MODAL ANALYSIS OF CRACKED BEAM WITH A PIEZOELECTRIC LAYER

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Abstract. Piezoelectric material was employed first as sensor/actuator for structural control and then it has got an effective use for structural health monitoring and repairing damaged structures. In this report, modal analysis of cracked beam with piezoelectric layer is carried out to investigate effect of crack and piezoelectric layer thickness on natural frequencies of the structure and output charge generated in the piezoelectric layer by vibration modes. Governing equations of the coupled structure are established using the double beam model and two-spring (translational and rotational) representation of crack and solved to obtain the modal parameters including the output charge associated with natural modes acknowledged as modal piezoelectric charge (MPC). Numerical examples have been examined for validation and illustration of the developed theory.

Keywords: cracked beam, piezoelectric layer, modal analysis, structural health monitoring.

1. INTRODUCTION

Piezoelectric material was employed first as sensor/actuator for structural control [1–4] and then it has got an effective use for structural health monitoring [5–11] and repairing damaged structures [12–17]. Using piezoelectric material for controlling or monitoring structural behavior is essentially leading to analysis of the structures with piezoelectric components such as beams or plates with layers or patches. Namely, Lee and Kim [18] first proposed to apply the spectral element method (SEM) for vibration analysis of Euler-Bernoulli beam bonded with a piezoelectric layer and declared that the method is consistent to study dynamic characteristics of the elastic-piezoelectric two-layer beams. Then, the SEM have been developed for modelling and analysis of homogeneous [19] and composite [20] Timoshenko beams with piezoelectric layers. Yang and Lee [21] used the stepped beam model for modal analysis of Timoshenko beam with piezoelectric patches symmetrically bonded onto both the top and bottom and demonstrated that stiffness and inertia of the piezoelectric material, as well as shear deformation and rotary inertia of the
host beam may make change in natural frequencies of the coupled beam. The model of multi-step beam was employed also by Maurini et al. [22] for modal analysis of classical beam with numerous pairs of piezoelectric patches using different techniques including the so-called assumed modes method proposed by themselves. Wang and Quek [23] used the sandwich beam model for modal analysis of a Euler-Bernoulli beam embedded with piezoelectric layers and they found that natural frequency of the sandwich beam is function of stiffness and thickness of the piezoelectric layers. Nguyen Tien Khiem et al. [24] investigated effect of piezoelectric patch on natural frequencies of beam made of functionally graded material. Recently, dynamics of cracked structures with piezoelectric patches [25, 26] has attracted a special attention of researchers to develop an efficient method for crack identification using piezoelectric material. Namely, Zhao et al. [27] proposed an interesting technique for crack identification in beam-type structures by natural frequencies using coupled pairs of piezoelectric sensor and actuator.

In this report, modal analysis of cracked beam with piezoelectric layer is carried out to investigate effect of crack and piezoelectric layer thickness on natural frequencies of the structure and output charge generated in the piezoelectric layer by vibration mode. Governing equations of the coupled structure are established using the double beam model and two-spring (translational and rotational) representation of crack and solved to obtain the modal parameters including the output charge associated with natural modes acknowledged as modal piezoelectric charge (MPC). Numerical examples have been examined for validation and illustration of the developed theory.

2. GOVERNING EQUATIONS

Let’s consider a Euler–Bernoulli beam of length $L$, cross section area $A_b = b \times h_b$, elastic modulus and mass density $E, \rho$ that is bonded with a piezoelectric layer of thickness $h_p$ and the same width as the beam (Fig. 1). Using the classical theory of beam and notations shown in Fig. 1, governing equations for the beam are

\[
\begin{align*}
  u(x, z, t) &= u_0(x, t) - zw_0(z, t), \quad w(x, z, t) = w_0(x, t), \\
  \sigma_x &= E\varepsilon_x, \quad \varepsilon_x = u_0' - zw_0''
\end{align*}
\] (1)

where $u(x, z, t), w(x, z, t)$ denote axial and transverse displacements at arbitrary point in the beam; $u_0(x, t), w_0(x, t)$ - the displacements in the beam’s mid-plane ($z = 0$) and $\varepsilon_x, \sigma_x$ are strain and stress in cross-section at $x$.

Fig. 1. Beam with piezoelectric layer model
Based on the governing equations (1) strain energy of the beam can be calculated as

\[ \Pi_b = \left( \frac{1}{2} \right) \iiint (\sigma_x \varepsilon_x) dV_b = \left( \frac{1}{2} \right) \iiint [E \varepsilon_x^2] dV_b = \left( \frac{1}{2} \right) \int_0^L \left\{ E A_b \dot{u}_0^2 + I_b \ddot{w}_0^2 \right\} dx, \]  

(2)

where commas at the variables denotes their derivative with respect to \( x \) and \( I_b = bh^3/12 \). Total kinetic energy is

\[ T_b = \left( \frac{1}{2} \right) \iiint \rho (\dot{u}^2 + \dot{w}^2) dV_b = \left( \frac{1}{2} \right) \int_0^L \left\{ \rho A_b \dot{u}_0^2 + \rho I_b \ddot{w}_0^2 + \rho A_p \ddot{w}_0^2 \right\} dx. \]  

(3)

Considering the piezoelectric layer also as a Euler–Bernoulli beam, governing equations of the layer are

\[ u_p (x, z, t) = u_{p0} (x, t) - z \ddot{w}_{p0} (x, t), \quad w_p (x, z, t) = w_{p0} (x, t), \]  

\[ \varepsilon_{px} = \dot{u}_{p0} - z \ddot{w}_{p0} (x, t), \]  

\[ \sigma_{px} = C_{11}^p \varepsilon_{px} - h_{13} D, \quad \varepsilon = -h_{13} \varepsilon_{px} + \beta_{33}^p D, \]  

(4)  

(5)

with \( C_{11}^p \), \( h_{13} \), \( \beta_{33}^p \) denoting respectively the elastic modulus, piezoelectric and dielectric constants of the piezoelectric material; \( \varepsilon \) and \( D \) are electric field and electric displacement in the piezoelectric layer.

Assuming perfect bonding between the piezoelectric layer and the host beam, the continuity conditions for the mechanical displacements can be obtained as

\[ u \left( x, -\frac{h_b}{2}, t \right) = u_p \left( x, \frac{h_p}{2}, t \right), \quad w \left( x, -\frac{h_b}{2}, t \right) = w_p \left( x, \frac{h_p}{2}, t \right), \]  

(6)

that lead to

\[ u_{p0} = u_0 + \dot{w}_0 h, \quad h = (h_b + h_p)/2, \quad w_{p0} = \dot{w}_0. \]  

(7)

Therefore, Eqs. (4) can be rewritten in the form

\[ u_p (x, z, t) = u_0 (x, t) - (z - h) \ddot{w}_0 (x, t), \quad \varepsilon_{px} = \dot{u}_0 - (z - h) \ddot{w}_0, \]  

(8)

that allow one to calculate the energies of the piezoelectric layer as

\[ \Pi_p = \left( \frac{1}{2} \right) \iiint (\sigma_{px} \varepsilon_{px} + \varepsilon_{dx}) dV_p = \left( \frac{1}{2} \right) \iiint \left[ C_{11}^p \varepsilon_{px}^2 - 2h_{13} D \varepsilon_{px} + \beta_{33}^p D^2 \right] dV_p \]  

\[ = \left( \frac{1}{2} \right) \int_0^L \left\{ C_{11}^p A_p u_0^2 + 2C_{11}^p A_p h_0 \ddot{w}_0 + C_{11}^p [I_p + A_p h_0^2] \ddot{w}_0^2 \right\} dx \]  

\[ + \left( \frac{1}{2} \right) \int_0^L \left\{ -2h_{13} A_p D \dot{u}_0 - 2h_{13} A_p D h \ddot{w}_0 + \beta_{33}^p A_p D^2 \right\} dx, \]  

(9)
the later expression into the remained equations in (15) gives the equations of motion

\[ T_p = \left( \frac{1}{2} \right) \int \rho_p \left( \dot{u}_{p}^2 + \dot{w}_{p}^2 \right) dV_p \]

\[ = \left( \frac{1}{2} \right) \int_{0}^{L} \left\{ \rho_p A_p \ddot{u}_0^2 + \rho_p A_p h \dot{u}_0 \dot{w}_0 + \rho_p [I_p + A_p h^2] \dot{w}_0^2 + \rho_p A_p \dot{w}_0^2 \right\} dx, \quad (10) \]

\[ A_p = bh_p, \quad I_p = bh_p^2 / 12. \]

Summing up the defined above energies of both the host beam and piezoelectric layer gives total strain and kinetic energies of the double beam calculated as

\[ \Pi = \Pi_b + \Pi_p \]

\[ = \left( \frac{1}{2} \right) \int_{0}^{L} \left\{ EA_b u_0'' + EI_b w_0'' + C_{11}^p A_p u_0'' + 2C_{11}^p A_p h u_0' w_0'' + C_{11}^p [I_p + A_p h^2/4] w_0''^2 - 2h_{13} A_p D u_0' - h_{13} A_p Dh w_0'' + \beta_{33} A_p D^2 \right\} dx \]

\[ = \left( \frac{1}{2} \right) \int_{0}^{L} \left\{ A_{11} u_0'' + 2A_{12} u_0' w_0'' + A_{22} w_0''^2 - 2h_{13} A_p (u_0' + h w_0'') D + \beta_{33} A_p D^2 \right\} dx, \quad (11) \]

\[ T = T_b + T_p = \left( \frac{1}{2} \right) \int_{0}^{L} \left\{ I_{11} \ddot{u}_0^2 + 2I_{12} \dot{u}_0 \dot{w}_0 + I_{22} \dot{w}_0^2 + I_{11} \dot{w}_0^2 \right\} dx, \quad (12) \]

where the following notations have been used

\[ A_{11} = EA_b + C_{11}^p A_p, \quad A_{12} = C_{11}^p A_p h, \quad A_{22} = EI_b + C_{11}^p (I_p + A_p h^2), \]

\[ I_{11} = \rho A_b + \rho_p A_p, \quad I_{12} = \rho_p A_p h, \quad I_{22} = \rho I_b + \rho_p I_p + \rho_p A_p h^2. \quad (13) \]

Putting expressions (11)–(12) into Hamilton’s principle

\[ \int_{t_1}^{t_2} \delta (T - \Pi) \, dt = 0, \quad (14) \]

allows general equations of motion of the system to be derived in the form

\[ \left( I_{11} \ddot{u}_0 - A_{11} u_0'' \right) + \left( I_{12} \ddot{w}_0 - A_{12} w_0'' \right) + h_{13} A_p D' = 0, \]

\[ I_{11} \ddot{w}_0 + A_{22} w_0'' + A_{12} u_0'' - I_{12} \ddot{u}_0 - I_{22} \ddot{w}_0 - h_{13} A_p h D'' / 2 = 0, \]

\[ h_{13} A_p \left( u_0' + h w_0'' / 2 \right) - \beta_{33} A_p D = 0. \quad (15) \]

Obviously, the last equation in (15) yields \( D = h_{13} \left( u_0' + h w_0'' \right) / \beta_{33} \) and substituting the later expression into the remained equations in (15) gives the equations of motion
reduced to the final form

\[
\begin{align*}
I_{11} \dddot{u}_0 - B_{11} \dddot{u}_0 &= 0, \\
I_{11} \dddot{w}_0 + B_{22} \dddot{w}_0 + B_{12} \dddot{u}_0 - I_{12} \dddot{u}_0 - I_{22} \dddot{w}_0 &= 0,
\end{align*}
\]  
(16)

where

\[
B_{11} = A_{11} - A_p h_{13}^2 / \beta_{33}^p = EA_b + E_p A_p, \\
B_{12} = A_{12} - A_p h_{13}^2 / \beta_{33}^p = E_p A_p h,
\]

\[
B_{22} = A_{22} - A_p h_{13}^2 / \beta_{33}^p = E l + C_{11}^p I_p + E_p A_p h^2, \\
E_p = C_{11}^p - h_{13}^2 / \beta_{33}^p.
\]

Using Fourier transform \(\{U(x, \omega), W(x, \omega)\} = \int_{-\infty}^{\infty} \{u_0(x, t), w_0(x, t)\} e^{-i\omega t} dt\), Eq. (16) are transformed to

\[
\begin{align*}
(\omega^2 I_{11} U + B_{11} U'') + (\omega^2 I_{12} W' + B_{12} W''') &= 0, \\
B_{22} W'''' + B_{12} U''' + \omega^2 I_{12} U' + \omega^2 I_{22} W'' - \omega^2 I_{11} W &= 0,
\end{align*}
\]

or

\[
[A_0] \{d^4z/dx^4\} + [A_1] \{d^3z/dx^3\} + [A_2] \{d^2z/dx^2\} + [A_3] \{dz/dx\} + [A_4] \{z\} = 0, 
(17)
\]

where there are introduced following notations \(\{z\} = \{U(x, \omega), W(x, \omega)\}^T\) and

\[
A_0 = \begin{bmatrix}
0 & 0 \\
0 & B_{22}
\end{bmatrix}, \\
A_1 = \begin{bmatrix}
0 & B_{12} \\
B_{12} & 0
\end{bmatrix}, \\
A_2 = \begin{bmatrix}
B_{11} & 0 \\
0 & \omega^2 I_{22}
\end{bmatrix}, \\
A_3 = \begin{bmatrix}
0 & \omega^2 I_{12} \\
\omega^2 I_{12} & 0
\end{bmatrix}, \\
A_4 = \begin{bmatrix}
\omega^2 I_{11} & 0 \\
0 & -\omega^2 I_{11}
\end{bmatrix}.
\]

After Eqs. (16) have been solved, the output charge in the piezoelectric layer is calculated as

\[
Q = b \int_0^L Ddx = (bh_{13}/\beta_{33}^p) \left. (u_0 + hw_0') \right|_0^L.
\]  
(18)

**Fig. 2.** Double spring model of crack in beam

Furthermore, if the beam is damaged at position \(e\) to a top edged crack of depth \(a\) and crack is represented by the double spring model with \(R\) and \(T\) being stiffness of the axial and rotational springs, as shown in Fig. 2. Therefore, conditions that should be satisfied at crack position are [28, 29].
where \( \gamma_a = EA/T \) and \( \gamma_b = EI/R \) are so-called crack magnitudes calculated as
\[
\gamma_a = 2\pi(1 - v_0^2)hf_a(z), \quad \gamma_b = 6\pi(1 - v_0^2)hf_b(z), \quad z = a/h,
\]
\[
f_a(z) = z^2(0.6272 - 0.17248z + 5.92134z^2 - 10.7054z^3 + 31.5685z^4 - 67.47z^5 + 139.123z^6 - 146.682z^7 + 92.3552z^8),
\]
\[
f_b(z) = z^2(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + 47.1063z^6 - 40.7556z^7 + 19.6z^8).
\]

3. GENERAL SOLUTION OF FREE VIBRATION PROBLEM

In this section, Eq. (17) is solved by seeking its solution in the form
\[
\{z_0\} = \{U_0, W_0\}^T e^{\lambda x},
\]
that leads the equation to
\[
\left[ \lambda^4A_0 + \lambda^3A_1 + \lambda^2A_2 + \lambda A_3 + A_4 \right] \{U_0, W_0\}^T = 0.
\]
The latter equation has non-trivial solution under the condition
\[
det \left[ \lambda^4A_0 + \lambda^3A_1 + \lambda^2A_2 + \lambda A_3 + A_4 \right] = 0,
\]
that is so-called characteristic equation for determining wave number \( \lambda \) as function of frequency, \( \lambda = \lambda(\omega) \). It is not difficult to show the characteristic equation can be obtained in the form
\[
a\lambda^6 + b\lambda^4 + c\lambda^2 + d = 0,
\]
with
\[
a = B_{11}B_{22} - B_{12}^2, \quad b = \omega^2(B_{11}I_{22} + B_{22}I_{11} - 2B_{12}I_{12}),
\]
\[
c = \omega^4(I_{11}I_{22} - I_{12}^2) - \omega^2B_{11}I_{11}, \quad d = -\omega^4I_{11}^2.
\]
As a cubic algebraic equation with respect to \( \eta = \lambda^2 \), \( a\eta^3 + b\eta^2 + c\eta + d = 0 \), that has three roots denoted by \( \eta_1, \eta_2, \eta_3 \), six roots of characteristic equation (23) can be obtained in the form
\[
\lambda_{1,4} = \pm k_1, \quad \lambda_{2,5} = \pm k_2, \quad \lambda_{3,6} = \pm k_3, \quad k_j = \sqrt{\eta_j}, \quad j = 1, 2, 3.
\]
Hence, general solution of Eq. (22) is expressed as
\[
\{U_0(x, \omega) W_0(x, \omega)\} = \left\{ \alpha_1C_1 e^{k_1x} + \alpha_2C_2 e^{k_2x} + \alpha_3C_3 e^{k_3x} - \alpha_1C_4 e^{-k_1x} - \alpha_2C_5 e^{-k_2x} - \alpha_3C_6 e^{-k_3x} \right\},
\]
where \( \{C\} = \{C_1, \ldots, C_6\}^T \) is vector of arbitrary constants and 
\[
\alpha_j = -k_j (\omega^2 I_{12} + k_j^2 B_{12}) / (\omega^2 I_{11} + k_j^2 B_{11}), \quad j = 1, 2, 3.
\]

For example, using expression (24), a particular solution denoted by \( z_1(x, \omega) = \{U_1(x, \omega), W_1(x, \omega)\}^T \), satisfying conditions

\[
U_1(0, \omega) = Z_a^0, \quad U'_1(0, \omega) = 0, \quad W_1(0, \omega) = 0, \quad W'_1(0, \omega) = Z_b^0, \quad W''_1(0) = 0 = W'''_1(0),
\]
can be found as

\[
U_1(x, \omega) = g_{ua}(x, \omega) Z_a^0 + g_{ub}(x, \omega) Z_b^0, \quad W_1(x, \omega) = g_{wa}(x, \omega) Z_a^0 + g_{wb}(x, \omega) Z_b^0, \quad (25)
\]
where

\[
g_{ua}(x, \omega) = \alpha_1 \delta_{1a} \cos k_1 x + \alpha_2 \delta_{2a} \cos k_2 x + \alpha_3 \delta_{3a} \cos k_3 x,
\]
\[
g_{ub}(x, \omega) = \alpha_1 \delta_{1b} \cos k_1 x + \alpha_2 \delta_{2b} \cos k_2 x + \alpha_3 \delta_{3b} \cos k_3 x,
\]
\[
g_{wa}(x, \omega) = \delta_{1a} \sin k_1 x + \delta_{2a} \sin k_2 x + \delta_{3a} \sin k_3 x,
\]
\[
g_{wb}(x, \omega) = \delta_{1b} \sin k_1 x + \delta_{2b} \sin k_2 x + \delta_{3b} \sin k_3 x,
\]
\[
\delta_{1a} = k_2 k_3 (k_3^2 - k_2^2) / \Delta, \quad \delta_{2a} = k_1 k_3 (k_1^2 - k_3^2) / \Delta, \quad \delta_{3a} = k_1 k_2 (k_2^2 - k_1^2) / \Delta,
\]
\[
\delta_{1b} = (\alpha_3 k_2^2 - \alpha_2 k_3^2) / \Delta, \quad \delta_{2b} = (\alpha_1 k_3^2 - \alpha_3 k_1^2) / \Delta, \quad \delta_{3b} = (\alpha_2 k_1^2 - \alpha_1 k_2^2) / \Delta,
\]
\[
\Delta = \alpha_1 k_2 k_3 (k_3^2 - k_2^2) + \alpha_2 k_1 k_3 (k_1^2 - k_3^2) + \alpha_3 k_1 k_2 (k_2^2 - k_1^2).
\]

Using particular solution (25) with \( Z_a^0 = \gamma_a U'_0(e, \omega), Z_b^0 = \gamma_b W''_0(e, \omega) \), it can be shown that general solution for free vibration of cracked beam that satisfies conditions (19) can be obtained in the form

\[
U_c(x, \omega) = U_0(x, \omega) + \left\{ \begin{array}{ll}
0: & x < e, \\
U_1(x - e, \omega): & x \geq e,
\end{array} \right.
\]

\[
W_c(x, \omega) = W_0(x, \omega) + \left\{ \begin{array}{ll}
0: & x < e, \\
W_1(x - e, \omega): & x \geq e.
\end{array} \right.
\]

Introducing the following vectors and matrices

\[
\{z_c(x, \omega)\} = \{z_c(x, \omega), W_c(x, \omega)\}^T, \quad \{z_0(x, \omega)\} = \{U_0(x, \omega), W_0(x, \omega)\}^T,
\]
\[
[G_c(x, \omega)] = \left[ \begin{array}{c}
\gamma_a g_{ua}(x, \omega) \\
\gamma_b g_{ub}(x, \omega)
\end{array} \right], \quad [K(x)] = \left\{ \begin{array}{cc}
[G_c(x, \omega)] & : x \geq 0, \\
0 & : x < 0,
\end{array} \right.
\]
\[
[K'(x)] = \left\{ \begin{array}{c}
[G_c(x, \omega)] & : x \geq 0, \\
0 & : x < 0,
\end{array} \right., \quad [K''(x)] = \left\{ \begin{array}{cc}
[G_c''(x, \omega)] & : x \geq 0, \\
0 & : x < 0,
\end{array} \right.
\]
\[
[G_0(x, \omega)] = \left[ \begin{array}{cccc}
\alpha_1 e^{k_1 x} & \alpha_2 e^{k_2 x} & \alpha_3 e^{k_3 x} & -\alpha_1 e^{-k_1 x} \\
e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} & -e^{-k_1 x} \\
\end{array} \right], \quad [\tilde{G}(x, \omega)] = \left[ \begin{array}{cccc}
\alpha_1 k_1 e^{k_1 x} & \alpha_2 k_2 e^{k_2 x} & \alpha_3 k_3 e^{k_3 x} & -\alpha_1 k_1 e^{-k_1 x} \\
k_1 e^{k_1 x} & k_2 e^{k_2 x} & k_3 e^{k_3 x} & -k_1 e^{-k_1 x} \\
\end{array} \right].
\]
\[
[\Phi(x, \omega)] = [G_0(x, \omega)] + [K(x - e, \omega)\hat{G}(e, \omega)] \tag{28}
\]
and using expressions (24), solution (27) can be rewritten in the form
\[
\{z_c(x, \omega)\} = [\Phi(x, \omega)] \{C\}. \tag{29}
\]
Applying boundary conditions for general solution (29) allows one to solve the free vibration problem of the coupled beam. For example, in case of cantilever beam with boundary conditions
\[
U(0) = W(0) = W'(0) = U'(L) = W''(L) = W'''(L) = 0, \tag{30}
\]
one obtains equation for determining the constant vector \(\{C\} = \{C_1, \ldots, C_6\}^T\) as
\[
[B(\omega)] \{C\} = 0, \tag{31}
\]
where
\[
[B_{cc}(\omega)] = \\
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\
\phi_{21}(L) & \phi_{22}(L) & \phi_{23}(L) & \phi_{24}(L) & \phi_{25}(L) & \phi_{26}(L) \\
\phi_{31}(L) & \phi_{32}(L) & \phi_{33}(L) & \phi_{34}(L) & \phi_{35}(L) & \phi_{36}(L) \\
\end{bmatrix},
\]
\[
[B_{cs}(\omega)] = \\
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\
\phi_{21}(L) & \phi_{22}(L) & \phi_{23}(L) & \phi_{24}(L) & \phi_{25}(L) & \phi_{26}(L) \\
\phi_{31}(L) & \phi_{32}(L) & \phi_{33}(L) & \phi_{34}(L) & \phi_{35}(L) & \phi_{36}(L) \\
\end{bmatrix},
\]
\[
[B_{ss}(\omega)] = \\
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{11}(L) & \phi_{12}(L) & \phi_{13}(L) & \phi_{14}(L) & \phi_{15}(L) & \phi_{16}(L) \\
\phi_{21}(L) & \phi_{22}(L) & \phi_{23}(L) & \phi_{24}(L) & \phi_{25}(L) & \phi_{26}(L) \\
\phi_{31}(L) & \phi_{32}(L) & \phi_{33}(L) & \phi_{34}(L) & \phi_{35}(L) & \phi_{36}(L) \\
\end{bmatrix},
\]
Obviously, so-called frequency equation of the beam with piezoelectric layer can be obtained from Eq. (31) as
\[
d(\omega) \equiv \det[B(\omega)] = 0, \tag{32}
\]
positive roots of which provide the desired natural frequencies \(\omega_k, k = 1, 2, 3, \ldots\) Every natural frequency \(\omega = \omega_k\) allows one to find corresponding solution of Eq. (31) as \(\{C_k\} = \theta_k\{\beta_{k1}, \ldots, \beta_{k6}\}^T\), where \(\theta_k\) is arbitrary constant and \(\{\beta_{k1}, \ldots, \beta_{k6}\}^T\) is normalized solution of equation
\[
[B(\omega_k)] \{C\} = 0.
Thus, mode shape associated with natural frequency $\omega_k$ would be calculated as

$$\phi_u (x, \omega_k) = \theta_k (a_1 \beta_1 e^{k_1 x} + a_2 \beta_2 e^{k_2 x} + a_3 \beta_3 e^{k_3 x} - a_1 \beta_4 e^{-k_1 x} - a_2 \beta_5 e^{-k_2 x} - a_3 \beta_6 e^{-k_3 x}),$$

$$\phi_w (x, \omega_k) = \theta_k (\beta_1 k_1 e^{k_1 x} + \beta_2 k_2 e^{k_2 x} + \beta_3 k_3 e^{k_3 x} + \beta_4 k_4 e^{-k_1 x} + \beta_5 k_5 e^{-k_2 x} + \beta_6 k_6 e^{-k_3 x}),$$

from that slope mode can be calculated as

$$\phi_\theta (x, \omega_k) = \theta_k (\beta_1 k_1 e^{k_1 x} + \beta_2 k_2 e^{k_2 x} + \beta_3 k_3 e^{k_3 x} - \beta_4 k_4 e^{-k_1 x} - \beta_5 k_5 e^{-k_2 x} - \beta_6 k_6 e^{-k_3 x}).$$

(33)

The arbitrary constant $\theta_k$ is determined form a chosen normalization condition, e.g.

$$\max_x |\phi_w (x, \omega_k)| = 1.$$  \hspace{1cm} (34)

It can be noted that the slope mode represented by expression (33) can be employed for calculating output charge of the piezoelectric layer by formula (18). Namely, since $U (0) = U (L) = 0$ for simply supported beam, the formula (18) is reduced to

$$Q_k = Q_k^0 - (bh_{13} / \beta_{33}^p) \{ \gamma_a \phi_u (e, \omega_k) + \gamma_b \phi_w (e, \omega_k) \}, \quad k = 1, 2, 3, \ldots$$

(35)

where $Q_k^0 = (bh_{13} / \beta_{33}^p) \{ [\phi_u (L, \omega_k) - \phi_u (0, \omega_k)] + h [\phi'_u (L, \omega_k) - \phi'_u (0, \omega_k)] \}$ is the charge in case uncracked beam. The latter quantities are acknowledged hereby Modal Piezoelectric Response (MPR) associated with natural vibration $k$-th modes and these characteristics of the piezoelectric layer are numerically examined below mutually with natural frequencies of the coupled beam in dependence upon crack.

### 4. NUMERICAL RESULTS AND DISCUSSION

Numerical analysis is completed with following data: equal length and width of both beam and piezoelectric layer: $L = 1$ m, $b = 0.1$ m; material and geometry parameters of the host beam are denoted with lower index $b$ and those of piezoelectric layer with $p$ index:

$$E_b = 210 \text{ MPa}; \rho_b = 7800 \text{ kg/m}^3; \mu_b = 0.31; h_b = 0.05 \text{ m}; A_b = bh_b; I_b = bh_b^3 / 12,$$

and piezoelectric constants

$$C_{11}^p = 69.0084 \text{ GPa}, C_{55}^p = 21.0526 \text{ GPa}; \rho_p = 7750 \text{ kg/m}^3, h_{13} = -7.70394 \times 10^8 \text{ V/m},$$

$$\beta_{33}^p = 7.3885 \times 10^7 \text{ m/F}.$$

The so-called frequency parameters $\lambda_k = (\rho_b A_b \omega_k^2 / E_b I_b)^{1/4}$ that represent natural frequencies $\omega_k$ are calculated herein as function of crack position along the beam span with different crack depth and thickness of piezoelectric layer $h_p$. The charge generated in the piezoelectric layer, $Q_k$ calculated from $k$-th mode shape by formula (18) acknowledged here as modal piezoelectric response (MPR) is examined below in dependence upon crack parameters.

First, effect of piezoelectric layer thickness on natural frequencies of the undamaged (intact) beam is studied and 10 natural frequencies calculated for various thickness of piezoelectric layer $h_p$ are presented in Tabs. 1–3. Excellent agreement of the natural frequencies obtained in case of beam without piezoelectric layer (corresponding to zero thickness $h_p = 0$) with the well-known natural frequencies of single beam in different
Table 1. Effect of piezoelectric layer thickness on natural frequencies of simply supported intact beam

<table>
<thead>
<tr>
<th>$h_p$ (m)</th>
<th>0.001</th>
<th>0.003</th>
<th>0.005</th>
<th>0.008</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.7080</td>
<td>15.5016</td>
<td>15.4935</td>
<td>15.5025</td>
<td>15.5451</td>
<td>15.5912</td>
<td>15.9868</td>
<td>16.5579</td>
<td>B5</td>
</tr>
</tbody>
</table>

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$ m.

Table 2. Effect of piezoelectric layer thickness on natural frequencies of uncracked beam with clamped ends

<table>
<thead>
<tr>
<th>$h_p$ (m)</th>
<th>0.001</th>
<th>0.003</th>
<th>0.005</th>
<th>0.008</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.7300</td>
<td>4.7259</td>
<td>4.7282</td>
<td>4.7361</td>
<td>4.7575</td>
<td>4.7777</td>
<td>4.9347</td>
<td>5.1547</td>
<td>B1</td>
</tr>
<tr>
<td>17.2788</td>
<td>17.0366</td>
<td>17.0265</td>
<td>17.0350</td>
<td>17.0796</td>
<td>17.1286</td>
<td>17.5527</td>
<td>18.1629</td>
<td>B5</td>
</tr>
<tr>
<td>23.5620</td>
<td>22.9499</td>
<td>22.9148</td>
<td>22.9030</td>
<td>22.9253</td>
<td>22.9642</td>
<td>23.3789</td>
<td>23.7572</td>
<td>B7</td>
</tr>
</tbody>
</table>

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$ m.

Cases of boundary conditions demonstrates validity of the above developed model. Furthermore, the data given in the Tables reveal also the ordering of bending and axial ( longitudinal) vibration modes. Namely, the axial mode of vibration in cantilever beam appears (as fourth) earlier than that of beams with clamped or simply supported ends (appeared as fifth). Generally, all natural frequencies of coupled beam first decrease with increasing thickness of piezoelectric layer and then become increasing when the thickness exceeds 10% host beam thickness. Moreover, in most cases, the natural frequencies overcome those of the host beam alone as the layer thickness gets to be more than 20% thickness of the host beam. This implies the fact that bonded piezoelectric layer of a given thickness could increase stiffness of beam and, consequently, it can be used for repairing the beam of reduced stiffness for some reason such as cracking.
Table 3. Effect of piezoelectric layer thickness on natural frequencies of uncracked cantilever beam

<table>
<thead>
<tr>
<th>$h_p$ (m)</th>
<th>0.001</th>
<th>0.003</th>
<th>0.005</th>
<th>0.008</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8750</td>
<td>1.8779</td>
<td>1.8791</td>
<td>1.8878</td>
<td>1.8959</td>
<td>1.9591</td>
<td>2.0475</td>
<td>B1</td>
<td></td>
</tr>
<tr>
<td>7.8550</td>
<td>7.8364</td>
<td>7.8481</td>
<td>7.8815</td>
<td>7.9134</td>
<td>8.1645</td>
<td>8.5171</td>
<td>B3</td>
<td></td>
</tr>
<tr>
<td>17.2788</td>
<td>17.0366</td>
<td>17.0266</td>
<td>17.0351</td>
<td>17.0800</td>
<td>17.1294</td>
<td>17.1181</td>
<td>16.7851</td>
<td>B6</td>
</tr>
<tr>
<td>18.0689</td>
<td>18.0082</td>
<td>17.8919</td>
<td>17.7819</td>
<td>17.6276</td>
<td>17.5310</td>
<td>17.5504</td>
<td>18.1633</td>
<td>A2</td>
</tr>
<tr>
<td>23.5620</td>
<td>22.9498</td>
<td>22.9141</td>
<td>22.8974</td>
<td>22.7543</td>
<td>22.6319</td>
<td>22.0995</td>
<td>21.6710</td>
<td>B8</td>
</tr>
</tbody>
</table>

Notice: B1-B8 bending vibration modes; A1-A2: axial (longitudinal) vibration modes; $h_b = 0.05$.

Furthermore, crack-induced variations of natural frequencies of coupled beam and output charge generated in the piezoelectric layer under natural vibration mode are examined in dependence upon position and depth of the crack. The output charge calculated from a given mode shape is called herein modal piezoelectric charge (MPC) of the piezoelectric layer. The crack-induced variations examined in this study are natural frequencies and MPCs of cracked structure normalized by those of undamaged one.

![Fig. 3. Crack-induced variation of first natural frequency for cantilever beam](image)

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\[ (a_k, b_k) = (0, 0) \]

of particular when

\[ (u_k, w_k) = (0, 0) \]


Fig. 4. Crack-induced variation of second natural frequency for cantilever beam

Fig. 5. Crack-induced variation of modal output charge at first mode for cantilever beam
Modal analysis of cracked beam with a piezoelectric layer

Namely, crack appeared in host beam decreases electric charge generated in piezoelectric layer under fundamental mode of vibration, but it may increase to 8 times the charge generated by vibration of second mode. Also, it can be seen in both cases of vibration modes a position on beam crack appeared at which makes no effect on the electric charge produced in the piezoelectric layer. It is position 0.9 for first mode and 0.67 for second one.

Existence of the so-called nodes of piezoelectric charge can be explained perhaps that the crack position may make the expression \( \gamma_a \phi_u'(e, \omega_k) + \gamma_b \phi_w''(e, \omega_k) = 0, \forall a \) that may be in particular when \( \phi_u'(e, \omega_k) = \phi_w''(e, \omega_k) = 0 \).

5. CONCLUSION

In the present report, the double beam model was employed for constructing a model of a cracked Euler-Bernoulli beam with piezoelectric layer. The established governing equations of vibration show that piezoelectric layer bonded to a homogeneous beam leads to coupling of longitudinal and bending vibration modes. Therefore, an open crack appeared at a cross-section of the host beam is represented by a pair of axial and rotational springs with stiffness calculated from the crack depth.

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General solution of free vibration of the doubled beam with single crack was obtained in the frequency domain that allows straightforward studying effect of crack on natural frequencies and mode shapes of the coupled beam structure. For consideration of electric response of the piezoelectric layer as a distributed sensor to the natural vibration modes, a conception of modal piezoelectric charge calculated from a given mode shape was introduced and investigated in dependence upon crack position and depth.

Numerical example carried out for different cases of boundary conditions shows that thickness of the piezoelectric layer may increase natural frequencies, i.e. increase stiffness of beam, but it has insignificant influence on variation of natural frequencies due to crack. However, crack makes significant effect on the modal piezoelectric charge that provides an efficient indicator for crack detection from signal gathered in a distributed sensor bonded to an elastic structure such as beams or frames.

Based on the model established in this study, numerous problems on analysis and identification of cracked piezoelectric structures such as smart structures could be addressed that are subject for further works of the authors.

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REFERENCES

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