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MODAL ANALYSIS OF CRACKED BEAM WITH A PIEZOELECTRIC LAYER

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Abstract. Piezoelectric material was employed first as sensor/actuator for structural control and then it has got an effective use for structural health monitoring and repairing damaged structures. In this report, modal analysis of cracked beam with piezoelectric layer is carried out to investigate effect of crack and piezoelectric layer thickness on natural frequencies of the structure and output charge generated in the piezoelectric layer by vibration modes. Governing equations of the coupled structure are established using the double beam model and two-spring (translational and rotational) representation of crack and solved to obtain the modal parameters including the output charge associated with natural modes acknowledged as modal piezoelectric charge (MPC). Numerical examples have been examined for validation and illustration of the developed theory.

Keywords: cracked beam, piezoelectric layer, modal analysis, structural health monitoring.

1. INTRODUCTION

Piezoelectric material was employed first as sensor/actuator for structural control [1–4] and then it has got an effective use for structural health monitoring [5–11] and repairing damaged structures [12–17]. Using piezoelectric material for controlling or monitoring structural behavior is essentially leading to analysis of the structures with piezoelectric components such as beams or plates with layers or patches. Namely, Lee and Kim [18] first proposed to apply the spectral element method (SEM) for vibration analysis of Euler-Bernoulli beam bonded with a piezoelectric layer and declared that the method is consistent to study dynamic characteristics of the elastic-piezoelectric two-layer beams. Then, the SEM have been developed for modelling and analysis of homogeneous [19] and composite [20] Timoshenko beams with piezoelectric layers. Yang and Lee [21] used the stepped beam model for modal analysis of Timoshenko beam with piezoelectric patches symmetrically bonded onto both the top and bottom and demonstrated that stiffness and inertia of the piezoelectric material, as well as shear deformation and rotary inertia of the host beam may make change in natural frequencies of the coupled beam. The model of multi-step beam was employed also by Maurini et al. [22] for modal analysis of classical beam with numerous pairs of piezoelectric patches using different techniques including the so-called assumed modes method proposed by themselves. Wang and Quek [23] used the sandwich beam model for modal analysis of a Euler-Bernoulli beam embedded with piezoelectric layers and they found that natural frequency of the sandwich beam is function of stiffness and thickness of the piezoelectric layers. Nguyen Tien Khiem et al. [24] investigated effect of piezoelectric patch on natural frequencies of beam made of functionally graded material. Recently, dynamics of cracked structures with piezoelectric patches [25, 26] has attracted a special attention of researchers to develop an efficient method for crack identification using piezoelectric material. Namely, Zhao et al. [27] proposed an interesting technique for crack identification in beam-type structures by natural frequencies using coupled pairs of piezoelectric sensor and actuator.

In this report, modal analysis of cracked beam with piezoelectric layer is carried out to investigate effect of crack and piezoelectric layer thickness on natural frequencies of the structure and output charge generated in the piezoelectric layer by vibration mode. Governing equations of the coupled structure are established using the double beam model and two-spring (translational and rotational) representation of crack and solved to obtain the modal parameters including the output charge associated with natural modes acknowledged as modal piezoelectric charge (MPC). Numerical examples have been examined for validation and illustration of the developed theory.

2. GOVERNING EQUATIONS

Let's consider a Euler–Bernoulli beam of length L, cross section area $A_b = b \times h_b$, elastic modulus and mass density E, ρ that is bonded with a piezoelectric layer of thickness h_p and the same width as the beam (Fig. 1). Using the classical theory of beam and notations shown in Fig. 1, governing equations for the beam are

$$\begin{gathered} u\left(x,z,t\right) = u_0\left(x,t\right) - zw_0'\left(x,t\right), \quad w\left(x,z,t\right) = w_0\left(x,t\right), \\ \sigma_x = E\varepsilon_x, \quad \varepsilon_x = u_0' - zw_0'', \end{gathered} \tag{1}$$

where $u(x,z,t), w(x,z,t)$ denote axial and transverse displacements at arbitrary point in the beam; $u_0(x,t), w_0(x,t)$ - the displacements in the beam's mid-plane ($z = 0$) and ε_x, σ_x are strain and stress in cross-section at x.

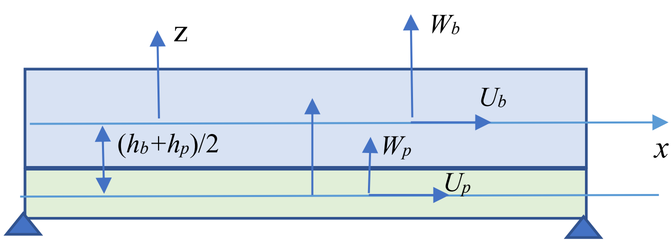

Fig. 1. Beam with piezoelectric layer model

Based on the governing equations (1) strain energy of the beam can be calculated as

$$\Pi_b = \left(\frac{1}{2}\right) \iiint (\sigma_x \varepsilon_x) dV_b = \left(\frac{1}{2}\right) \iiint [E\varepsilon_x^2] dV_b = \left(\frac{1}{2}\right) \int_0^L \left\{ EA_b u_0'^2 + I_b w_0''^2 \right\} dx, \quad (2)$$

where commas at the variables denotes their derivative with respect to x and $I_b = bh^3/12$. Total kinetic energy is

$$T_b = \left(\frac{1}{2}\right) \iiint \rho \left(\dot{u}^2 + \dot{w}^2\right) dV_b = \left(\frac{1}{2}\right) \int_0^L \left\{ \rho A_b \dot{u}_0^2 + \rho I_b \dot{w}_0'^2 + \rho A_b \dot{w}_0^2 \right\} dx. \quad (3)$$

Considering the piezoelectric layer also as a Euler–Bernoulli beam, governing equations of the layer are

$$\begin{gathered} u_p(x, \bar{z}, t) = u_{p0}(x,t) - \bar{z} w_{p0}'(x,t), \quad w_p(x,\bar{z},t) = w_{p0}(x,t), \\ \varepsilon_{px} = u_{p0}' - \bar{z} w_{p0}''(x,t), \end{gathered} \quad (4)$$

$$\sigma_{px} = C_{11}^p \varepsilon_{px} - h_{13} D, \quad \in = -h_{13} \varepsilon_{px} + \beta_{33}^p D, \quad (5)$$

with C_{11}^p, h_{13}, β_{33}^p denoting respectively the elastic modulus, piezoelectric and dielectric constants of the piezoelectric material; $\in$ and D are electric field and electric displacement in the piezoelectric layer.

Assuming perfect bonding between the piezoelectric layer and the host beam, the continuity conditions for the mechanical displacements can be obtained as

$$u\left(x, -\frac{h_b}{2}, t\right) = u_p\left(x, \frac{h_p}{2}, t\right), \quad w\left(x, -\frac{h_b}{2}, t\right) = w_p\left(x, \frac{h_p}{2}, t\right), \quad (6)$$

that lead to

$$u_{p0} = u_0 + w_0' h, \quad h = (h_b + h_p)/2, \quad w_{p0} = w_0. \quad (7)$$

Therefore, Eqs. (4) can be rewritten in the form

$$u_p(x, \bar{z}, t) = u_0(x,t) - (\bar{z} - h) w_0'(x,t), \quad \varepsilon_{px} = u_0' - (\bar{z} - h) w_0'', \quad (8)$$

that allow one to calculate the energies of the piezoelectric layer as

$$\begin{aligned} \Pi_p &= \left(\frac{1}{2}\right) \iiint (\sigma_{px}\varepsilon_{px} + \in D) dV_p = \left(\frac{1}{2}\right) \iiint [C_{11}^p \varepsilon_{px}^2 - 2h_{13} D \varepsilon_{px} + \beta_{33}^p D^2] dV_p \\ &= \left(\frac{1}{2}\right) \int_0^L \left\{ C_{11}^p A_p u_0'^2 + 2C_{11}^p A_p h u_0' w_0'' + C_{11}^p \left[I_p + A_p h^2\right] w_0''^2 \right\} dx \\ &+ \left(\frac{1}{2}\right) \int_0^L \left\{ -2h_{13} A_p D u_0' - 2h_{13} A_p D h w_0'' + \beta_{33}^p A_p D^2 \right\} dx, \end{aligned} \quad (9)$$

$$
\begin{aligned}
T_p &= \left(\frac{1}{2}\right) \iiint \rho_p \left(\dot{u}_p^2 + \dot{w}_p^2\right) dV_p \\
&= \left(\frac{1}{2}\right) \int_0^L \left\{\rho_p A_p \dot{u}_0^2 + \rho_p A_p h \dot{u}_0 \dot{w}_0' + \rho_p [I_p + A_p h^2] \dot{w}_0'^2 + \rho_p A_p \dot{w}_0^2\right\} dx, \\
A_p &= bh_p, \quad I_p = bh_p^3/12.
\end{aligned}
\tag{10}
$$

Summing up the defined above energies of both the host beam and piezoelectric layer gives total strain and kinetic energies of the double beam calculated as

$$
\begin{aligned}
\Pi &= \Pi_b + \Pi_p \\
&= \left(\frac{1}{2}\right) \int_0^L \left\{EA_b u_0'^2 + EI_b w_0''^2 + C_{11}^p A_p u_0'^2 + 2C_{11}^p A_p h u_0' w_0'' \right.\\
&\quad \left. + C_{11}^p [I_p + A_p h^2/4] w_0''^2 - 2h_{13} A_p D u_0' - h_{13} A_p D h w_0'' + \beta_{33}^p A_p D^2 \right\} dx \\
&= \left(\frac{1}{2}\right) \int_0^L \left\{A_{11} u_0'^2 + 2A_{12} u_0' w_0'' + A_{22} w_0''^2 - 2h_{13} A_p (u_0' + h w_0'') D + \beta_{33}^p A_p D^2\right\} dx,
\end{aligned}
\tag{11}
$$

$$
T = T_b + T_p = \left(\frac{1}{2}\right) \int_0^L \left\{I_{11} \dot{u}_0^2 + 2I_{12} \dot{u}_0 \dot{w}_0' + I_{22} \dot{w}_0'^2 + I_{11} \dot{w}_0^2\right\} dx, \tag{12}
$$

where the following notations have been used

$$
\begin{gathered}
A_{11} = EA_b + C_{11}^p A_p, \quad A_{12} = C_{11}^p A_p h, \quad A_{22} = EI_b + C_{11}^p \left(I_p + A_p h^2\right), \\
I_{11} = \rho A_b + \rho_p A_p, \quad I_{12} = \rho_p A_p h, \quad I_{22} = \rho I_b + \rho_p I_p + \rho_p A_p h^2.
\end{gathered}
\tag{13}
$$

Putting expressions (11)–(12) into Hamilton's principle

$$
\int_{t_1}^{t_2} \delta (T - \Pi)\, dt = 0, \tag{14}
$$

allows general equations of motion of the system to be derived in the form

$$
\begin{gathered}
\left(I_{11} \ddot{u}_0 - A_{11} u_0''\right) + \left(I_{12} \ddot{w}_0' - A_{12} w_0'''\right) + h_{13} A_p D' = 0, \\
I_{11} \ddot{w}_0 + A_{22} w_0'''' + A_{12} u_0''' - I_{12} \ddot{u}_0' - I_{22} \ddot{w}_0'' - h_{13} A_p h D''/2 = 0, \\
h_{13} A_p \left(u_0' + h w_0''/2\right) - \beta_{33}^p A_p D = 0.
\end{gathered}
\tag{15}
$$

Obviously, the last equation in (15) yields $D = h_{13}\left(u_0' + h w_0''\right)/\beta_{33}^p$ and substituting the later expression into the remained equations in (15) gives the equations of motion reduced to the final form

$$\left(I_{11}\ddot{u}_0 - B_{11}u_0''\right) + \left(I_{12}\ddot{w}_0' - B_{12}w_0'''\right) = 0,$$
$$I_{11}\ddot{w}_0 + B_{22}w_0'''' + B_{12}u_0''' - I_{12}\ddot{u}_0' - I_{22}\ddot{w}_0'' = 0, \tag{16}$$

where

$$B_{11} = A_{11} - A_p h_{13}^2/\beta_{33}^p = EA_b + E_p A_p, \quad B_{12} = A_{12} - A_p h h_{13}^2/\beta_{33}^p = E_p A_p h,$$
$$B_{22} = A_{22} - A_p h^2 h_{13}^2/\beta_{33}^p = EI_b + C_{11}^p I_p + E_p A_p h^2, \quad E_p = C_{11}^p - h_{13}^2/\beta_{33}^p.$$

Using Fourier transform $\{U(x,\omega), W(x,\omega)\} = \int\limits_{-\infty}^{\infty} \{u_0(x,t), w_0(x,t)\} e^{-i\omega t} dt$, Eq. (16) are transformed to

$$(\omega^2 I_{11} U + B_{11} U'') + (\omega^2 I_{12} W' + B_{12} W''') = 0,$$
$$B_{22} W'''' + B_{12} U''' + \omega^2 I_{12} U' + \omega^2 I_{22} W'' - \omega^2 I_{11} W = 0,$$

or

$$[\mathbf{A}_0]\{d^4\mathbf{z}/dx^4\} + [\mathbf{A}_1]\{d^3\mathbf{z}/dx^3\} + [\mathbf{A}_2]\{d^2\mathbf{z}/dx^2\} + [\mathbf{A}_3]\{d\mathbf{z}/dx\} + [\mathbf{A}_4]\{\mathbf{z}\} = 0, \tag{17}$$

where there are introduced following notations $\{\mathbf{z}\} = \{U(x,\omega), W(x,\omega)\}^T$ and

$$[\mathbf{A}_0] = \begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix}, \quad [\mathbf{A}_1] = \begin{bmatrix} 0 & B_{12} \\ B_{12} & 0 \end{bmatrix}, \quad [\mathbf{A}_2] = \begin{bmatrix} B_{11} & 0 \\ 0 & \omega^2 I_{22} \end{bmatrix},$$
$$[\mathbf{A}_3] = \begin{bmatrix} 0 & \omega^2 I_{12} \\ \omega^2 I_{12} & 0 \end{bmatrix}, \quad [\mathbf{A}_4] = \begin{bmatrix} \omega^2 I_{11} & 0 \\ 0 & -\omega^2 I_{11} \end{bmatrix}.$$

After Eqs. (16) have been solved, the output charge in the piezoelectric layer is calculated as

$$Q = b\int\limits_0^L D dx = (bh_{13}/\beta_{33}^p)\left(u_0 + hw_0'\right)\Big|_0^L. \tag{18}$$

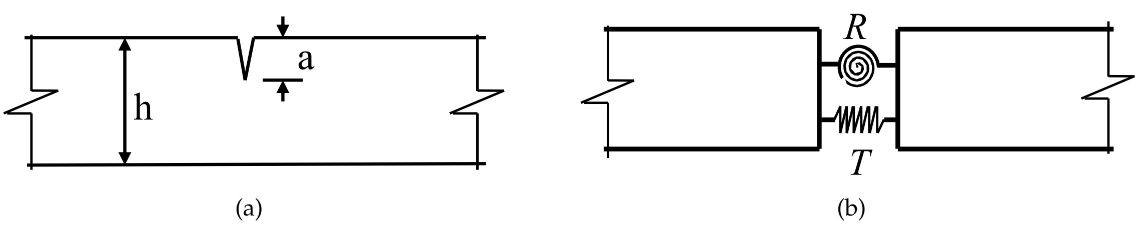

Fig. 2. Double spring model of crack in beam

Furthermore, if the beam is damaged at position e to a top edged crack of depth a and crack is represented by the double spring model with R and T being stiffness of the axial and rotational springs, as shown in Fig. 2. Therefore, conditions that should be satisfied at crack position are [28,29].

$$U\left(e+0,\omega\right)-U\left(e-0,\omega\right)=\gamma_a U'\left(e,\omega\right),\quad U'\left(e+0,\omega\right)=U'\left(e-0,\omega\right)=U'\left(e,\omega\right),$$
$$W\left(e+0,\omega\right)=W\left(e-0,\omega\right),\quad W'\left(e+0,\omega\right)-W'\left(e-0,\omega\right)=\gamma_b W''\left(e,\omega\right)$$
$$W''\left(e+0,\omega\right)=W''\left(e-0,\omega\right)=W''\left(e,\omega\right),W'''\left(e+0,\omega\right)=W'''\left(e-0,\omega\right)=W'''\left(e,\omega\right),\tag{19}$$

where $\gamma_a = EA/T$ and $\gamma_b = EI/R$ are so-called crack magnitudes calculated as

$$\begin{aligned}&\gamma_a = 2\pi(1-\nu_0^2)hf_a(z),\quad \gamma_b = 6\pi\left(1-\nu_0^2\right)hf_b\left(z\right),\quad z=a/h,\\ &f_a(z) = z^2(0.6272-0.17248z+5.92134z^2-10.7054z^3+31.5685z^4-67.47z^5\\ &\qquad +139.123z^6-146.682z^7+92.3552z^8),\\ &f_b(z) = z^2(0.6272-1.04533z+4.5948z^2-9.9736z^3+20.2948z^4-33.0351z^5\\ &\qquad +47.1063z^6-40.7556z^7+19.6z^8).\end{aligned}\tag{20}$$

3. GENERAL SOLUTION OF FREE VIBRATION PROBLEM

In this section, Eq. (17) is solved by seeking its solution in the form

$$\{\mathbf{z}_0\} = \{U_0, W_0\}^T e^{\lambda x},\tag{21}$$

that leads the equation to

$$\left[\lambda^4\mathbf{A}_0+\lambda^3\mathbf{A}_1+\lambda^2\mathbf{A}_2+\lambda\mathbf{A}_3+\mathbf{A}_4\right]\{U_0,W_0\}^T=0.\tag{22}$$

The latter equation has non-trivial solution under the condition

$$\det\left[\lambda^4\mathbf{A}_0+\lambda^3\mathbf{A}_1+\lambda^2\mathbf{A}_2+\lambda\mathbf{A}_3+\mathbf{A}_4\right]=0,$$

that is so-called characteristic equation for determining wave number λ as function of frequency, $\lambda = \lambda(\omega)$. It is not difficult to show the characteristic equation can be obtained in the form

$$a\lambda^6+b\lambda^4+c\lambda^2+d=0,\tag{23}$$

with

$$a = B_{11}B_{22}-B_{12}^2,\quad b=\omega^2(B_{11}I_{22}+B_{22}I_{11}-2B_{12}I_{12}),$$

$$c=\omega^4\left(I_{11}I_{22}-I_{12}^2\right)-\omega^2B_{11}I_{11},\quad d=-\omega^4I_{11}^2.$$

As a cubic algebraic equation with respect to $\eta=\lambda^2$, $a\eta^3+b\eta^2+c\eta+d=0$, that has three roots denoted by η_1,η_2,η_3, six roots of characteristic equation (23) can be obtained in the form

$$\lambda_{1,4}=\pm k_1,\quad \lambda_{2,5}=\pm k_2,\quad \lambda_{3,6}=\pm k_3,\quad k_j=\sqrt{\eta_j},\quad j=1,2,3.$$

Hence, general solution of Eq. (22) is expressed as

$$\begin{aligned}\{U_0\left(x,\omega\right)W_0\left(x,\omega\right)\} = &\Big\{\alpha_1C_1e^{k_1x}+\alpha_2C_2e^{k_2x}+\alpha_3C_3e^{k_3x}-\alpha_1C_4e^{-k_1x}-\alpha_2C_5e^{-k_2x}\\ &-\alpha_3C_6e^{-k_3x}C_1e^{k_1x}+C_2e^{k_2x}+C_3e^{k_3x}+C_4e^{-k_1x}+C_5e^{-k_2x}+C_6e^{-k_3x}\Big\},\end{aligned}\tag{24}$$

where $\{\mathbf{C}\} = \{C_1, \ldots, C_6\}^T$ is vector of arbitrary constants and

$$\alpha_j = -k_j(\omega^2 I_{12} + k_j^2 B_{12})/(\omega^2 I_{11} + k_j^2 B_{11}), \quad j = 1, 2, 3.$$

For example, using expression (24), a particular solution denoted by $\mathbf{z}_1(x,\omega) = \{U_1(x,\omega), W_1(x,\omega)\}^T$, satisfying conditions

$$U_1(0,\omega) = Z_a^0,\ U_1'(0,\omega) = 0,\ W_1(0,\omega) = 0,\ W_1'(0,\omega) = Z_b^0,\ W_1''(0) = W_1'''(0) = 0,$$

can be found as

$$U_1(x,\omega) = g_{ua}(x,\omega) Z_a^0 + g_{ub}(x,\omega) Z_b^0,\ W_1(x,\omega) = g_{wa}(x,\omega) Z_a^0 + g_{wb}(x,\omega) Z_b^0, \quad (25)$$

where

$$\begin{gathered} g_{ua}(x,\omega) = \alpha_1\delta_{1a}\cos k_1 x + \alpha_2\delta_{2a}\cos k_2 x + \alpha_3\delta_{3a}\cos k_3 x, \\ g_{ub}(x,\omega) = \alpha_1\delta_{1b}\cos k_1 x + \alpha_2\delta_{2b}\cos k_2 x + \alpha_3\delta_{3b}\cos k_3 x, \\ g_{wa}(x,\omega) = \delta_{1a}\sin k_1 x + \delta_{2a}\sin k_2 x + \delta_{3a}\sin k_3 x, \\ g_{wb}(x,\omega) = \delta_{1b}\sin k_1 x + \delta_{2b}\sin k_2 x + \delta_{3b}\sin k_3 x, \\ \delta_{1a} = k_2k_3\left(k_3^2 - k_2^2\right)/\Delta, \quad \delta_{2a} = k_1k_3\left(k_1^2 - k_3^2\right)/\Delta, \quad \delta_{3a} = k_1k_2\left(k_2^2 - k_1^2\right)/\Delta, \\ \delta_{1b} = \left(\alpha_3k_2^3 - \alpha_2k_3^3\right)/\Delta, \quad \delta_{2b} = \left(\alpha_1k_3^3 - \alpha_3k_1^3\right)/\Delta, \quad \delta_{3b} = \left(\alpha_2k_1^3 - \alpha_1k_2^3\right)/\Delta, \\ \Delta = \alpha_1k_2k_3\left(k_3^2 - k_2^2\right) + \alpha_2k_1k_3\left(k_1^2 - k_3^2\right) + \alpha_3k_1k_2\left(k_2^2 - k_1^2\right). \end{gathered} \quad (26)$$

Using particular solution (25) with $Z_a^0 = \gamma_a U_0'(e,\omega), Z_b^0 = \gamma_b W_0''(e,\omega)$, it can be shown that general solution for free vibration of cracked beam that satisfies conditions (19) can be obtained in the form

$$\begin{aligned} U_c(x,\omega) &= U_0(x,\omega) + \begin{cases} 0: & x < e, \\ U_1(x-e,\omega): & x \geq e, \end{cases} \\ W_c(x,\omega) &= W_0(x,\omega) + \begin{cases} 0: & x < e, \\ W_1(x-e,\omega): & x \geq e. \end{cases} \end{aligned} \quad (27)$$

Introducing the following vectors and matrices

$$\{\mathbf{z}_c(x,\omega)\} = \{U_c(x,\omega), W_c(x,\omega)\}^T, \quad \{\mathbf{z}_0(x,\omega)\} = \{U_0(x,\omega), W_0(x,\omega)\}^T,$$

$$[\mathbf{G}_c(x,\omega)] = \begin{bmatrix} \gamma_a g_{ua}(x,\omega) & \gamma_b g_{ub}(x,\omega) \\ \gamma_a g_{wa}(x,\omega) & \gamma_b g_{wb}(x,\omega) \end{bmatrix}, \quad [\mathbf{K}(x)] = \begin{cases} [\mathbf{G}_c(x,\omega)]: & x \geq 0, \\ [\mathbf{0}]: & x < 0, \end{cases}$$

$$\left[\mathbf{K}'(x)\right] = \begin{cases} \left[\mathbf{G}_c'(x,\omega)\right]: & x \geq 0, \\ [\mathbf{0}]: & x < 0, \end{cases} \quad \left[\mathbf{K}''(x)\right] = \begin{cases} \left[\mathbf{G}_c''(x,\omega)\right]: & x \geq 0, \\ [\mathbf{0}]: & x < 0, \end{cases}$$

$$[\mathbf{G}_0(x,\omega)] = \begin{bmatrix} \alpha_1 e^{k_1x} & \alpha_2 e^{k_2x} & \alpha_3 e^{k_3x} & -\alpha_1 e^{-k_1x} & -\alpha_2 e^{-k_2x} & -\alpha_3 e^{-k_3x} \\ e^{k_1x} & e^{k_2x} & e^{k_3x} & e^{-k_1x} & e^{-k_2x} & e^{-k_3x} \end{bmatrix},$$

$$[\widehat{\mathbf{G}}(x,\omega)] = \begin{bmatrix} \alpha_1k_1e^{k_1x} & \alpha_2k_2e^{k_2x} & \alpha_3k_3e^{k_3x} & \alpha_1k_1e^{-k_1x} & \alpha_2k_2e^{-k_2x} & \alpha_3k_3e^{-k_3x} \\ k_1^2e^{k_1x} & k_2^2e^{k_2x} & k_3^2e^{k_3x} & k_1^2e^{-k_1x} & k_2^2e^{-k_2x} & k_3^2e^{-k_3x} \end{bmatrix},$$

$$[\mathbf{\Phi}(x,\omega)] = [\mathbf{G}_0(x,\omega)] + [\mathbf{K}(x-e,\omega)\widehat{\mathbf{G}}(e,\omega)] \tag{28}$$

and using expressions (24), solution (27) can be rewritten in the form

$$\{\mathbf{z}_c\,(x,\omega)\} = [\mathbf{\Phi}\,(x,\omega)]\,\{\mathbf{C}\}\,. \tag{29}$$

Applying boundary conditions for general solution (29) allows one to solve the free vibration problem of the coupled beam. For example, in case of cantilever beam with boundary conditions

$$U\,(0) = W\,(0) = W'\,(0) = U'\,(L) = W''\,(L) = W'''\,(L) = 0, \tag{30}$$

one obtains equation for determining the constant vector $\{\mathbf{C}\} = \{C_1,\ldots,C_6\}^T$ as

$$[\mathbf{B}\,(\omega)]\,\{\mathbf{C}\} = 0, \tag{31}$$

where

$$[\mathbf{B}_{can}(\omega)] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & -k_1 & -k_2 & -k_3 \\ \phi'_{11}(L) & \phi'_{12}(L) & \phi'_{13}(L) & \phi'_{14}(L) & \phi'_{15}(L) & \phi'_{16}(L) \\ \phi''_{21}(L) & \phi''_{22}(L) & \phi''_{23}(L) & \phi''_{24}(L) & \phi''_{25}(L) & \phi''_{26}(L) \\ \phi'''_{21}(L) & \phi'''_{22}(L) & \phi'''_{23}(L) & \phi'''_{24}(L) & \phi'_{25}(L) & \phi'''_{26}(L) \end{bmatrix},$$

$\phi_{ij}(x), \phi'_{ij}(x), \phi''_{ij}(x), \phi'''_{ij}(x), i = 1,2; j = 1,\ldots,6$ are elements of the matrix $[\mathbf{\Phi}\,(x,\omega)]$ and their derivatives. Similarly, for clamped and simply supported beams, the matrix $[\mathbf{B}(\omega)]$ get respectively the form

$$[\mathbf{B}_{cc}(\omega)] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & -k_1 & -k_2 & -k_3 \\ \phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\ \phi_{21}(L) & \phi_{22}(L) & \phi_{23}(L) & \phi_{24}(L) & \phi_{25}(L) & \phi_{26}(L) \\ \phi'_{21}(L) & \phi'_{22}(L) & \phi'_{23}(L) & \phi'_{24}(L) & \phi'_{25}(L) & \phi'_{26}(L) \end{bmatrix},$$

$$[\mathbf{B}_{ss}(\omega)] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ k_1^2 & k_2^2 & k_3^2 & k_1^2 & k_2^2 & k_3^2 \\ \phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\ \phi_{21}(L) & \phi_{22}(L) & \phi_{23}(L) & \phi_{24}(L) & \phi_{25}(L) & \phi_{26}(L) \\ \phi''_{21}(L) & \phi''_{22}(L) & \phi''_{23}(L) & \phi''_{24}(L) & \phi''_{25}(L) & \phi''_{26}(L) \end{bmatrix}.$$

Obviously, so-called frequency equation of the beam with piezoelectric layer can be obtained from Eq. (31) as

$$d\,(\omega) \equiv \det\,[\mathbf{B}(\omega)] = 0, \tag{32}$$

positive roots of which provide the desired natural frequencies $\omega_k, k = 1,2,3,\ldots$ Every natural frequency $\omega = \omega_k$ allows one to find corresponding solution of Eq. (31) as $\{\mathbf{C}_k\} = \vartheta_k\{\beta_{k1},\ldots,\beta_{k6}\}^T$, where ϑ_k is arbitrary constant and $\{\beta_{k1},\ldots,\beta_{k6}\}^T$ is normalized solution of equation

$$[\mathbf{B}\,(\omega_k)]\,\{\mathbf{C}\} = 0.$$

Thus, mode shape associated with natural frequency ω_k would be calculated as

$$\phi_u(x,\omega_k) = \vartheta_k(\alpha_1\beta_{k1}e^{k_1x} + \alpha_2\beta_{k2}e^{k_2x} + \alpha_3\beta_{k3}e^{k_3x} - \alpha_1\beta_{k4}e^{-k_1x} - \alpha_2\beta_{k5}e^{-k_2x} - \alpha_3\beta_{k6}e^{-k_3x}),$$

$$\phi_w(x,\omega_k) = \vartheta_k(\beta_{k1}e^{k_1x} + \beta_{k2}e^{k_2x} + \beta_{k3}e^{k_3x} + \beta_{k4}e^{-k_1x} + \beta_{k5}e^{-k_2x} + \beta_{k6}e^{-k_3x}),$$

from that slope mode can be calculated as

$$\phi_\theta(x,\omega_k) = \vartheta_k(\beta_1k_1e^{k_1x} + \beta_2k_2e^{k_2x} + \beta_3k_3e^{k_3x} - \beta_4k_1e^{-k_1x} - \beta_5k_2e^{-k_2x} - \beta_6k_3e^{-k_3x}). \quad (33)$$

The arbitrary constant ϑ_k is determined form a chosen normalization condition, e.g.

$$\max_x |\phi_w(x,\omega_k)| = 1. \quad (34)$$

It can be noted that the slope mode represented by expression (33) can be employed for calculating output charge of the piezoelectric layer by formula (18). Namely, since $U(0) = U(L) = 0$ for simply supported beam, the formula (18) is reduced to

$$Q_k = Q_k^0 - \left(bh_{13}/\beta_{33}^p\right)\left\{\gamma_a\phi_u'(e,\omega_k) + \gamma_b\phi_w''(e,\omega_k)\right\}, \quad k = 1,2,3,\ldots \quad (35)$$

where $Q_k^0 = \left(bh_{13}/\beta_{33}^p\right)\left\{\left[\phi_u(L,\omega_k) - \phi_u(0,\omega_k)\right] + h\left[\phi_w'(L,\omega_k) - \phi_w'(0,\omega_k)\right]\right\}$ is the charge in case uncracked beam. The latter quantities are acknowledged hereby Modal Piezoelectric Response (MPR) associated with natural vibration k-th modes and these characteristics of the piezoelectric layer are numerically examined below mutually with natural frequencies of the coupled beam in dependence upon crack.

4. NUMERICAL RESULTS AND DISCUSSION

Numerical analysis is completed with following data: equal length and width of both beam and piezoelectric layer: $L = 1$ m, $b = 0.1$ m; material and geometry parameters of the host beam are denoted with lower index b and those of piezoelectric layer with p index:

$$E_b = 210\,\text{MPa}; \rho_b = 7800\,\text{kg/m}^3; \mu_b = 0.31; h_b = 0.05\text{ m}; A_b = bh_b; I_b = bh_b^3/12,$$

and piezoelectric constants

$$C_{11}^p = 69.0084\,\text{GPa}, C_{55}^p = 21.0526\,\text{GPa}, \rho_p = 7750\,\text{kg/m}^3, h_{13} = -7.70394\times10^8\text{ V/m},$$

$$\beta_{33}^p = 7.3885\times10^7\text{ m/F}.$$

The so-called frequency parameters $\lambda_k = (\rho_bA_b\omega_k^2/E_bI_b)^{1/4}$ that represent natural frequencies ω_k are calculated herein as function of crack position along the beam span with different crack depth and thickness of piezoelectric layer h_p. The charge generated in the piezoelectric layer, Q_k calculated from k-th mode shape by formula (18) acknowledged here as modal piezoelectric response (MPR) is examined below in dependence upon crack parameters.

First, effect of piezoelectric layer thickness on natural frequencies of the undamaged (intact) beam is studied and 10 natural frequencies calculated for various thickness of piezoelectric layer h_p are presented in Tabs. 1–3. Excellent agreement of the natural frequencies obtained in case of beam without piezoelectric layer (corresponding to zero thickness $h_p = 0$) with the well-known natural frequencies of single beam in different

Table 1. Effect of piezoelectric layer thickness on natural frequencies of simply supported intact beam

$h_p = 0$	0.001 (m)	0.003 (m)	0.005 (m)	0.008 (m)	0.01 (m)	0.02 (m)	0.03 (m)	Mode
3.1416	3.1393	3.1408	3.1461	3.1604	3.1738	3.2783	3.4245	B1
6.2832	6.2686	6.2708	6.2804	6.3075	6.3332	6.5355	6.8195	B2
9.4248	9.3781	9.3794	9.3915	9.4284	9.4642	9.7506	10.1543	B3
12.5664	12.4586	12.4567	12.4689	12.5114	12.5545	12.9095	13.4403	B4
14.7531	**14.7036**	**14.6086**	**14.5186**	**14.3923**	**14.3133**	**13.9677**	**13.6606**	**A1**
15.7080	15.5016	15.4935	15.5025	15.5451	15.5912	15.9868	16.5579	B5
18.8496	18.4994	18.4818	18.4838	18.5204	18.5651	18.9770	19.3746	B6
20.8641	**20.7941**	**20.6600**	**20.5334**	**20.3559**	**20.2450**	**19.7638**	**19.5842**	**A2**
21.9912	21.4455	21.4145	21.4052	21.4289	21.4673	21.8677	22.4805	B7
25.1328	24.3344	24.2866	24.2630	24.2694	24.2997	24.2337	23.7583	B8

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$ m.

Table 2. Effect of piezoelectric layer thickness on natural frequencies of uncracked beam with clamped ends

$h_p = 0$	0.001 (m)	0.003 (m)	0.005 (m)	0.008 (m)	0.01 (m)	0.02 (m)	0.03 (m)	Mode
4.7300	4.7259	4.7282	4.7361	4.7575	4.7777	4.9347	5.1547	B1
7.8530	7.8322	7.8348	7.8465	7.8799	7.9118	8.1626	8.5149	B2
10.9960	10.9353	10.9364	10.9500	10.9921	11.0333	11.3637	11.8300	B3
14.1377	14.0059	14.0029	14.0156	14.0615	14.1081	13.9716	13.6982	B4
14.7531	**14.7036**	**14.6087**	**14.5189**	**14.3933**	**14.3151**	**14.5002**	**15.0547**	**A1**
17.2788	17.0366	17.0265	17.0350	17.0796	17.1286	17.5527	18.1629	B5
20.4204	20.0201	19.9994	19.9998	20.0365	20.0828	19.7658	19.3863	B6
20.8641	**20.7940**	**20.6599**	**20.5331**	**20.3554**	**20.2445**	**20.5158**	**21.1547**	**A2**
23.5620	22.9499	22.9148	22.9030	22.9253	22.9642	23.3789	23.7572	B7
26.7036	25.4674	25.3030	25.1478	24.9308	24.7957	24.2150	24.0148	B8

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$.

cases of boundary conditions demonstrates validity of the above developed model. Furthermore, the data given in the Tables reveal also the ordering of bending and axial (longitudinal) vibration modes. Namely, the axial mode of vibration in cantilever beam appears (as fourth) earlier than that of beams with clamped or simply supported ends (appeared as fifth). Generally, all natural frequencies of coupled beam first decrease with increasing thickness of piezoelectric layer and then become increasing when the thickness exceeds 10% host beam thickness. Moreover, in most cases, the natural frequencies overcome those of the host beam alone as the layer thickness gets to be more than 20% thickness of the host beam. This implies the fact that bonded piezoelectric layer of a given thickness could increase stiffness of beam and, consequently, it can be used for repairing the beam of reduced stiffness for some reason such as cracking.

Table 3. Effect of piezoelectric layer thickness on natural frequencies of uncracked cantilever beam

$h_p = 0$	0.001 (m)	0.003 (m)	0.005 (m)	0.008 (m)	0.01 (m)	0.02 (m)	0.03 (m)	Mode
1.8750	1.8748	1.8758	1.8791	1.8878	1.8959	1.9591	2.0475	B1
4.6940	4.6898	4.6920	4.6998	4.7210	4.7410	4.8967	5.1147	B2
7.8550	7.8338	7.8364	7.8481	7.8815	7.9134	8.1645	8.5171	B3
10.4321	**10.3970**	**10.3298**	**10.2663**	**10.1771**	**10.1214**	**9.8797**	**9.6853**	**A1**
10.9956	10.9352	10.9363	10.9500	10.9921	11.0333	11.3640	11.8305	B4
14.1377	14.0059	14.0029	14.0157	14.0618	14.1089	14.4981	15.0512	B5
17.2788	17.0366	17.0266	17.0351	17.0800	17.1294	17.1181	16.7851	B6
18,0689	**18.0082**	**17.8919**	**17.7819**	**17.6276**	**17.5310**	**17.5504**	**18.1633**	**A2**
20.4204	20.0201	19.9994	19.9998	20.0367	20.0832	20.5173	21.1634	B7
23.5620	22.9498	22.9141	22.8974	22.7543	22.6319	22.0995	21.6710	B8

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$.

Furthermore, crack-induced variations of natural frequencies of coupled beam and output charge generated in the piezoelectric layer under natural vibration mode are examined in dependence upon position and depth of the crack. The output charge calculated from a given mode shape is called herein modal piezoelectric charge (MPC) of the piezoelectric layer. The crack-induced variations examined in this study are natural frequencies and MPCs of cracked structure normalized by those of undamaged one.

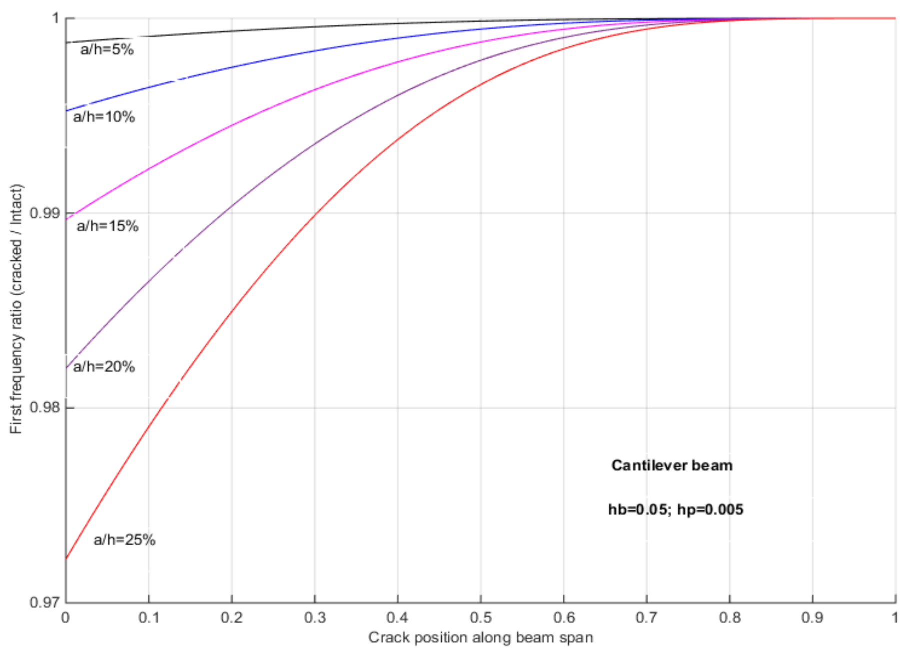

Fig. 3. Crack-induced variation of first natural frequency for cantilever beam

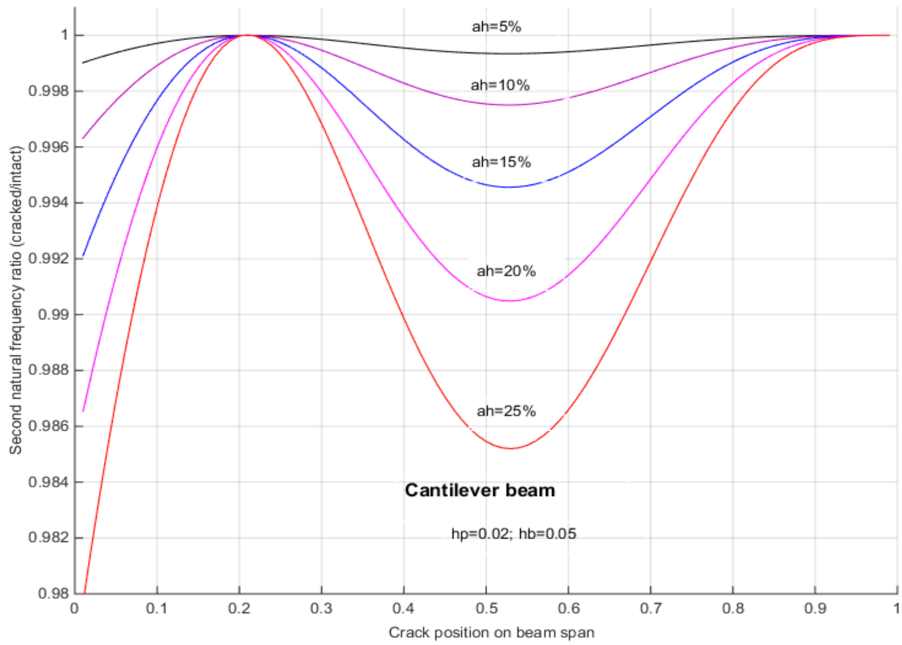

Fig. 4. Crack-induced variation of second natural frequency for cantilever beam

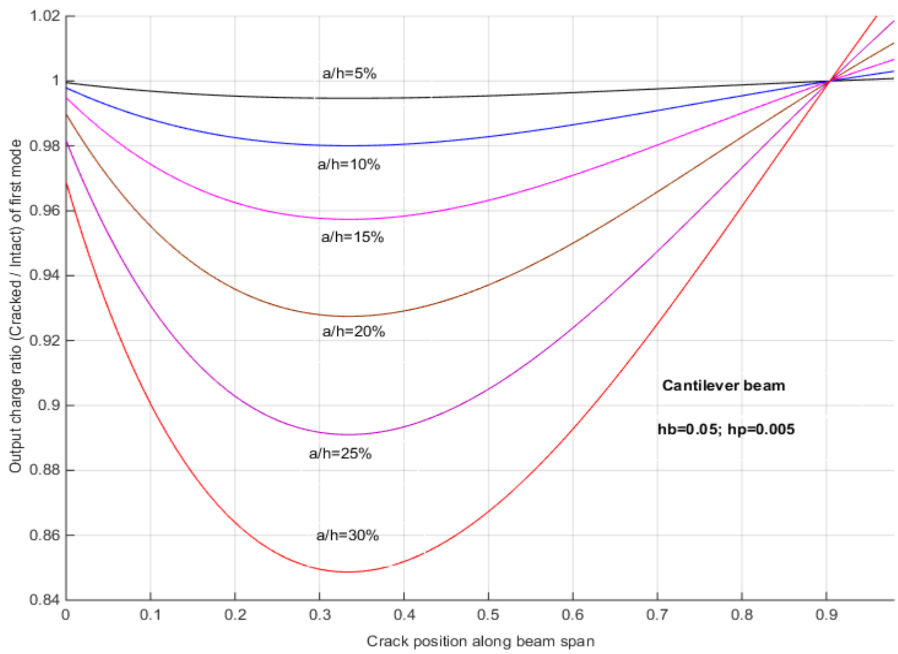

Fig. 5. Crack-induced variation of modal output charge at first mode for cantilever beam

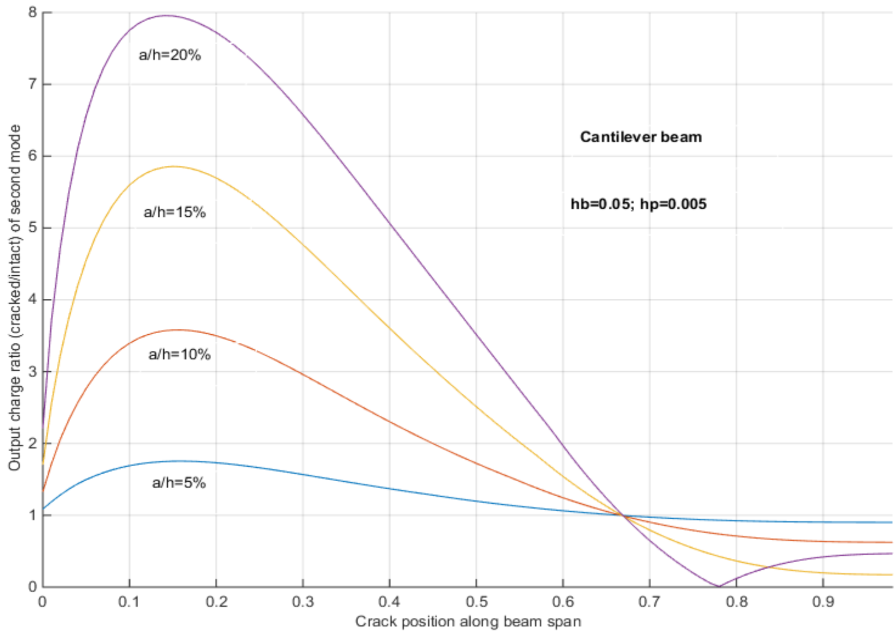

Fig. 6. Crack-induced variation of modal output charge at second mode for cantilever beam

The normalized modal parameters of cantilever beam are shown in Figs. 3–6 as function of crack position in different crack depth. Graphs given in the Figures show that piezoelectric layer makes insignificant contribution to the variation of natural frequencies due to crack, but crack may significantly change the output charge generated in the piezoelectric layer under vibration modes. Namely, crack appeared in host beam decreases electric charge generated in piezoelectric layer under fundamental mode of vibration, but it may increase to 8 times the charge generated by vibration of second mode. Also, it can be seen in both cases of vibration modes a position on beam crack appeared at which makes no effect on the electric charge produced in the piezoelectric layer. It is position 0.9 for first mode and 0.67 for second one. Existence of the so-called nodes of piezoelectric charge can be explained perhaps that the crack position may make the expression $\gamma_a \phi'_u(e, \omega_k) + \gamma_b \phi''_w(e, \omega_k) = 0, \forall a$ that may be in particular when $\phi'_u(e, \omega_k) = \phi''_w(e, \omega_k) = 0$.

5. CONCLUSION

In the present report, the double beam model was employed for constructing a model of a cracked Euler-Bernoulli beam with piezoelectric layer. The established governing equations of vibration show that piezoelectric layer bonded to a homogeneous beam leads to coupling of longitudinal and bending vibration modes. Therefore, an open crack appeared at a cross-section of the host beam is represented by a pair of axial and rotational springs with stiffness calculated from the crack depth.

General solution of free vibration of the doubled beam with single crack was obtained in the frequency domain that allows straightforward studying effect of crack on natural frequencies and mode shapes of the coupled beam structure. For consideration of electric response of the piezoelectric layer as a distributed sensor to the natural vibration modes, a conception of modal piezoelectric charge calculated from a given mode shape was introduced and investigated in dependence upon crack position and depth.

Numerical example carried out for different cases of boundary conditions shows that thickness of the piezoelectric layer may increase natural frequencies, i.e. increase stiffness of beam, but it has insignificant influence on variation of natural frequencies due to crack. However, crack makes significant effect on the modal piezoelectric charge that provides an efficient indicator for crack detection from signal gathered in a distributed sensor bonded to an elastic structure such as beams or frames.

Based on the model established in this study, numerous problems on analysis and identification of cracked piezoelectric structures such as smart structures could be addressed that are subject for further works of the authors.

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