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Dedicated to Professor J.N. Reddy on the Occasion of His 75^{th} Birthday

A SIMPLE SIZE-DEPENDENT ISOGEOMETRIC APPROACH FOR BENDING ANALYSIS OF FUNCTIONALLY GRADED MICROPLATES USING THE MODIFIED STRAIN GRADIENT ELASTICITY THEORY

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Abstract. In this study, a simple size-dependent isogeometric approach for bending analysis of functionally graded (FG) microplates using the modified strain gradient theory (MSGT), simple first-order shear deformation theory (sFSDT) and isogeometric analysis is presented for the first time. The present approach reduces one variable when comparing with the original first-order shear deformation theory (FSDT) within five variables and only considers three material length scale parameters (MLSPs) to capture size effects. Effective material properties as Young's modulus, Poisson's ratio and density mass are computed by a rule of mixture. Thanks to the principle of virtual work, the essential equations which are solved by the isogeometric analysis method, are derived. Rectangular and circular FG microplates with different boundary conditions, volume fraction and material length scale parameter are exampled to evaluate the deflections of FG microplates.

Keywords: isogeometric analysis, functionally graded microplate, modified strain gradient theory, simple first-order shear deformation theory.

1. INTRODUCTION

In few years, thanks to the owner of the outstanding mechanical, physical and electronic properties, microstructures in the form of beams, plates and shells have been widely used in several devices such as micro/nano-electromechanical systems (MEMS/NEMS), biosensors, nano-wires, micro-actuators and microscopes, etc. In addition, experimental observations show that the small scale effect exists in the microstructures. Thus, to understand material behaviors at microscale level, the development of new theoretical model is the utmost importance.

The classical continuum theories cannot accurately capture behaviors of microstructures due to ignoring the material length scale parameter. To overcome those disadvantages, several continuum theories with additional considering size effects such as: the nonlocal elasticity [1], modified nonlocal elasticity [2], couple stress [3], modified couple stress [4], surface elasticity [5] and strain gradient [6] have been extended and developed to analyze behaviors of microstructures. The stiffness-softening and stiffness-hardening mechanisms can be shown when using the nonlocal elasticity theory and the strain gradient elasticity theory, respectively. Among them, a theory proposed by Mindlin [7] is widely used for microstructures due to considering all components of the higher-order deformation. However, for engineering practices, it is too difficult to use because of containing five MLSPs. For that reason, the modified strain gradient theory (MSGT) of Lam et al. [8] is a suitable choice due to reducing of two MLSPs. The MSGT has been applied and developed in many studies for microbeams and microplates. Generally, three primary groups consisting of analytical, semi analytical and numerical solutions have been applied into MSGT. The first and second solutions are suitable for the simple problems and can be considered as the exact solution. And their results can be used to verify the accuracy of numerical solutions. Wang et al. [9] introduced a size-dependent Kirchhoff model to analyze the static bending, stability and free vibration behaviors of the isotropic rectangular microplates. Then, that model was extended and developed for isotropic microplates [10], FG microplates [11] and multi-layer microplates [12]. The size-dependent first-order shear deformation model was presented by Ansari et al. [13] to investigate bending, buckling and free vibration behaviors of FG microplates. Zhang et al. [14,15] developed a size-dependent refined higher-order shear deformation model for analysis of FG microplates. That model combined with isogeometric analysis (IGA) was also reported by Thai et al. [16] and Farzam et al. [17] for FG microplates under mechanical and thermal loads, respectively. In addition, a size-dependent higher-order shear deformation model with five variables were presented in [18–20] for analyzing of microplates. Besides, a size-dependent three-dimensional elasticity model was presented by Salehipour and Shahsavar [21] to study free vibration behaviors of FG microplates.

In this study, a novel size-dependent isogeometric approach based on the MSGT, sFSDT and isogeometric analysis is presented to analyze the bending behavior of FG microplates. The present model reduces one variable when comparing with original FSDT and is simple and free of shear locking. This is novel topic and not studied and published in the literature so far. That is a great motivation for us to perform this study. Numerical examples are studied to show advantages of the present model when comparing to different referenced models for analysis of FG microplates.

2. BASIC EQUATIONS

2.1. Problem description

The functionally graded plate is made from a mixture of ceramic and metal, in which ceramic-rich and metal-rich surfaces are distributed at the top $(z = h/2)$ and bottom $(z = -h/2)$, respectively. Effective material properties are computed by the rule of mixture, in which the volume fraction of the ceramic and metal phases are assumed continuous change through thickness as

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n, \quad z \in \left[-\frac{h}{2}, \frac{h}{2}\right], \quad V_m = 1 - V_c\,, \tag{1}$$

where the symbols c, m and n are the ceramic, metal and power index, respectively. Effective material properties based on the rule of mixture are defined by

$$E_e = E_c V_c(z) + E_m V_m(z), \quad \nu_e = \nu_c V_c(z) + \nu_m V_m(z), \quad \rho_e = \rho_c V_c(z) + \rho_m V_m(z). \tag{2}$$

2.2. Modified strain gradient elasticity theory

The virtual strain energy δU for an isotropic linearly elastic material according to the modified strain gradient theory [8] is described by

$$\delta U = \int\limits_V \left(\sigma_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ij} + p_i\delta\zeta_i + \tau_{ijk}\delta\eta_{ijk}\right)dV, \tag{3}$$

where ε, χ, ζ and η are the strain, symmetric rotation gradient, dilatation gradient and deviatoric stretch gradient, respectively; σ is Cauchy stress; m, p and τ are high-order stresses corresponding with strain gradient χ, ζ and η, respectively.

The components of strain and strain gradient are expressed as follows

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{4}$$

$$\chi_{ij} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right), \quad \theta_i = \frac{1}{2}e_{ijk}u_{k,j}, \tag{5}$$

$$\zeta_j = \varepsilon_{mm,j}, \tag{6}$$

$$\eta_{ijk} = \frac{1}{3}\left(\varepsilon_{ij,k} + \varepsilon_{jk,i} + \varepsilon_{ik,j}\right) - \frac{1}{15}\left(\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right) + \delta_{ik}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right) - \frac{1}{15}\left(\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right)\right), \tag{7}$$

where u_i and θ_i are components of displacement and rotation vectors, respectively. And δ_{ij} and e_{ijk} are the Kronecker's delta and permutation tensor, respectively.

The Cauchy stress and high-order stress components are defined by

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{8}$$

$$m_{ij} = 2\mu l_1^2\chi_{ij}, \tag{9}$$

$$p_j = 2\mu l_2^2\zeta_j, \tag{10}$$

$$\tau_{ijk} = 2\mu l_3^2\eta_{ijk}, \tag{11}$$

where λ and μ are Lame constants; l_1, l_2 and l_3 are three length scale parameters.

2.3. Kinematics of FG microplate

Let us consider a plate of total thickness h. The mid-plane surface denoted by Ω is a function of x and y coordinates and, the z-axis is taken normal to the plate. The displacement field of any points in the plate according to the first-order shear deformation theory is described by

$$\begin{aligned} \bar{u}(x,y,z) &= u(x,y) + z\beta_x(x,y), \\ \bar{v}(x,y,z) &= v(x,y) + z\beta_y(x,y), \\ \bar{w}(x,y,z) &= w(x,y), \end{aligned} \tag{12}$$

where u, v, w, β_x and β_y are in-plane, transverse displacements and two rotation components in the y-z, x-z planes, respectively.

To eliminate the shear locking phenomenon in FSDT, a hypothesis is introduced as follows

$$w = w^b + w^s, \quad \beta_x = -w^b_{,x}, \quad \beta_y = -w^b_{,y}. \tag{13}$$

Inserting Eq. (13) into Eq. (12), the displacement fields of the FSDT which is called the simple first-order shear deformation theory (sFSDT), are described by

$$\begin{aligned} \bar{u} &= u - zw^b_{,x} \\ \bar{v} &= v - zw^b_{,y} \\ \bar{w} &= w^b + w^s \end{aligned} \quad \text{or} \quad \bar{\mathbf{u}} = \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w^b + w^s \end{Bmatrix} + z\begin{Bmatrix} -w^b_{,x} \\ -w^b_{,y} \\ 0 \end{Bmatrix} = \mathbf{u}_1 + z\mathbf{u}_2. \tag{14}$$

Inserting Eq. (14) into Eq. (4), strain components according to sFSDT are presented by

$$\begin{gathered} \varepsilon_{xx} = u_{,x} - zw^b_{,xx}, \quad \varepsilon_{yy} = v_{,y} - zw^b_{,yy}, \quad \varepsilon_{zz} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(u_{,y} + v_{,x}\right) - zw^b_{,xy}, \\ \varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = \frac{1}{2}w^s_{,x}, \quad \varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}w^s_{,y}. \end{gathered} \tag{15}$$

Bending and shear strains can be described as

$$\boldsymbol{\varepsilon} = \left\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\right\}^{\mathrm{T}} = \boldsymbol{\varepsilon}^1 + z\boldsymbol{\varepsilon}^2 \quad \text{and} \quad \boldsymbol{\gamma} = \left\{\gamma_{xz} \quad \gamma_{yz}\right\}^T = \boldsymbol{\varepsilon}^s, \tag{16}$$

where

$$\boldsymbol{\varepsilon}^1 = \left\{u_{,x} \quad v_{,y} \quad u_{,y} + v_{,x}\right\}^T, \quad \boldsymbol{\varepsilon}^2 = -\left\{w^b_{,xx} \quad w^b_{,yy} \quad 2w^b_{,xy}\right\}^T, \quad \boldsymbol{\varepsilon}^s = \left\{w^s_{,x} \quad w^s_{,y}\right\}^T. \tag{17}$$

Similarly, the rotation vector is rewritten by inserting Eq. (14) into Eq. (5) by

$$\theta_x = \frac{1}{2}\left(2w^b_{,y} + w^s_{,y}\right), \quad \theta_y = \frac{1}{2}\left(-2w^b_{,x} - w^s_{,x}\right), \quad \theta_z = \frac{1}{2}\left(v_{,x} - u_{,y}\right). \tag{18}$$

Substituting Eq. (18) into Eq. (5), the rotation gradient components are described as

$$\begin{gathered} \chi_{xx} = \frac{1}{2}\left(2w^b_{,xy} + w^s_{,xy}\right), \quad \chi_{yy} = \frac{1}{2}\left(-2w^b_{,xy} - w^s_{,xy}\right), \quad \chi_{xy} = \frac{1}{2}\left(w^b_{,yy} - w^b_{,xx} + \frac{1}{2}\left(w^s_{,yy} - w^s_{,xx}\right)\right), \\ \chi_{xz} = \frac{1}{4}\left(v_{,xx} - u_{,xy}\right), \quad \chi_{yz} = \frac{1}{4}\left(v_{,xy} - u_{,yy}\right), \quad \chi_{zz} = 0. \end{gathered} \tag{19}$$

The rotation gradient components can be rewritten under a compact form by

$$\boldsymbol{\chi}^b = \begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} w^b_{,xy} + \frac{1}{2} w^s_{,xy} \\ -w^b_{,xy} - \frac{1}{2} w^s_{,xy} \\ \frac{1}{2}\left(w^b_{,yy} - w^b_{,xx}\right) + \frac{1}{4}\left(w^s_{,yy} - w^s_{,xx}\right) \end{Bmatrix}, \quad \boldsymbol{\chi}^s = \begin{Bmatrix} \chi_{xz} \\ \chi_{yz} \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} v_{,xx} - u_{,xy} \\ v_{,xy} - u_{,yy} \end{Bmatrix}. \quad (20)$$

Substituting the strains in Eq. (15) into Eq. (6), the dilatation tensor is expressed by

$$\zeta_x = u_{,xx} + v_{,xy} - z\left(w^b_{,xxx} + w^b_{,xyy}\right), \quad \zeta_y = v_{,yy} + u_{,xy} - z\left(w^b_{,yyy} + w^b_{,xxy}\right), \quad \zeta_z = -\left(w^b_{,xx} + w^b_{,yy}\right). \quad (21)$$

The dilatation tensor can be rewritten under a compact form by

$$\boldsymbol{\zeta} = \{\zeta_x \quad \zeta_y \quad \zeta_z\}^T = \boldsymbol{\zeta}^1 + z\boldsymbol{\zeta}^2, \quad (22)$$

where

$$\boldsymbol{\zeta}^1 = \left\{u_{,xx} + v_{,xy} \quad v_{,yy} + u_{,xy} \quad -w^b_{,xx} - w^b_{,yy}\right\}^T, \quad \boldsymbol{\zeta}^2 = -\left\{w^b_{,xxx} + w^b_{,xyy} \quad w^b_{,yyy} + w^b_{,xxy} \quad 0\right\}^T. \quad (23)$$

Similarly, substituting the strains in Eq. (15) into Eq. (7), the deviatoric stretch gradient components are defined as

$$\begin{aligned}
\eta_{xxx} &= \frac{2}{5}u_{,xx} - \frac{1}{5}u_{,yy} - \frac{2}{5}v_{,xy} + z\left(-\frac{2}{5}w^b_{,xxx} + \frac{3}{5}w^b_{,xyy}\right), \\
\eta_{yyy} &= \frac{2}{5}v_{,yy} - \frac{1}{5}v_{,xx} - \frac{2}{5}u_{,xy} + z\left(-\frac{2}{5}w^b_{,yyy} + \frac{3}{5}w^b_{,xxy}\right), \\
\eta_{yyx} = \eta_{yxy} = \eta_{xyy} &= -\frac{3}{15}u_{,xx} + \frac{4}{15}u_{,yy} + \frac{8}{15}v_{,xy} + z\left(\frac{3}{15}w^b_{,xxx} - \frac{12}{15}w^b_{,xyy}\right), \\
\eta_{xxy} = \eta_{xyx} = \eta_{yxx} &= -\frac{3}{15}v_{,yy} + \frac{4}{15}v_{,xx} + \frac{8}{15}u_{,xy} + z\left(\frac{3}{15}w^b_{,yyy} - \frac{12}{15}w^b_{,xxy}\right), \\
\eta_{zzx} = \eta_{zxz} = \eta_{xzz} &= -\frac{3}{15}u_{,xx} - \frac{1}{15}u_{,yy} - \frac{2}{15}v_{,xy} + z\left(\frac{3}{15}w^b_{,xxx} + \frac{3}{15}w^b_{,xyy}\right), \\
\eta_{zzy} = \eta_{zyz} = \eta_{yzz} &= -\frac{3}{15}v_{,yy} - \frac{1}{15}v_{,xx} - \frac{2}{15}u_{,xy} + z\left(\frac{3}{15}w^b_{,yyy} + \frac{3}{15}w^b_{,xxy}\right), \\
\eta_{zzz} &= \frac{1}{5}\left(w^b_{,xx} + w^b_{,yy}\right) - \frac{1}{5}\left(w^s_{,xx} + w^s_{,yy}\right), \\
\eta_{xxz} = \eta_{xzx} = \eta_{zxx} &= -\frac{4}{15}w^b_{,xx} + \frac{1}{15}w^b_{,yy} + \frac{4}{15}w^s_{,xx} - \frac{1}{15}w^s_{,yy}, \\
\eta_{yyz} = \eta_{yzy} = \eta_{zyy} &= -\frac{4}{15}w^b_{,yy} + \frac{1}{15}w^b_{,xx} + \frac{4}{15}w^s_{,yy} - \frac{1}{15}w^s_{,xx}, \\
\eta_{xyz} = \eta_{yzx} = \eta_{zxy} = \eta_{xzy} = \eta_{zyx} = \eta_{yxz} &= -\frac{1}{3}w^b_{,xy} + \frac{1}{3}w^s_{,xy}.
\end{aligned} \quad (24)$$

These components are also rewritten under compact forms by

$$\bar{\boldsymbol{\eta}} = \{\eta_{xxx} \quad \eta_{yyy} \quad \eta_{yyx} \quad \eta_{xxy} \quad \eta_{zzx} \quad \eta_{zzy}\}^T = \bar{\boldsymbol{\eta}}_1 + z\bar{\boldsymbol{\eta}}_2, \quad \tilde{\boldsymbol{\eta}} = \{\eta_{zzz} \quad \eta_{xxz} \quad \eta_{yyz} \quad \eta_{xyz}\}^T, \quad (25)$$

where

$$\tilde{\boldsymbol{\eta}} = \begin{Bmatrix} \frac{1}{5}w^b_{,xx} + \frac{1}{5}w^b_{,yy} - \frac{1}{5}w^s_{,xx} - \frac{1}{5}w^s_{,yy} \\ -\frac{4}{15}w^b_{,xx} + \frac{1}{15}w^b_{,yy} + \frac{4}{15}w^s_{,xx} - \frac{1}{15}w^s_{,yy} \\ -\frac{4}{15}w^b_{,yy} + \frac{1}{15}w^b_{,xx} + \frac{4}{15}w^s_{,yy} - \frac{1}{15}w^s_{,xx} \\ -\frac{1}{3}w^b_{,xy} + \frac{1}{3}w^s_{,xy} \end{Bmatrix},$$

$$\bar{\boldsymbol{\eta}}_1 = \begin{Bmatrix} \frac{2}{5}u_{,xx} - \frac{1}{5}u_{,yy} - \frac{2}{5}v_{,xy} \\ \frac{2}{5}v_{,yy} - \frac{1}{5}v_{,xx} - \frac{2}{5}u_{,xy} \\ -\frac{3}{15}u_{,xx} + \frac{4}{15}u_{,yy} + \frac{8}{15}v_{,xy} \\ -\frac{3}{15}v_{,yy} + \frac{4}{15}v_{,xx} + \frac{8}{15}u_{,xy} \\ -\frac{3}{15}u_{,xx} - \frac{1}{15}u_{,yy} - \frac{2}{15}v_{,xy} \\ -\frac{3}{15}v_{,yy} - \frac{1}{15}v_{,xx} - \frac{2}{15}u_{,xy} \end{Bmatrix}, \quad \bar{\boldsymbol{\eta}}_2 = \begin{Bmatrix} -\frac{2}{5}w^b_{,xxx} + \frac{3}{5}w^b_{,xyy} \\ -\frac{2}{5}w^b_{,yyy} + \frac{3}{5}w^b_{,xxy} \\ \frac{3}{15}w^b_{,xxx} - \frac{12}{15}w^b_{,xyy} \\ \frac{3}{15}w^b_{,yyy} - \frac{12}{15}w^b_{,xxy} \\ \frac{3}{15}w^b_{,xxx} + \frac{3}{15}w^b_{,xyy} \\ \frac{3}{15}w^b_{,yyy} + \frac{3}{15}w^b_{,xxy} \end{Bmatrix}. \tag{26}$$

The classical and higher-order stress elastic constitutive relations are expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \varsigma_{xy} \\ \varsigma_{xz} \\ \varsigma_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{27}$$

$$\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \\ m_{xz} \\ m_{yz} \end{Bmatrix} = 2G{l_1}^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ \chi_{xy} \\ \chi_{xz} \\ \chi_{yz} \end{Bmatrix}, \tag{28}$$

$$\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = 2G{l_2}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \zeta_x \\ \zeta_y \\ \zeta_z \end{Bmatrix}, \tag{29}$$

$$\begin{Bmatrix} \tau_{xxx} \\ \tau_{yyy} \\ \tau_{yyx} \\ \tau_{xxy} \\ \tau_{zzx} \\ \tau_{zzy} \end{Bmatrix} = 2G{l_3}^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_{xxx} \\ \eta_{yyy} \\ \eta_{yyx} \\ \eta_{xxy} \\ \eta_{zzx} \\ \eta_{zzy} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{zzz} \\ \tau_{xxz} \\ \tau_{yyz} \\ \tau_{xyz} \end{Bmatrix} = 2G{l_3}^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_{zzz} \\ \eta_{xxz} \\ \eta_{yyz} \\ \eta_{xyz} \end{Bmatrix}, \tag{30}$$

where

$$\begin{gathered} Q_{11} = Q_{22} = \frac{E_e}{1-\nu_e^2}, \quad Q_{12} = Q_{21} = \frac{\nu_e E_e}{1-\nu_e^2}, \quad Q_{66} = \frac{E_e}{2(1+\nu_e)}, \quad Q_{55} = Q_{44} = \frac{k^s E_e}{2(1+\nu_e)}, \\ G = \frac{E_e}{2(1+\nu_e)}, \quad k^s = \frac{5}{6}, \end{gathered} \tag{31}$$

in which E_e and ν_e are the effective Young modulus and Poisson's ratio, respectively. The discrete Galerkin weak form for the bending analysis of the FG microplate are described by

$$\begin{aligned} &\int_\Omega \int_{-h/2}^{h/2} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \varsigma_{xy}\delta\gamma_{xy} + \varsigma_{xz}\delta\gamma_{xz} + \varsigma_{yz}\delta\gamma_{yz}\right) d\Omega dz + \ldots \\ &+ \int_\Omega \int_{-h/2}^{h/2} \left(m_{xx}\delta\chi_{xx} + m_{yy}\delta\chi_{yy} + 2m_{xy}\delta\chi_{xy} + 2m_{xz}\delta\chi_{xz} + 2m_{yz}\delta\chi_{yz}\right) d\Omega dz + \ldots \\ &+ \int_\Omega \int_{-h/2}^{h/2} \left(\tau_{xxx}\delta\eta_{xxx} + \tau_{yyy}\delta\eta_{yyy} + 3\tau_{yyx}\delta\eta_{yyx} + 3\tau_{xxy}\delta\eta_{xxy} + 3\tau_{zzx}\delta\eta_{zzx} + 3\tau_{zzy}\delta\eta_{zzy}\right) d\Omega dz + \ldots \end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega}\int_{-h/2}^{h/2}\left(\tau_{zzz}\delta\eta_{zzz}+3\tau_{xxz}\delta\eta_{xxz}+3\tau_{yyz}\delta\eta_{yyz}+6\tau_{xyz}\delta\eta_{xyz}\right)d\Omega dz+\ldots \\
&+\int_{\Omega}\int_{-h/2}^{h/2}\left(p_x\delta\zeta_x+p_y\delta\zeta_y+p_z\delta\zeta_z\right)d\Omega dz=\int_{\Omega}\int_{-h/2}^{h/2}\delta\left(w^b+w^s\right)q_0 d\Omega dz.
\end{aligned}
\tag{32}
$$

Eq. (32) can split into two independent integrals following to middle surface and z-axis direction. Substituting Eq. (27)–(30) into Eq. (32), the discrete Galerkin weak form can be rewritten as follows

$$
\begin{aligned}
&\int_{\Omega}\delta\hat{\varepsilon}^T\hat{\mathbf{D}}\hat{\varepsilon}\mathrm{d}\Omega+\int_{\Omega}\delta(\varepsilon^s)^T\mathbf{D}^s\varepsilon^s\mathrm{d}\Omega+\int_{\Omega}\delta\left(\chi^b\right)^T\mathbf{D}_c^b\mathbf{\Gamma}_c^b\chi^b\mathrm{d}\Omega+\int_{\Omega}\delta(\chi^s)^T\mathbf{D}_c^s\mathbf{\Gamma}_c^s\chi^s\mathrm{d}\Omega+\ldots \\
&+\int_{\Omega}\delta\hat{\zeta}^T\hat{\mathbf{D}}^{di}\hat{\zeta}\mathrm{d}\Omega+\int_{\Omega}\delta\hat{\bar{\eta}}^T\hat{\mathbf{D}}^{de}\hat{\mathbf{\Gamma}}^{de}\hat{\bar{\eta}}\mathrm{d}\Omega+\int_{\Omega}\delta\tilde{\eta}^T\mathbf{D}^{dev}\mathbf{\Gamma}^{dev}\tilde{\eta}\mathrm{d}\Omega=\int_{\Omega}\delta\left(w^b+w^s\right)q_0\mathrm{d}\Omega,
\end{aligned}
\tag{33}
$$

where

$$
\begin{gathered}
\hat{\varepsilon}=\left\{\varepsilon^1\quad\varepsilon^2\right\}^T,\quad\hat{\zeta}=\left\{\zeta^1\quad\zeta^2\right\}^T,\quad\hat{\bar{\eta}}=\left\{\bar{\eta}_1\quad\bar{\eta}_2\right\}^T, \\
\hat{\mathbf{D}}=\begin{bmatrix}\mathbf{A}^b & \mathbf{B}^b\\ \mathbf{B}^b & \mathbf{D}^b\end{bmatrix},\quad\hat{\mathbf{D}}^{di}=\begin{bmatrix}\mathbf{A}^{di} & \mathbf{B}^{di}\\ \mathbf{B}^{di} & \mathbf{D}^{di}\end{bmatrix},\quad\hat{\mathbf{D}}^{de}=\begin{bmatrix}\mathbf{A}^{de} & \mathbf{B}^{de}\\ \mathbf{B}^{de} & \mathbf{D}^{de}\end{bmatrix},\quad\hat{\mathbf{\Gamma}}^{de}=\begin{bmatrix}\mathbf{\Gamma}^{de} & 0\\ 0 & \mathbf{\Gamma}^{de}\end{bmatrix},
\end{gathered}
\tag{34}
$$

in which

$$
\begin{gathered}
\left(\mathbf{A}^b,\mathbf{B}^b,\mathbf{D}^b\right)=\int_{-h/2}^{h/2}\left(1,z,z^2\right)\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\mathrm{d}z,\quad\mathbf{D}^s=\int_{-h/2}^{h/2}\begin{bmatrix}Q_{44} & 0\\ 0 & Q_{55}\end{bmatrix}\mathrm{d}z, \\
\mathbf{D}_c^b=\int_{-h/2}^{h/2}2Gl_1{}^2\mathbf{I}_{3\times3}\mathrm{d}z,\quad\mathbf{D}_c^s=\int_{-h/2}^{h/2}2Gl_1{}^2\mathbf{I}_{2\times2}\mathrm{d}z,\quad\left(\mathbf{A}^{di},\mathbf{B}^{di},\mathbf{D}^{di}\right)=\int_{-h/2}^{h/2}2Gl_2{}^2\left(1,z,z^2\right)\mathbf{I}_{3\times3}\mathrm{d}z, \\
\mathbf{D}^{dev}=\int_{-h/2}^{h/2}2Gl_3{}^2\mathbf{I}_{4\times4}\mathrm{d}z,\quad\left(\mathbf{A}^{de},\mathbf{B}^{de},\mathbf{D}^{de}\right)=\int_{-h/2}^{h/2}2Gl_3{}^2\left(1,z,z^2\right)\mathbf{I}_{6\times6}\mathrm{d}z, \\
\mathbf{\Gamma}_c^b=\mathrm{diag}\left(1,1,2\right),\quad\mathbf{\Gamma}_c^s=\mathrm{diag}\left(2,2\right),\quad\mathbf{\Gamma}^{dev}=\mathrm{diag}\left(1,3,3,6\right),\quad\mathbf{\Gamma}^{de}=\mathrm{diag}\left(1,1,3,3,3,3\right),
\end{gathered}
\tag{35}
$$

in which $\mathbf{I}_{2\times2},\mathbf{I}_{3\times3},\mathbf{I}_{4\times4},\mathbf{I}_{6\times6}$ are the identity matrices of size $2\times2, 3\times3, 4\times4$ and 6×6, respectively.

3. FG MICROPLATE FORMULATION USING NURBS BASIS FUNCTIONS

For a knot vector $\Xi=\left\{\xi_1,\xi_2,\ldots,\xi_{n+p+1}\right\}$, B-spline basis functions can be recursively built by the Cox-de Boor algorithm as follows

$$
\bar{N}_{i,0}\left(\xi\right)=\begin{cases}1 & \text{if}\quad\xi_i\le\xi<\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}\ (p=0),
\tag{36}
$$

and

$$
\bar{N}_{i,p}\left(\xi\right)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}\bar{N}_{i,p-1}\left(\xi\right)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}\bar{N}_{i+1,p-1}\left(\xi\right)\ (p>1).
\tag{37}
$$

Similarly, the two-dimensional B-splines basis functions are calculated by using the tensor product of two knot vectors $\Xi=\left\{\xi_1,\xi_2,\ldots,\xi_{n+p+1}\right\}$ and $\mathbf{H}=\left\{\eta_1,\eta_2,\ldots,\eta_{m+q+1}\right\}$ as

$$
R_{i,j}\left(\xi,\eta\right)=\bar{N}_{i,p}\left(\xi\right)\bar{M}_{j,q}\left(\eta\right).
\tag{38}
$$

NURBS basic functions can be defined by a linear combination of B-spline basis functions and their corresponding weights as follows

$$
N_{i,j}\left(\xi,\eta\right)=N_I\left(\xi\right)=\frac{\bar{N}_{i,p}\left(\xi\right)\bar{M}_{j,q}\left(\eta\right)w_{i,j}}{\sum_{\hat{i}}^{n}\sum_{\hat{j}}^{m}\bar{N}_{\hat{i},p}\left(\xi\right)\bar{M}_{\hat{j},q}\left(\eta\right)w_{\hat{i},\hat{j}}}.
\tag{39}
$$

The displacements using NURBS basis functions [22] can be expressed as

$$\mathbf{u}^h(x,y)=\sum_{I=1}^{m\times n}\begin{bmatrix}N_I(x,y)&0&0&0\\0&N_I(x,y)&0&0\\0&0&N_I(x,y)&0\\0&0&0&N_I(x,y)\end{bmatrix}\mathbf{q}_I,\tag{40}$$

where $\mathbf{q}_I=\left\{u_I\quad v_I\quad w_I^b\quad w_I^s\right\}^T$ are degrees of freedom of control point I.

Substituting Eq. (40) into Eq. (17), the strain components can be rewritten as

$$\hat{\varepsilon}=\left\{\varepsilon^1\quad\varepsilon^2\right\}^T=\sum_{I=1}^{n}\left\{\mathbf{B}_I^1\quad\mathbf{B}_I^2\right\}^T\mathbf{q}_I=\sum_{I=1}^{n}\hat{\mathbf{B}}_I\mathbf{q}_I\quad\text{and}\quad\varepsilon^s=\sum_{I=1}^{n}\mathbf{B}_I^s\mathbf{q}_I,\tag{41}$$

in which,

$$\mathbf{B}_I^1=\begin{bmatrix}N_{I,x}&0&0&0\\0&N_{I,y}&0&0\\N_{I,y}&N_{I,x}&0&0\end{bmatrix},\quad\mathbf{B}_I^2=-\begin{bmatrix}0&0&N_{I,xx}&0\\0&0&N_{I,yy}&0\\0&0&2N_{I,xy}&0\end{bmatrix},\quad\mathbf{B}_I^s=\begin{bmatrix}0&0&0&N_{I,x}\\0&0&0&N_{I,y}\end{bmatrix}.\tag{42}$$

Similarly, the curvatures are obtained by substituting Eq. (40) into Eq. (20) as follows

$$\chi^b=\sum_{I=1}^{n}\mathbf{B}_{cI}^b\mathbf{q}_I,\quad\chi^s=\sum_{I=1}^{n}\mathbf{B}_{cI}^s\mathbf{q}_I,\tag{43}$$

in which,

$$\mathbf{B}_{cI}^b=\begin{bmatrix}0&0&N_{I,xy}&\frac{1}{2}N_{I,xy}\\0&0&-N_{I,xy}&-\frac{1}{2}N_{I,xy}\\0&0&\frac{1}{2}\left(N_{I,yy}-N_{I,xx}\right)&\frac{1}{4}\left(N_{I,yy}-N_{I,xx}\right)\end{bmatrix},\quad\mathbf{B}_{cI}^s=\frac{1}{4}\begin{bmatrix}-N_{I,xy}&N_{I,xx}&0&0\\-N_{I,yy}&N_{I,xy}&0&0\end{bmatrix}.\tag{44}$$

Substituting Eq. (40) into Eq. (22), the dilatation components can be rewritten as

$$\hat{\zeta}=\left\{\zeta^1\quad\zeta^2\right\}^T=\sum_{I=1}^{n}\left\{\mathbf{B}_I^{\zeta 1}\quad\mathbf{B}_I^{\zeta 2}\right\}^T\mathbf{q}_I=\sum_{I=1}^{n}\hat{\mathbf{B}}_I^{\zeta}\mathbf{q}_I,\tag{45}$$

where

$$\mathbf{B}_I^{\zeta 1}=\begin{bmatrix}N_{I,xx}&N_{I,xy}&0&0\\N_{I,xy}&N_{I,yy}&0&0\\0&0&-N_{I,xx}-N_{I,yy}&0\end{bmatrix},\quad\mathbf{B}_I^{\zeta 2}=\begin{bmatrix}0&0&-N_{I,xxx}-N_{I,xyy}&0\\0&0&-N_{I,yyy}-N_{I,xxy}&0\\0&0&0&0\end{bmatrix}.\tag{46}$$

Substituting Eq. (40) into Eq. (25), the dilatation stretch gradient components can be rewritten as

$$\hat{\eta}=\left\{\bar{\eta}_1\quad\bar{\eta}_2\right\}^T=\sum_{I=1}^{n}\left\{\mathbf{B}_I^{\bar{\eta}_1}\quad\mathbf{B}_I^{\bar{\eta}_2}\right\}^T\mathbf{q}_I=\sum_{I=1}^{n}\hat{\mathbf{B}}_I^{\bar{\eta}}\mathbf{q}_I,\quad\tilde{\eta}=\sum_{I=1}^{n}\mathbf{B}_I^{\tilde{\eta}}\mathbf{q}_I,\tag{47}$$

where

$$\mathbf{B}_I^{\bar{\eta}_1}=\begin{bmatrix}\frac{2}{5}N_{I,xx}-\frac{1}{5}N_{I,yy}&-\frac{2}{5}N_{I,xy}&0&0\\-\frac{2}{5}N_{I,xy}&\frac{2}{5}N_{I,yy}-\frac{1}{5}N_{I,xx}&0&0\\-\frac{3}{15}N_{I,xx}+\frac{4}{15}N_{I,yy}&\frac{8}{15}N_{I,xy}&0&0\\\frac{8}{15}N_{I,xy}&-\frac{3}{15}N_{I,yy}+\frac{4}{15}N_{I,xx}&0&0\\-\frac{3}{15}N_{I,xx}-\frac{1}{15}N_{I,yy}&-\frac{2}{15}N_{I,xy}&0&0\\-\frac{2}{15}N_{I,xy}&-\frac{3}{15}N_{I,yy}-\frac{1}{15}N_{I,xx}&0&0\end{bmatrix},$$

$$\mathbf{B}_I^{\bar{\eta}_2} = \begin{bmatrix} 0 & 0 & -\frac{2}{5}N_{I,xxx} + \frac{3}{5}N_{I,xyy} & 0 \\ 0 & 0 & -\frac{2}{5}N_{I,yyy} + \frac{3}{5}N_{I,xxy} & 0 \\ 0 & 0 & \frac{3}{15}N_{I,xxx} - \frac{12}{15}N_{I,xyy} & 0 \\ 0 & 0 & \frac{3}{15}N_{I,yyy} - \frac{12}{15}N_{I,xxy} & 0 \\ 0 & 0 & \frac{3}{15}N_{I,xxx} + \frac{3}{15}N_{I,xyy} & 0 \\ 0 & 0 & \frac{3}{15}N_{I,yyy} + \frac{3}{15}N_{I,xxy} & 0 \end{bmatrix},$$

$$\mathbf{B}_I^{\tilde{\eta}} = \begin{bmatrix} 0 & 0 & \frac{1}{5}N_{I,xx} + \frac{1}{5}N_{I,yy} & -\frac{1}{5}N_{I,xx} - \frac{1}{5}N_{I,yy} \\ 0 & 0 & -\frac{4}{15}N_{I,xx} + \frac{1}{15}N_{I,yy} & \frac{4}{15}N_{I,xx} - \frac{1}{15}N_{I,yy} \\ 0 & 0 & -\frac{4}{15}N_{I,yy} + \frac{1}{15}N_{I,xx} & \frac{4}{15}N_{I,yy} - \frac{1}{15}N_{I,xx} \\ 0 & 0 & -\frac{1}{3}N_{I,xy} & \frac{1}{3}N_{I,xy} \end{bmatrix}.$$

Substituting Eqs. (41), (43), (45) and (47) into Eq. (33), the weak form of the bending analysis of the FG microplate is rewritten as

$$\mathbf{Kq} = \mathbf{F}, \tag{48}$$

where $\mathbf{K}$ ($\mathbf{K} = \mathbf{K}^{\varepsilon} + \mathbf{K}^{\chi} + \mathbf{K}^{\zeta} + \mathbf{K}^{\eta}$), $\mathbf{M}$ and $\mathbf{F}$ are the global stiffness matrix and force vector, respectively, in which

$$\begin{gathered} \mathbf{K}^{\varepsilon} = \int_{\Omega} \hat{\mathbf{B}}^T \hat{\mathbf{D}} \hat{\mathbf{B}} \mathrm{d}\Omega + \int_{\Omega} (\mathbf{B}^s)^T \mathbf{D}^s \mathbf{B}^s \mathrm{d}\Omega, \quad \mathbf{K}^{\chi} = \int_{\Omega} \left(\mathbf{B}_c^b\right)^T \mathbf{D}_c^b \boldsymbol{\Gamma}_c^b \mathbf{B}_c^b \mathrm{d}\Omega + \int_{\Omega} (\mathbf{B}_c^s)^T \mathbf{D}_c^s \boldsymbol{\Gamma}_c^s \mathbf{B}_c^s \mathrm{d}\Omega, \\ \mathbf{K}^{\zeta} = \int_{\Omega} \left(\hat{\mathbf{B}}^{\zeta}\right)^T \hat{\mathbf{D}}^{di} \hat{\mathbf{B}}^{\zeta} \mathrm{d}\Omega, \quad \mathbf{K}^{\eta} = \int_{\Omega} \left(\hat{\mathbf{B}}^{\bar{\eta}}\right)^T \hat{\mathbf{D}}^{de} \hat{\boldsymbol{\Gamma}}^{de} \hat{\mathbf{B}}^{\bar{\boldsymbol{\eta}}} \mathrm{d}\Omega + \int_{\Omega} \left(\mathbf{B}^{\tilde{\eta}}\right)^T \mathbf{D}^{dev} \boldsymbol{\Gamma}^{dev} \mathbf{B}^{\tilde{\eta}} \mathrm{d}\Omega, \\ \mathbf{F} = \int_{\Omega} q_0 \{0 \quad 0 \quad N_I \quad N_I\}^T \mathrm{d}\Omega. \end{gathered} \tag{49}$$

4. NUMERICAL EXAMPLES AND DISCUSSIONS

In this study, three length scale parameters which assumed to be equal through the thickness direction $l_1 = l_2 = l_3 = l = 15 \times 10^{-6}$ m, are taken the same as in [8]. The FG microplate is made from the bottom metal surface (Aluminum-Al) to the top ceramic surface (alumina-Al_2O_3). The material properties for Al are $E_m = 70$ GPa, $v_m = 0.3$, $\rho_m = 2702$ kg/m^3 and Al_2O_3 are $E_c = 380$ GPa, $v_c = 0.3$, $\rho_c = 3800$ kg/m^3. Without loss of generality, to compare results, the cubic NURBS basis function of 17×17 element [22] is only used in the numerical examples.

4.1. FG square microplate

Firstly, we consider the simply supported and fully clamped FG rectangular microplates with the length (a) and the weight (b) under sinusoidally distributed load ($q_0 = \bar{q}_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$). The deflection of the FG microplate are computed by: $\hat{w} = \frac{10h^3 E_c}{\bar{q}_0 a^4} w\left(\frac{a}{2}, \frac{b}{2}\right)$, in which E_c are the Young's modulus of ceramic. Different values of the length-to-thickness ratio (a/h), power index (n), and material length scale ratio (l/h) are studied to investigate the non-dimensional deflection of FG microplate. The central non-dimensional deflection is tabulated in Tab. 1. For comparison purpose, referenced results reported by Zhang et al. [15] using the analytical solution based on the refined plate theory (RPT) with 4 degrees of freedom (DOFs), Thai et al. [18] using the IGA based on the third-order shear deformation theory (TSDT) with 5 DOFs and Thai et al. [16] using the IGA based on RPT are presented. It can be seen that an excellent agreement is achieved in the case of $l/h = 0$. In addition, obtained results are slightly larger than referenced ones corresponding to the values of power index from 0 to 1, while an opposite phenomenon is shown for different values of power index. For the case of $l/h \neq 0$, most of obtained results are slightly larger than referenced ones. Especially, a slight difference of present results and referenced ones is shown in the case of $l/h = 1$ because of using the sFSDT instead of HSDT as in referenced solutions. In addition, the results of HSDT are smaller than the results of sFSDT due to considering the higher-order shear term in the MSGT. Besides, as observed from Tab. 1, the non-dimensional deflection increases when increasing of the power index and the length-to-thickness ratio. While, the non-dimensional deflection decreases by a rise of the material length scale ratio l/h.

Table 1. Non-dimensional displacement $\hat{w}$ of FG rectangular microplates $(a = b)$ under sinusoidally distributed load

a/h	n	Method	l/h					
			0	0.05	0.1	0.2	0.5	1.0
Simply supported								
5	0.1	Exact-RPT [15]	0.3883	0.3731	0.3341	0.2359	0.0778	0.0230
		IGA-TSDT [18]	0.3785	0.3648	0.3290	0.2366	0.0803	0.0240
		IGA-RPT [16]	0.3785	0.3305	0.2973	0.2125	0.0717	0.0214
		Present	0.3792	0.3671	0.3359	0.2564	0.1206	0.0582
	0.5	Exact-RPT [15]	0.5198	0.4983	0.4435	0.3086	0.0997	0.0293
		IGA-TSDT [18]	0.5177	0.4975	0.4457	0.3153	0.1045	0.0310
		IGA-RPT [16]	0.5176	0.4965	0.4426	0.3098	0.1018	0.0303
		Present	0.5192	0.5010	0.4546	0.3397	0.1551	0.0742
	1	Exact-RPT [15]	0.6688	0.6396	0.5658	0.3879	0.1223	0.0357
		IGA-TSDT [18]	0.6688	0.6412	0.5709	0.3977	0.1286	0.0378
		IGA-RPT [16]	0.6688	0.6399	0.5670	0.3908	0.1252	0.0369
		Present	0.6691	0.6442	0.5812	0.4285	0.1919	0.0914
	2	Exact-RPT [15]	0.8671	0.8286	0.7313	0.4980	0.1544	0.0447
		IGA-TSDT [18]	0.8671	0.8307	0.7379	0.5107	0.1627	0.0475
		IGA-RPT [16]	0.8671	0.8292	0.7332	0.5021	0.1580	0.0460
		Present	0.8592	0.8277	0.7477	0.5530	0.2488	0.1186
	4	Exact-RPT [15]	1.0408	0.9967	0.8843	0.6095	0.1921	0.0558
		IGA-TSDT [18]	1.0409	0.9994	0.8927	0.6263	0.2034	0.0597
		IGA-RPT [16]	1.0409	0.9977	0.8875	0.6159	0.1964	0.0573
		Present	1.0101	0.9781	0.8950	0.6831	0.3215	0.1550
	10	Exact-RPT [15]	1.2269	1.1790	1.0557	0.7455	0.2454	0.0724
		IGA-TSDT [18]	1.2276	1.1829	1.0668	0.7678	0.2614	0.0781
		IGA-RPT [16]	1.2276	1.1811	1.0609	0.7548	0.2510	0.0743
		Present	1.1802	1.1492	1.0671	0.8457	0.4225	0.2072
10	0.1	IGA-RPT [16]	0.3278	0.3157	0.2842	0.2033	0.0682	0.0203
		Present	0.3280	0.3164	0.2864	0.2093	0.0805	0.0336
	0.5	IGA-RPT [16]	0.4537	0.4355	0.3887	0.2723	0.0884	0.0260
		Present	0.4540	0.4365	0.3917	0.2799	0.1040	0.0428
	1	IGA-RPT [16]	0.5890	0.5640	0.5004	0.3453	0.1095	0.0320
		Present	0.5890	0.5650	0.5039	0.3550	0.1290	0.0528
	2	IGA-RPT [16]	0.7573	0.7253	0.6439	0.4446	0.1407	0.0409
		Present	0.7552	0.7249	0.6474	0.4576	0.1671	0.0685
	4	IGA-RPT [16]	0.8815	0.8480	0.7614	0.5405	0.1784	0.0526
		Present	0.8736	0.8429	0.7630	0.5576	0.2147	0.0895
	10	IGA-RPT [16]	1.0087	0.9755	0.8879	0.6535	0.2298	0.0694
		Present	0.9966	0.9672	0.8891	0.6767	0.2796	0.1194

a/h	n	Method	l/h					
			0	0.05	0.1	0.2	0.5	1.0
Fully clamped								
5	0	IGA-TSDT [18]	0.1647	-	0.1404	0.0984	0.0322	0.0095
		IGA-RPT [16]	0.1606	0.1535	0.1361	0.0945	0.0308	0.0091
		Present	0.1615	0.1567	0.1446	0.1145	0.0588	0.0258
	0.5	IGA-TSDT [18]	0.2429	-	0.2050	0.1420	0.0459	0.0135
		IGA-RPT [16]	0.2362	0.2253	0.1986	0.1363	0.0440	0.0130
		Present	0.2387	0.2308	0.2111	0.1635	0.0817	0.0355
	1	IGA-TSDT [18]	0.3113	-	0.2603	0.1774	0.0559	0.0163
		IGA-RPT [16]	0.3029	0.2881	0.2523	0.1704	0.0537	0.0158
		Present	0.3045	0.2938	0.2673	0.2049	0.1010	0.0438
	2	IGA-TSDT [18]	0.4086	-	0.3383	0.2267	0.0693	0.0200
		IGA-RPT [16]	0.3976	0.3773	0.3284	0.2183	0.0663	0.0192
		Present	0.3919	0.3783	0.3447	0.2648	0.1310	0.0569
	5	IGA-TSDT [18]	0.5437	-	0.4476	0.2968	0.0893	0.0256
		IGA-RPT [16]	0.5303	0.5025	0.4362	0.2877	0.0856	0.0244
		Present	0.4963	0.4819	0.4456	0.3545	0.1832	0.0804
	10	IGA-TSDT [18]	0.6304	-	0.5214	0.3505	0.1081	0.0313
		IGA-RPT [16]	0.6134	0.5820	0.5071	0.3378	0.1019	0.0292
		Present	0.5721	0.5573	0.5195	0.4217	0.2244	0.0992
10	0	IGA-TSDT [18]	0.1170	-	0.1016	0.0730	0.0247	0.0074
		IGA-RPT [16]	0.1151	0.1108	0.0997	0.0712	0.0240	0.0071
		Present	0.1152	0.1114	0.1017	0.0766	0.0339	0.0161
	0.5	IGA-TSDT [18]	0.1773	-	0.1521	0.1068	0.0349	0.0103
		IGA-RPT [16]	0.1747	0.1675	0.1492	0.1042	0.0340	0.0100
		Present	0.1751	0.1686	0.1522	0.1115	0.0473	0.0222
	1	IGA-TSDT [18]	0.2295	-	0.1951	0.1349	0.0430	0.0126
		IGA-RPT [16]	0.2261	0.2163	0.1915	0.1318	0.0419	0.0123
		Present	0.2262	0.2174	0.1951	0.1409	0.0586	0.0274
	2	IGA-TSDT [18]	0.2967	-	0.2517	0.1733	0.0547	0.0159
		IGA-RPT [16]	0.2922	0.2794	0.2472	0.1694	0.0532	0.0155
		Present	0.2903	0.2792	0.2508	0.1818	0.0759	0.0355
	5	IGA-TSDT [18]	0.3676	-	0.3161	0.2233	0.0734	0.0216
		IGA-RPT [16]	0.3609	0.3466	0.3100	0.2182	0.0712	0.0209
		Present	0.3515	0.3403	0.3113	0.2359	0.1054	0.0503
	10	IGA-TSDT [18]	0.4121	-	0.3582	0.2584	0.0884	0.0265
		IGA-RPT [16]	0.4041	0.3893	0.3510	0.2523	0.0854	0.0254
		Present	0.3927	0.3817	0.3527	0.2745	0.1282	0.0620

4.2. FG circular microplate

Let us consider a FG circular microplate of the radius R and thickness h subjected to a uniform load q_0. The deflection of the FG circular microplate are calculated by: $\hat{w} = \dfrac{64D_c}{q_0 R^4} w\,(0,0)$, where $D_c = \dfrac{E_c h^3}{12\,(1-\upsilon_c^2)}$. Different types of boundary condition as roller, simply supported and fully clamped are studied. Similar to the previous example, different values of thickness-to-radius ratio (h/R) and material length scale ratio (l/h) are investigated. The central non-dimensional deflection of FG plates is given in Tab. 2. The non-dimensional deflection proposed by Zhang et al. [14] based analytical solution using TSDT, Thai et al. [18] based on IGA-TSDT and Thai et al. [16] based on IGA-RPT are provided to compare results. It can be seen that obtained results are in good agreement with those referenced ones Basically, most of obtained results are slightly larger than compared to referenced solutions within considering of size effects. Again, we can clearly see that the non-dimensional displacement of the microplate is decreased when material length scale ratio increases. For reason, it can be concluded that the size-dependent microplate model based on MSGT can be increased the stiffness of microplate leading to a decline of the displacement.

Table 2. Comparison of normalized displacement of FG circular microplate with a power index $n = 1.5$

h/R	Reference	l/h				
		0.0	0.1	0.2	0.5	1.0
Roller						
0.2	Exact-TSDT [14]	9.9266	8.6275	5.9775	1.9019	0.5556
	IGA-TSDT [18]	9.9230	8.4748	5.8988	1.8917	0.5529
	IGA-RPT [16]	9.9230	7.8347	5.8880	1.9045	0.5464
	Present	9.9132	8.4943	6.0249	2.1636	0.9185
0.15	Exact-TSDT [14]	9.8090	8.4931	5.9080	1.8746	0.5464
	IGA-TSDT [18]	9.7621	8.3364	5.7993	1.8557	0.5417
	IGA-RPT [16]	9.7620	8.3281	5.7898	1.9669	0.5419
	Present	9.7565	8.3551	5.8680	2.0294	0.7498
0.1	Exact-TSDT [14]	9.7734	8.4464	5.8141	1.8519	0.5399
	IGA-TSDT [18]	9.6471	8.2372	5.7278	1.8299	0.5336
	IGA-RPT [16]	9.6470	8.2315	5.7138	1.8262	0.5317
	Present	9.6446	8.2434	5.7478	1.9036	0.6267
0.05	Exact-TSDT [14]	9.7669	8.3167	5.7794	1.8406	0.5363
	IGA-TSDT [18]	9.5781	8.1776	5.6846	1.8143	0.5288
	IGA-RPT [16]	9.5780	8.1729	5.6751	1.8101	0.5268
	Present	9.5714	8.1757	5.6848	1.8285	0.5512
0.01	Exact-TSDT [14]	9.7669	8.3050	5.7673	1.8370	0.5349
	IGA-TSDT [18]	9.5560	8.1584	5.6706	1.8092	0.5272
	IGA-RPT [16]	9.5560	8.1542	5.6625	1.8041	0.5255
	Present	9.5559	8.1543	5.6629	1.8050	0.5265
Simply supported						
0.2	Exact-TSDT [14]	8.5551	7.4543	5.3811	1.8359	0.5497
	IGA-TSDT [18]	8.3939	7.3241	5.3088	1.8251	0.5470
	IGA-RPT [16]	8.3938	7.2072	5.3635	1.8280	0.5451
	Present	8.3830	7.3015	5.4968	2.0036	8.9012
0.15	Exact-TSDT [14]	8.3839	7.3358	5.3121	1.8074	0.5405
	IGA-TSDT [18]	8.2325	7.1847	5.2085	1.7892	0.5358
	IGA-RPT [16]	8.2325	7.1624	4.7928	1.7867	0.5297
	Present	8.2262	7.2219	5.3664	1.6518	0.6591
0.1	Exact-TSDT [14]	8.2621	7.2448	5.2204	1.7849	0.5339
	IGA-TSDT [18]	8.1171	7.0847	5.1359	1.7632	0.5278
	IGA-RPT [16]	8.1171	7.0787	5.1863	1.7648	0.5228
	Present	8.1144	7.0867	5.1405	1.8033	0.6392
0.05	Exact-TSDT [14]	8.2303	7.1476	5.1764	1.7733	0.5304
	IGA-TSDT [18]	8.0479	7.0245	5.0920	1.7474	0.5229
	IGA-RPT [16]	8.0479	7.0203	5.0833	1.7486	0.5183
	Present	8.0472	7.0225	5.0920	1.7616	0.5453
0.01	Exact-TSDT [14]	8.2119	7.1440	5.1735	1.7693	0.5289
	IGA-TSDT [18]	8.0258	7.0052	5.0778	1.7422	0.5213
	IGA-RPT [16]	8.0257	7.0009	5.0696	1.7372	0.5196
	Present	8.0257	7.0010	5.0700	1.7380	0.5206

h/R	Reference	l/h				
		0.0	0.1	0.2	0.5	1.0
Fully clamped						
0.2	Exact-TSDT [14]	2.6947	2.2749	1.5768	0.4877	0.1415
	IGA-TSDT [18]	2.7015	2.2896	1.5768	0.4994	0.1456
	IGA-RPT [16]	2.6997	2.2808	1.5526	0.4910	0.1442
	Present	2.6935	2.3255	1.6870	0.7106	0.3422
0.15	Exact-TSDT [14]	2.5628	2.1482	1.4934	0.4666	0.1353
	IGA-TSDT [18]	2.5459	2.1632	1.4934	0.4733	0.1378
	IGA-RPT [16]	2.5447	2.1558	1.4797	0.4670	0.1354
	Present	2.5404	2.1802	1.5515	0.5993	0.2646
0.1	Exact-TSDT [14]	2.4566	2.0878	1.4309	0.4512	0.1307
	IGA-TSDT [18]	2.4340	2.0705	1.4308	0.4532	0.1319
	IGA-RPT [16]	2.4332	2.0647	1.4204	0.4472	0.1301
	Present	2.4311	2.0752	1.4525	0.5101	0.1968
0.05	Exact-TSDT [14]	2.3910	2.0320	1.3918	0.4410	0.1278
	IGA-TSDT [18]	2.3664	2.0137	1.3918	0.4404	0.1280
	IGA-RPT [16]	2.3661	2.0090	1.3831	0.4352	0.1263
	Present	2.3655	2.0116	1.3912	0.4518	0.1452
0.01	Exact-TSDT [14]	2.3765	2.0226	1.3785	0.4381	0.1270
	IGA-TSDT [18]	2.3446	1.9953	1.3790	0.4361	0.1267
	IGA-RPT [16]	2.3446	1.9910	1.3709	0.4311	0.1250
	Present	2.3445	1.9911	1.3712	0.4318	0.1258

5. CONCLUSIONS

In this paper, a novel simple size-dependent isogeometric approach using MSGT, sFSDT and isogeometric analysis was presented to analyze the bending behavior of FG microplates. The advantages of the present approach are only to contain four unknowns and three material length scale parameters. Thus, it is suitable for computation of size-dependent practical problems. In addition, material properties as Young's modulus, Poison's ratio and density mass were computed by the power rule. In addition, the classical sFSDT model was retrieved from the present model by taking zero of all material length scale parameters. Numerical results pointed out that the non-dimensional deflection obtained from the present solution are slightly larger than those referenced ones in the case of $l/h \neq 0$. In addition, the stiffness of microplate can be increased as considering small scale effects leading to a decrease of the deflection.

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