numerical algorithm, dynamical systems.

1. INTRODUCTION

A differential equation is called the fractional differential equation if it includes at least one fractional-order derivative in the expression. Ordinary differential equations involving fractional differential operators of Riemann–Liouville's type or of Caputor's type are known to have many potential applications in mathematical modeling, in areas like mechanics, and the life sciences [1–9].

Among the approximate methods for finding a solution of nonlinear fractional differential equations, the decomposition method and the numerical method are often used. The decomposition method is a nonnumerical method for solving nonlinear differential equations [10–15]. The method was developed by George Adomian in 1984. Essentially, it approximates the solution of a non-linear differential equation with a series of functions. This method is getting into use for the solution of fractional differential equations. By using the decomposition method, one needs to express nonlinear terms in the form of power series. That is a difficult problem for many nonlinear fractional differential equations.

The use of numerical algorithms for the solution of differential equations involving fractional derivatives has been discussed in several works [16–28]. Yuan and Agrawal [22] have rewritten the definition of a fractional derivative and turned a fractional differential equation into a system of linear differential equations. However, Schmidt and Gaul [23] have shown that in some cases, this method loses the advantages of fractional calculus over integer-order calculus.

Zhang and Shimizu [28] presented the numerical method for dynamic problems involving fractional operators. Using the idea of Zhang and Shimizu, a new algorithm is developed by incorporating one-step schemes of well-known Newmark types [29] into its formula. Further, based on Riemann–Liouville's definition of fractional derivatives and the well-known Runge–Kutta–Nyström numerical method for calculating the solution of differential equations [30], we present a new algorithm for calculating nonlinear fractional differential equations. It is shown that the proposed algorithm is very efficient in many cases.

This study is organized into four sections. In Section 2, we present three numerical algorithms for solving fractional differential equations, including a well-known algorithm and two new algorithms. In Section 3, the effectiveness of the numerical algorithms is studied in detail. Section 4 includes some concluding remarks of the study.

2. SOME NUMERICAL ALGORITHMS FOR CALCULATING RESPONSES OF MECHANICAL SYSTEMS INVOLVING FRACTIONAL DERIVATIVES

2.1. Preliminaries

Fractional integrals and derivatives are deduced from the generalization of the integer-order operations. It is usual to define the integral operator D^{-q} as

$$ {}_aD_t^{-q}x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} x(\tau) d\tau, \tag{1}$$

where $q > 0$ and $\Gamma(x)$ is the Gamma function

$$ \Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz. \tag{2}$$

For a continuous $x(t)$,

$$ D^{-p} D^{-q} x(t) = D^{-(p+q)} x(t), \tag{3}$$

as given in [3] (if both p and q are non-negative).

With the fractional integral operator, fractional derivatives are easily introduced. For a real $\alpha > 0$, D^α is defined by the Riemann–Liouville definition [3], using the above fractional integral operator

$$ {}_aD_t^{\alpha}x(t) = \frac{d^n}{dt^n}\left(\frac{d^{-(n-\alpha)}x(t)}{dt^{-(n-\alpha)}}\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)d\tau. \tag{4} $$

Another choice is the Caputo definition

$$ {}_a^CD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-q)}\int_a^x (t-\tau)^{n-\alpha-1}\left[\frac{d^n}{d\tau^n}x(\tau)\right]d\tau. \tag{5} $$

In both cases $(n-1) < \alpha < n$.

Actually, the two definitions only differ in the consideration of conditions at the start of the interval

$$ {}_aD_t^{\alpha}x(t) = {}_a^CD_t^{\alpha}x(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}x^{(k)}(a). \tag{6} $$

The differential equation to be solved is the vibration equation with fractional damping, with one degree of freedom

$$ m\ddot{x}(t) + b\dot{x}(t) + \mu c(x)D^{\alpha}x(t) + g(x) = f(t), \quad 0 < \alpha < 1, \tag{7} $$

with the initial conditions

$$ x(0) = x_0, \dot{x}(0) = \dot{x}_0. \tag{8} $$

The existence and uniqueness of the solutions of Eq. (7) are presented in [4]. Note that this study focuses on developing numerical algorithms for solving this equation.

In the applications, D practically always means ${}_0D_t$, and most authors use the Riemann–Liouville, or the mathematically equivalent Gruenwald–Letnikov definition (see [3] for precise conditions of equivalence). Also, since the Riemann–Liouville definition has a singularity for non-zero initial conditions, the initial conditions are often considered zero. For a physical interpretation of this singularity, see [5–7].

Using the step-size

$$ h = \Delta t = t_i - t_{i-1}, \tag{9} $$

we have

$$ t_n = t_0 + nh = t_0 + n\Delta t, \quad n = 1, 2, 3, \ldots \tag{10} $$

Using the notations $x(t_i) = x_i$, from Eq. (7) we have the following iterative computational scheme at the time t_n as follows

$$ m\ddot{x}_n + b\dot{x}_n + \mu c(x_n)D^q x_n + kx_n = f(t_n), \quad n = 1, 2, 3, \ldots \tag{11} $$

2.2. The Newmark-based algorithm proposed by Zhang and Shimizu: A review

Based on the single-step integration method by Newmark [29], Zhang and Shimizu (1998) have obtained the following approximation formulas [28]

$$ \ddot{x}_n = \frac{1}{\beta\Delta t^2}(x_n - x_{n-1}) - \frac{1}{\beta\Delta t}\dot{x}_{n-1} - \left(\frac{1}{2\beta} - 1\right)\ddot{x}_{n-1} = \psi_2(\beta, \dot{x}_{n-1}, \ddot{x}_{n-1}, x_{n-1}, x_n), \tag{12} $$

$$\dot{x}_n = \dot{x}_{n-1} + (1-\alpha)\Delta t\ddot{x}_{n-1} + \alpha\Delta t\ddot{x}_n = \left(1-\frac{\alpha}{\beta}\right)\dot{x}_{n-1} + \left(1-\frac{\alpha}{2\beta}\right)\Delta t\ddot{x}_{n-1} + \frac{\alpha}{\beta\Delta t}\left(x_n - x_{n-1}\right) = \psi_1\left(\alpha,\beta,\dot{x}_{n-1},\ddot{x}_{n-1},x_{n-1},x_n\right). \tag{13}$$

Constants α, β are parameters associated with the quadrature scheme [28, 29]. The numerical algorithm to calculate the fractional derivative at $t = t_n$ of Eq. (4) is

$$D^q x\left(t_n\right) = \frac{x(t_0)}{\Gamma(1-q)}t_n^{-q} + \frac{1}{\Gamma(1-q)}\left[\int\limits_0^{t_{n-1}}\frac{\dot{x}(\tau)d\tau}{(t_n-\tau)^q} + \int\limits_{t_{n-1}}^{t_n}\frac{\dot{x}(\tau)d\tau}{(t_n-\tau)^q}\right] = \frac{1}{\Gamma(1-q)}\left(I_0 + I_{n-1} + \Delta I_n\right), \tag{14}$$

where we denote

$$I_{n-1} = \int\limits_0^{t_{n-1}}\frac{\dot{x}(\tau)d\tau}{(t_n-\tau)^q} \approx \frac{h}{2}\left[\frac{\dot{x}_0}{t_n^q} + \frac{\dot{x}_{n-1}}{h^q} + 2\sum_{i=1}^{n-2}\frac{\dot{x}(ih)}{(t_n-ih)^q}\right], \tag{15}$$

$$\Delta I_n = \int\limits_{t_{n-1}}^{t_n}\frac{\dot{x}(\tau)d\tau}{(t_n-\tau)^q} = \frac{\Delta t^{1-q}}{1-q}\dot{x}_{n-1} + (1-\alpha)\frac{\Delta t^{2-q}}{(1-q)(2-q)}\ddot{x}_{n-1} + \alpha\frac{\Delta t^{2-q}}{(1-q)(2-q)}\ddot{x}_n. \tag{16}$$

By substituting $\ddot{x}_n$ in Eq. (12) into Eq. (16) we obtain

$$\Delta I_n = \int\limits_{t_{n-1}}^{t_n}\frac{\dot{x}(\tau)d\tau}{(t_n-\tau)^q} = \frac{\Delta t^{1-q}}{(1-q)(2-q)}\left[\frac{\alpha}{\beta\Delta t}\left(x_n - x_{n-1}\right) + \left(2-q-\frac{\alpha}{\beta}\right)\dot{x}_{n-1} + \left(1-\frac{\alpha}{2\beta}\right)\Delta t\ddot{x}_{n-1}\right]. \tag{17}$$

Then, from Eq. (14) we have

$$D^q x\left(t_n\right) = \frac{1}{\Gamma(1-p)}\left(I_0 + I_{n-1} + \Delta I_n\right) = \psi_q\left(\alpha,\beta,\dot{x}_0,\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_{n-1},\ddot{x}_{n-1},x_{n-1},x_n\right). \tag{18}$$

From Eqs. (12), (13) and (18) we can rewrite Eq. (11) in the following form

$$m\psi_2\left(\beta,\dot{x}_{n-1},\ddot{x}_{n-1},x_{n-1},x_n\right) + b\psi_1\left(\alpha,\beta,\dot{x}_{n-1},\ddot{x}_{n-1},x_{n-1},x_n\right) + \mu c(x_n)\psi_q\left(\alpha,\beta,\dot{x}_0,\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_{n-1},\ddot{x}_{n-1},x_{n-1},x_n\right) + kx_n = f(t_n). \tag{19}$$

Eq. (19) is a nonlinear algebraic equation to find unknown x_n. We can then calculate $\dot{x}_n, \ddot{x}_n$ according to the following formulas

$$\begin{aligned} \dot{x}_n &= \dot{x}_{n-1} + (1-\alpha)\Delta t\ddot{x}_{n-1} + \alpha\Delta t\ddot{x}_n, \\ \ddot{x}_n &= \frac{1}{\beta\Delta t^2}\left(x_n - x_{n-1}\right) - \frac{1}{\beta\Delta t}\dot{x}_{n-1} - \left(\frac{1}{2\beta}-1\right)\ddot{x}_{n-1}. \end{aligned} \tag{20}$$

2.3. The improved Runge–Kutta–Nyström (RKN) algorithm

From the definition of the Liouville–Riemann's fractional derivative, Eq. (4), we can apply the composition rule to $D^q x(t)$ [1–3], that is

$$\begin{aligned}
D^q x(t_n) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{t_0}^{t_n}\frac{x(\tau)}{(t_n-\tau)^q}d\tau = \frac{x(t_0)}{\Gamma(1-q)}t_n^{-q} + \frac{1}{\Gamma(1-q)}\int_0^{t_n}\frac{\dot{x}(\tau)}{(t_n-\tau)^q}d\tau \\
&= \frac{x(t_0)}{\Gamma(1-q)}t_n^{-q} + \frac{1}{\Gamma(2-q)}\left[-\dot{x}(\tau)(t_n-\tau)^{1-q}\Big|_0^{t_n} + \int_0^{t_n}\ddot{x}(\tau)(t_n-\tau)^{1-q}d\tau\right] \\
&= \frac{x(t_0)}{\Gamma(1-q)}t_n^{-q} + \frac{1}{\Gamma(2-q)}\left[\dot{x}(t_0)t_n^{1-q} + \int_0^{t_n}\ddot{x}(\tau)(t_n-\tau)^{1-q}d\tau\right] \\
&= \frac{1}{\Gamma(1-q)}I_0 + \frac{1}{\Gamma(2-q)}\left(J_0 + J(t_n)\right),
\end{aligned} \tag{21}$$

where we denote

$$J_0 = \dot{x}(t_0)t_n^{1-q}, \quad J(t_n) = \int_0^{t_n}\ddot{x}(\tau)(t_n-\tau)^{1-q}d\tau = \int_0^{t_n} y_{t_n}(\tau)d\tau. \tag{22}$$

We approximate the integrals according to Eq. (22) for every instance tn by trapezoid numerical integration with an accuracy of $O(t^3)$ as follows

$$\begin{aligned}
&J(t_0) = 0, \\
&J(t_n) \approx \sum_{j=0}^{n-1}\frac{h}{2}\left[y_{t_n}(\tau_j) + y_{t_n}(\tau_{j+1})\right] = \sum_{j=0}^{n-2}\frac{h}{2}\left[y_{t_n}(\tau_j) + y_{t_n}(\tau_{j+1})\right] + \frac{h}{2}y_{t_n}(t_{n-1}), \quad (n \geq 1), \\
&J\left(t_n + \frac{h}{2}\right) \approx \sum_{j=0}^{n-1}\frac{h}{2}\left[y_{t_n+h/2}(\tau_j) + y_{t_n+h/2}(\tau_{j+1})\right] + \frac{h}{4}y_{t_n+h/2}(t_n), \quad (n \geq 0),
\end{aligned} \tag{23}$$

Thus, the formulas for determining the level fractional derivatives at $t_n, t_{n+h/2}$ and t_{n+h} have the following forms

$$\begin{aligned}
D^q x(t_n) &= \psi_1(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_{n-1}), \\
D^q x(t_n + h/2) &= \psi_2(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_{n-1}), \\
D^q x(t_n + h) = D^q x(t_{n+1}) &= \psi_3(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n).
\end{aligned} \tag{24}$$

From the approximation formula determining $D^q x(t_n)$ at step $t = t_n$, Eq. (7) can be rewritten in the following form

$$\begin{aligned}
&m\ddot{x}_n + b\dot{x}_n + \mu c(x_n)\psi_1(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_{n-1}) + kx_n = f(t_n), \\
&\ddot{x}_n = \frac{1}{m}\left(f(t_n) - b\dot{x}_n - \mu c(x_n)\psi_1(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_{n-1}) - kx_n\right) = g(t_n, x_n, \dot{x}_n),
\end{aligned} \tag{25}$$

It should be noted that when implementing expansion (21) we have used the assumption on convergence of integral $I = \int\limits_0^t \frac{\dot{x}(\tau)}{(t-\tau)^q} d\tau$ if $0 < q < 1$. Indeed, since the function $\dot{x}(t)$ is continuous on each finite time interval, so we have

$$|\dot{x}(\tau)| \le M \Rightarrow I = \int\limits_0^t \frac{\dot{x}(\tau)}{(t-\tau)^q} d\tau \le M \int\limits_0^t \frac{d\tau}{(t-\tau)^q}. \tag{26}$$

From the above inequality we deduce that the integral $I = \int\limits_0^t \frac{\dot{x}(\tau)}{(t-\tau)^q} d\tau$ converges if $0 < q < 1$. Then we can develop

$$\begin{aligned} I = \int\limits_0^{t_n} \frac{\dot{x}(\tau)}{(t_n-\tau)^q} d\tau &= \frac{1}{1-q} \left[-\dot{x}(\tau)(t_n-\tau)^{1-q} \Big|_0^{t_n} + \int\limits_0^{t_n} \ddot{x}(\tau)(t_n-\tau)^{1-q} d\tau \right] \\ &= \frac{1}{1-q} \left[\dot{x}(t_0) t_n^{1-q} + \int\limits_0^{t_n} \ddot{x}(\tau)(t_n-\tau)^{1-q} d\tau \right]. \end{aligned} \tag{27}$$

Applying the Runge–Kutta–Nyström algorithm with an accuracy of $O(t^4)$ to the differential equation (25), we have a straightforward schema [30] as below.

$$\ddot{x}_n = g(t_n, x_n, \dot{x}_n), \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \tag{28}$$

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + h\dot{\mathbf{x}}_n + \frac{h}{3}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \\ \dot{\mathbf{x}}_{n+1} &= \dot{\mathbf{x}}_n + \frac{1}{3}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \\ \ddot{\mathbf{x}}_{n+1} &= g(t_n + h, \mathbf{x}_{n+1}, \dot{\mathbf{x}}_{n+1}). \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathbf{k}_1 &= \frac{h}{2} g(t_n, \mathbf{x}_n, \dot{\mathbf{x}}_n), \\ \mathbf{k}_2 &= \frac{h}{2} g\left(t_n + \frac{h}{2}, \mathbf{x}_n + \frac{h}{2}\dot{\mathbf{x}}_n + \frac{h}{4}\mathbf{k}_1, \dot{\mathbf{x}}_n + \mathbf{k}_1\right), \\ \mathbf{k}_3 &= \frac{h}{2} g\left(t_n + \frac{h}{2}, \mathbf{x}_n + \frac{h}{2}\dot{\mathbf{x}}_n + \frac{h}{4}\mathbf{k}_1, \dot{\mathbf{x}}_n + \mathbf{k}_2\right), \\ \mathbf{k}_4 &= \frac{h}{2} g(t_n + h, \mathbf{x}_n + h\dot{\mathbf{x}}_n + h\mathbf{k}_3, \dot{\mathbf{x}}_n + 2\mathbf{k}_3). \end{aligned} \tag{30}$$

2.4. The improved Newmark algorithm

Using the second-order derivative by numerical integral we propose an algorithm based on the well-known Newmark algorithm [28, 29] to find the solution of Eq. (7).

Firstly, we have rewritten the Eqs. (12) and (13) as follows

$$\ddot{x}_n = \frac{1}{\beta \Delta t^2}\left(x_n - x_{n-1}\right) - \frac{1}{\beta \Delta t}\dot{x}_{n-1} - \left(\frac{1}{2\beta} - 1\right)\ddot{x}_{n-1} = \psi_2\left(\beta, \dot{x}_{n-1}, \ddot{x}_{n-1}, x_{n-1}, x_n\right), \quad (31)$$

$$\begin{aligned} \dot{x}_n &= \dot{x}_{n-1} + (1-\alpha)\Delta t \ddot{x}_{n-1} + \alpha \Delta t \ddot{x}_n \\ &= \left(1 - \frac{\alpha}{\beta}\right)\dot{x}_{n-1} + \left(1 - \frac{\alpha}{2\beta}\right)\Delta t \ddot{x}_{n-1} + \frac{\alpha}{\beta \Delta t}\left(x_n - x_{n-1}\right) \\ &= \psi_1\left(\alpha, \beta, \dot{x}_{n-1}, \ddot{x}_{n-1}, x_{n-1}, x_n\right). \end{aligned} \quad (32)$$

The numerical algorithm to calculate the fractional derivative at $t = t_n$ of Eq. (7) is

$$\begin{aligned} D^q x\left(t_n\right) &= \frac{x(t_0)}{\Gamma(1-q)} t_n^{-q} + \frac{1}{(1-q)\Gamma(1-q)}\left[\dot{x}(t_0) t_n^{1-q} + \int_0^{t_n} \ddot{x}(\tau)(t_n - \tau)^{1-q} d\tau\right] \\ &= \frac{1}{\Gamma(1-q)} I_0 + \frac{1}{(1-q)\Gamma(1-q)}\left(J_0 + J(t_n)\right), \end{aligned} \quad (33)$$

where

$$J_0 = \dot{x}(t_0) t_n^{1-q}, \;\; J(t_n) = \int_0^{t_n} \ddot{x}(\tau)(t_n - \tau)^{1-q} d\tau = \int_0^{t_n} y_{t_n}(\tau) d\tau. \quad (34)$$

Similarly, we approximate the integrals in Eq. (34) for every instance t_n by trapezoid numerical integration as follows

$$\begin{aligned} & J(t_0) = 0, \\ & J\left(t_n\right) \approx \sum_{j=0}^{n-1} \frac{h}{2}\left[y_{t_n}(\tau_j) + y_{t_n}(\tau_{j+1})\right] = \sum_{j=0}^{n-2} \frac{h}{2}\left[y_{t_n}(\tau_j) + y_{t_n}(\tau_{j+1})\right] + \frac{h}{2} y_{t_n}(t_{n-1}), \quad (n \geq 1) \\ & \Rightarrow D^q x\left(t_n\right) = \psi_q\left(x_0, \dot{x}_0, \ddot{x}_0, \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_{n-1}\right). \end{aligned} \quad (35)$$

From Eqs. (31), (32) and (35), we can rewrite Eq. (7) in the following form

$$\begin{aligned} & m\psi_2\left(\beta, \dot{x}_{n-1}, \ddot{x}_{n-1}, x_{n-1}, x_n\right) + b\psi_1\left(\alpha, \beta, \dot{x}_{n-1}, \ddot{x}_{n-1}, x_{n-1}, x_n\right) \\ & \qquad + \mu c(x_n)\psi_q\left(\dot{x}_0, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{n-1}, x_{n-1}\right) + k x_n = f(t_n). \end{aligned} \quad (36)$$

Eq. (36), a nonlinear algebraic equation of an unknown x_n, can be solved by the Newton–Raphson method of iteration. The variables $\dot{x}_n, \ddot{x}_n$ can then be determined by

$$\begin{aligned} \dot{x}_n &= \dot{x}_{n-1} + (1-\alpha)\Delta t \ddot{x}_{n-1} + \alpha \Delta t \ddot{x}_n, \\ \ddot{x}_n &= \frac{1}{\beta \Delta t^2}\left(x_n - x_{n-1}\right) - \frac{1}{\beta \Delta t}\dot{x}_{n-1} - \left(\frac{1}{2\beta} - 1\right)\ddot{x}_{n-1}. \end{aligned} \quad (37)$$

3. NUMERICAL RESULTS

To compare the accuracy of the numerical algorithms, these motion equations of a one-degree-of-freedom oscillator would be considered to evaluate.

Example 1: Consider the following system

$$\ddot{x}(t) + 0.8D^{0.5}x(t) + x^3 = f(t), \tag{38}$$

where

$$f(t) = 2\left(t - \frac{9}{10}\right)\left(t - \frac{7}{10}\right) + 4t\left(t - \frac{7}{10}\right) + 4t\left(t - \frac{9}{10}\right) + 2t^2 + \frac{8}{10\Gamma(0.5)}\left(\frac{128}{35}\sqrt{t^7} - \frac{128}{25}\sqrt{t^5} + \frac{42}{25}\sqrt{t^3}\right) + \left[t^2\left(t - \frac{9}{10}\right)\left(t - \frac{7}{10}\right)\right]^3, \tag{39}$$

and the initial conditions are

$$x(0) = 0, \quad \dot{x}(0) = 0. \tag{40}$$

The exact solution of this equation is known as [11,13,25] (see Fig. 1).

$$x_{\text{exact}} = t^2\left(t - \frac{9}{10}\right)\left(t - \frac{7}{10}\right). \tag{41}$$

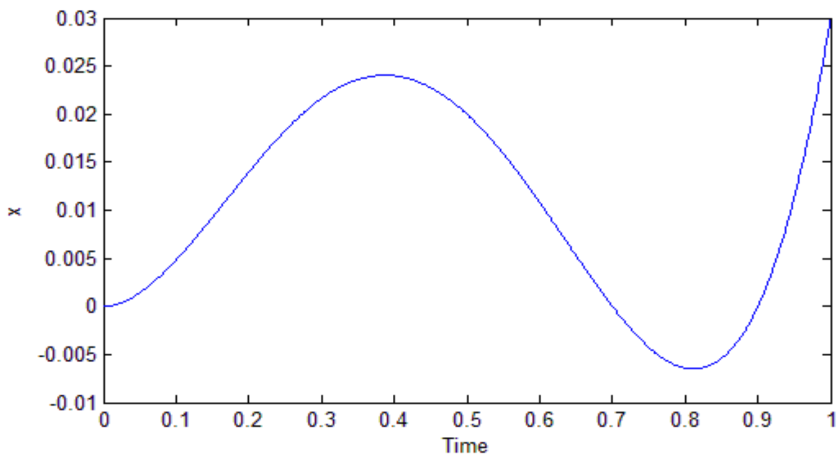

Fig. 1. Exact solution x of Eq. (38)

Three numerical algorithms considered in Section 2 are then applied to find the approximate solution of Eq. (38) to evaluate their accuracy. Table 1 and Table 2 show numerical values of the solution to Eq. (38) by our algorithms, exact solution, and the solution obtained by Ray et al. [13], and Atanackovic et al. [25]. Table 1 shows the five numerical solutions in comparison with the exact solution and their relative errors in percentage. It can be seen that the results of the improved Newmark and Newmark method in [28] are quite similar. However, the improved Runge–Kutta–Nyström algorithm shows the finest results with the highest accuracy over other algorithms.

Table 1. The exact solution, numerical solutions, and error in percentage of Eq. (38) (time step size $\Delta t = 0.001$)

Time	x_{exact}	$x_{\text{Atanackovic}}$ [25]	x_{Ray} [13]	$x_{\text{Zhang–Shimizu}}$ [28]	$x_{\text{improved Newmark}}$	$x_{\text{improved RKN}}$
0.25	0.01828125	0.018253191 (0.153486%)	0.0182813 (0.000274%)	0.018507464 (1.237407%)	0.018508641 (1.243849%)	0.018281311 (0.000332%)
0.5	0.02	0.019851524 (0.742382%)	0.0200026 (0.013000%)	0.020643625 (3.218126%)	0.020647787 (3.238937%)	0.020000016 (0.000080%)
0.75	−0.00421875	−0.004492587 (6.490951%)	−0.00419593 (0.540919%)	−0.003348193 (20.635433%)	−0.003343554 (20.745378%)	−0.004218918 (0.003975%)
1	0.03	0.029380011 (2.066632%)	0.0300995 (0.331667%)	0.030526185 (1.753949%)	0.030530651 (1.768835%)	0.029999869 (0.000437%)

Noted: The parentheses (.) indicate the relative errors (i.e., $\frac{|x_{method} - x_{\text{exact}}|}{x_{\text{exact}}} \times 100\%$) of the numerical results, in which the lowest error (highest accuracy) is underlined.

Example 2: To further evaluate these methods, let's consider the below system [31]

$$\ddot{x}(t) + 0.8D^{0.5}x(t) + x(t)^2 = f(t), \tag{42}$$

where

$$f(t) = 2\left(t - \frac{3}{10}\right)\left(t - \frac{8}{10}\right) + 4t\left(t - \frac{3}{10}\right) + 4t\left(t - \frac{8}{10}\right) + 2t^2 + \frac{8}{10\sqrt{\pi}}\left(\frac{128}{35}\sqrt{t^7} - \frac{88}{25}\sqrt{t^5} + \frac{16}{25}\sqrt{t^3}\right) + \left[t^2\left(t - \frac{3}{10}\right)\left(t - \frac{8}{10}\right)\right]^2, \tag{43}$$

and the initial conditions are

$$x(0) = 0, \quad \dot{x}(0) = 0. \tag{44}$$

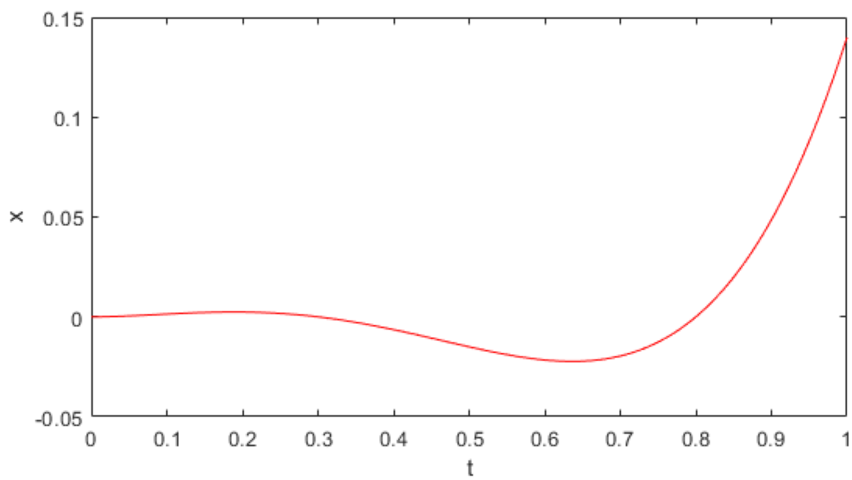

Fig. 2. Exact solution x of Eq. (42)

The exact solution of this equation as indicated in [31] is (see Fig. 2)

$$x_{\text{exact}} = t^2 \left(t - \frac{3}{10}\right)\left(t - \frac{8}{10}\right). \tag{45}$$

Table 2. The exact solution, numerical solutions, and error in percentage of Eq. (42) (time step size $\Delta t = 0.001$)

Time	x_{exact}	$x_{\text{Zhang–Shimizu}}$	$x_{\text{improved Newmark}}$	$x_{\text{improved RKN}}$
0.25	0.00171875	0.001858277 (8.1179263%)	0.001858493 (8.1305000%)	0.001718750 (<u>0.0000077%</u>)
0.5	−0.015	−0.014694001 (2.0399913%)	−0.014694632 (2.0357850%)	−0.015000092 (<u>0.0006150%</u>)
0.75	−0.01265625	−0.012534713 (0.9602941%)	−0.012537936 (0.9348281%)	−0.012656366 (<u>0.0009155%</u>)
1	0.14	0.139197963 (0.5728838%)	0.139202096 (0.5699317%)	0.140000451 (<u>0.0009155%</u>)

Similar to the results of Example 1, Table 2 shows the five numerical solutions in comparison with the exact solution and the relative error between them. The improved RKN also indicates robustness with the highest calculating accuracy. Therefore, the above comparison of the calculation accuracy leads to the following remark: the accuracy of the Runge–Kutta–Nyström algorithm using the second-order derivative is very good and better than the other aforementioned methods.

4. CONCLUSIONS

Using the idea of Zhang and Shimizu [28], two new numerical algorithms for finding the solution of nonlinear fractional differential equations are introduced in this paper. Based on Liouville–Riemann definition of fractional derivatives, and using the well-known Newmark numerical integration method [29], the well-known Runge–Kutta –Nyström integration method [30] for differential equations, we proposed two new numerical algorithms for solving the second-order systems containing fractional derivative components, namely, the improved Newmark algorithm and the improved Runge–Kutta–Nyström algorithm.

Compared to the aforementioned algorithms, two new algorithms have advantages in simplicity in the approximation of the fractional derivative components (see Eqs. (23) and (35)) and calculating scheme (see Eqs. (28)–(30)). The accuracy of these methods was verified by two examples in Section 3. It reveals that the improved Runge–Kutta–Nyström is very accurate for the vibration analysis of systems involving fractional derivatives. Noted that the improved RKN method could be easily extended for other systems with higher degrees of freedom as indicated in [32], and Duffing and Vander Pol systems as implemented in [33]. However, the problem of convergence and error in the calculating procedure is required for further investigation.

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