

## ASYMPTOTIC THEORY FOR THIN TWO-PLY SHELLS

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**Abstract.** We develop an asymptotic model for the finite-deformation, small-strain response of thin laminated shells composed of two perfectly bonded laminae that exhibit reflection symmetry of the material properties with respect to an interfacial surface. No *a priori* hypotheses are made concerning the kinematics of deformation. The asymptotic procedure culminates in a generalization of Koiter's well-known shell theory to accommodate the laminated structure, and incorporates a rigorous limit model for pure bending.

*Keywords:* laminated shells; asymptotic expansion.

### 1. INTRODUCTION

The development of tractable two-dimensional models for laminated plates and shells has been the focus of significant activity lying at the intersection of continuum mechanics, differential geometry and mathematics. The achievements of the engineering community toward this objective are summarized in [1, 2]. This body of work entails through-thickness integration of the three-dimensional equations combined with *a priori* hypotheses of the Kirchhoff–Love type, and generalizations thereof, concerning the variation of the deformation through the thickness of the shell. On the mathematical side, recent work, summarized in [3], emphasizes asymptotic expansion of the three-dimensional equations and the related method of gamma convergence, concerned with the limiting variational problem as shell thickness tends to zero.

In the present work we aim to strike a balance between these schools by deriving a small-thickness estimate of the (three-dimensional) strain energy of the shell without any *a priori* kinematical hypotheses apart from the requisite degree of smoothness. This estimate is terminated at order  $h^3$ , where  $h$  is the uniform shell thickness prior to deformation. The model thus derived accounts for stretching and leading-order bending effects and reduces to Koiter's energy [4] if the laminae have identical properties, and if the interface between them coincides with the midsurface of the shell. We use this energy in a virtual-power statement to extract the differential equations and boundary conditions of the two-dimensional theory. Though our attention is directed entirely to the modeling of two-ply laminates, our procedure is extendable, albeit with substantial effort, to multi-layer shells.

Standard notation is used throughout. Thus boldface is used for vectors and tensors and a dot between bold symbols is used to denote the standard Euclidean inner product; for example, if  $A_1$  and  $A_2$  are second-order tensors, then their inner product is  $A_1 \cdot A_2 = tr(A_1 A_2^t)$ , where  $tr(\cdot)$  is the trace and the superscript  $t$  is used to denote the transpose. The associated norm is  $|A| = \sqrt{A \cdot A}$ . The linear operator  $Sym(\cdot)$  yields the symmetric part of its second-order tensor argument, and the notation  $\otimes$  identifies the standard tensor product of vectors. We also use  $Orth^+$  to denote the group of rotation tensors. The notations  $A^{-1}$ ,  $A^*$  and  $\det A$  respectively are used to denote the inverse, the cofactor and

the determinant of a tensor  $A$ ; they are connected by  $A^* = (\det A)A^{-t}$  if  $A$  is invertible. For a fourth-order tensor  $\mathcal{A}$ , the notation  $\mathcal{A}[B]$  stands for the second-order tensor resulting from the linear action of  $\mathcal{A}$  on the second-order tensor  $B$  (see Truesdell and Noll [5]; Eq. (7.10)). Its transpose  $\mathcal{A}^t$  is defined by  $B \cdot \mathcal{A}^t[A] = A \cdot \mathcal{A}[B]$ , and  $\mathcal{A}$  is said to possess major symmetry if  $\mathcal{A}^t = \mathcal{A}$ . If  $A \cdot \mathcal{A}[B] = A^t \cdot \mathcal{A}[B]$  and  $A \cdot \mathcal{A}[B] = A \cdot \mathcal{A}[B^t]$ , then  $\mathcal{A}$  is said to possess the minor symmetries. Finally, the notation  $(\cdot)_A$  stands for the derivative with respect to tensor  $A$ .

Superposed tildes or carets are used to identify three-dimensional variables, whereas variables appearing without these are the restrictions of their three-dimensional counterparts to the surface  $\Omega$  forming the interface of the laminae of the thin shell-like body. The body itself occupies the volume  $\kappa = \Omega \times C$ , where  $C$  is the collection of generators of the shell. Our basic assumption is that the shell is thin in the sense that its thickness  $h$  satisfies  $h/l \ll 1$ , where  $l$  is a characteristic length associated with the geometry of  $\Omega$ , i.e., its minimum local principal radius of curvature or its smallest spanwise dimension. For the sake of notational convenience we adopt  $l$  as the unit of length, so that  $h \ll 1$ , and seek a model for the shell valid to order  $h^3$ , the scaling associated with leading-order bending effects.

In Section 2 we set the stage for the work by summarizing the relevant aspects of the three-dimensional theory of nonlinear elasticity. This forms the parent theory for the ensuing dimension-reduction procedure. This is followed, in Section 3, by an account of the relevant differential geometry and associated kinematical formulae [6]. The basic expansion procedure used to derive the two-dimensional shell energy - based on Leibniz' Rule combined with Taylor expansions - is described in Section 4. Following a brief discussion of reflection symmetry in Section 5, in Section 6 we use a virtual-power statement to derive restrictions on various through-thickness derivatives of the deformations of the laminae. Solutions to these restrictions in terms of the deformation of the interface lead to a final expression for the energy in terms of the interfacial deformation field alone. We close, in Section 7, with a detailed discussion of the relevant differential equations of equilibrium and attendant boundary conditions.

## 2. THREE-DIMENSIONAL PARENT THEORY

The Piola stress of three-dimensional elasticity theory is given by

$$\tilde{P} = \tilde{\Psi}_{\tilde{F}}, \tag{1}$$

the derivative with respect to the deformation gradient  $\tilde{F}$  of the strain energy,  $\tilde{\Psi}(\tilde{F}; \tilde{x})$ , per unit reference volume. Here  $\tilde{x}$  is the position of a material point in a reference configuration  $\kappa$ . We assume that this configuration could, in principle, be occupied by the body and thus impose the restriction  $\det \tilde{F} > 0$ .

We seek the optimal expression for the term  $E$  in the expansion

$$\mathcal{E} = E + o(h^3), \tag{2}$$

of the strain energy

$$\mathcal{E} = \int_{\kappa} \tilde{\Psi} dv. \tag{3}$$

Galilean invariance imposes the restriction  $\tilde{\Psi}(\tilde{F}; \tilde{x}) = \tilde{\Psi}(\tilde{Q}\tilde{F}; \tilde{x})$  for every rotation  $\tilde{Q}$ . As is well known, this implies that  $\tilde{\Psi}$  depends on  $\tilde{F}$  through the strain

$$\tilde{E} = \frac{1}{2}(\tilde{F}^t \tilde{F} - I), \tag{4}$$

where  $I$  is the (referential) identity for 3-space. Accordingly,

$$\tilde{\Psi}(\tilde{F}; \tilde{x}) = \tilde{U}(\tilde{E}; \tilde{x}), \tag{5}$$

say, yielding

$$\tilde{P} = \tilde{F}\tilde{S}, \tag{6}$$

where

$$\tilde{S} = \tilde{U}_{\tilde{E}}, \tag{7}$$

is the (symmetric) 2nd Piola–Kirchhoff stress.

Application of the chain rule furnishes the useful connection [6]

$$\tilde{\Psi}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\tilde{\mathbf{F}}; \tilde{\mathbf{x}})[\mathbf{B}] = \mathbf{B}\tilde{\mathbf{S}} + \tilde{\mathbf{F}}\tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\tilde{\mathbf{E}}; \tilde{\mathbf{x}})[\tilde{\mathbf{F}}^t \mathbf{B}], \quad (8)$$

for any tensor  $\mathbf{B}$ , where  $\tilde{\Psi}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}$  enjoys major symmetry and  $\tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}$  possesses major symmetry and both minor symmetries.

We normalize the energy such that  $\tilde{\Psi}(\mathbf{I}; \tilde{\mathbf{x}}) = 0$ . Galilean invariance then implies that  $\tilde{\Psi}(\tilde{\mathbf{F}}; \tilde{\mathbf{x}}) = 0$  if  $\tilde{\mathbf{F}} \in Orth^+$ . We also assume the converse to be true, i.e.,  $\tilde{\Psi}(\tilde{\mathbf{F}}; \tilde{\mathbf{x}}) = 0$  if and only if  $\tilde{\mathbf{F}} \in Orth^+$ . We further suppose that  $\tilde{\mathbf{S}} = \mathbf{0}$  if and only if  $\tilde{\mathbf{E}} = \mathbf{0}$ , and that  $\tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\mathbf{0}; \tilde{\mathbf{x}})$  - the classical elasticity tensor - is positive definite in the sense that  $\mathbf{A} \cdot \tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\mathbf{0}; \tilde{\mathbf{x}})[\mathbf{A}] > 0$  for all non-zero symmetric  $\mathbf{A}$ . Then,

$$\tilde{\mathbf{S}} = \tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\mathbf{0}; \tilde{\mathbf{x}})[\tilde{\mathbf{E}}] + o(|\tilde{\mathbf{E}}|), \quad (9)$$

and it follows from (8) that

$$\tilde{\Psi}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\mathbf{I}; \tilde{\mathbf{x}})[\mathbf{B}] = \tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\mathbf{0}; \tilde{\mathbf{x}})[\mathbf{B}]. \quad (10)$$

To accommodate energy-minimizing states we restrict our considerations to deformations satisfying the strong-ellipticity condition [5]

$$\mathbf{a} \otimes \mathbf{b} \cdot \tilde{\Psi}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\tilde{\mathbf{F}}; \tilde{\mathbf{x}})[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \forall \quad \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}. \quad (11)$$

Our assumptions then yield the classical strong-ellipticity condition at zero strain

$$\mathbf{a} \otimes \mathbf{b} \cdot \tilde{\mathbf{U}}_{\tilde{\mathbf{E}}\tilde{\mathbf{E}}}(\mathbf{0}; \tilde{\mathbf{x}})[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \forall \quad \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}, \quad (12)$$

which play an essential role in this work.

### 3. GEOMETRIC AND KINEMATIC FORMULAE

We seek a model for the shell in which the primary field to be determined is the deformation of the surface  $\Omega$  forming the interface between the two laminae. This is facilitated by use of the standard *normal-coordinate* parametrization of 3-space in the vicinity of the interface [6, 7]. Thus,  $\tilde{\mathbf{x}} = \hat{\mathbf{x}}(\theta^\alpha, \zeta)$ , where

$$\hat{\mathbf{x}}(\theta^\alpha, \zeta) = \mathbf{x}(\theta^\alpha) + \zeta \mathbf{N}(\theta^\alpha), \quad (13)$$

in which  $\mathbf{x}(\theta^\alpha)$ ;  $\alpha = 1, 2$  is the parametrization of  $\Omega$  with unit-normal field  $\mathbf{N}(\theta^\alpha)$  and  $\zeta$  is the coordinate in the direction perpendicular to  $\Omega$ , the latter corresponding to  $\zeta = 0$ . The lateral surfaces of the thin three-dimensional body are assumed to correspond to constant values of  $\zeta$ . These are separated by the small distance  $h$ , the thickness of the shell. For the sake of simplicity we assume the thickness to be uniform.

The orientation of  $\Omega$  is induced by the assumed right-handedness of the coordinate system  $(\theta^\alpha, \zeta)$ ; thus,  $\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{N} > 0$ , where  $\mathbf{A}_\alpha = \mathbf{x}_{,\alpha} \equiv \partial \mathbf{x} / \partial \theta^\alpha$  span the tangent plane  $T_{\Omega(p)}$  to  $\Omega$  at the material point  $p$  with coordinates  $\theta^\alpha$ . The *curvature* tensor  $\mathbf{B}$  of  $\Omega$  is the symmetric linear map from  $T_{\Omega(p)}$  to itself defined by the Weingarten equation [8]

$$d\mathbf{N} = -\mathbf{B}d\mathbf{x}, \quad (14)$$

where  $d\mathbf{x} = \mathbf{A}_\alpha d\theta^\alpha$  and  $d\mathbf{N} = \mathbf{N}_{,\alpha} d\theta^\alpha$ . Accordingly,

$$d\tilde{\mathbf{x}} = d\mathbf{x} + \zeta d\mathbf{N} + \mathbf{N}d\zeta = \mathbf{G}(d\mathbf{x} + \mathbf{N}d\zeta), \quad (15)$$

where

$$\mathbf{G} = \boldsymbol{\mu} + \mathbf{N} \otimes \mathbf{N}, \quad \boldsymbol{\mu} = \mathbf{1} - \zeta \mathbf{B}, \quad (16)$$

and

$$\mathbf{1} = \mathbf{I} - \mathbf{N} \otimes \mathbf{N} = \mathbf{A}_\alpha \otimes \mathbf{A}^\alpha, \quad (17)$$

is the projection onto - and the identity map for -  $T_{\Omega(p)}$ , on which the basis  $\{\mathbf{A}^\alpha\}$  is reciprocal to  $\{\mathbf{A}_\alpha\}$ .

We require the volume measure induced by the coordinates. This is [6, 7]  $dv = \mu d\zeta da$ , where  $da$  is the area measure on  $\Omega$ , and

$$\mu = 1 - 2H\zeta + \zeta^2 K, \quad (18)$$

is the (two-dimensional) determinant of  $\mu$  in which

$$H = \frac{1}{2}tr\mathbf{B} \quad \text{and} \quad K = \det \mathbf{B}, \tag{19}$$

are the mean and Gaussian curvatures of  $\Omega$ , respectively. The transformation from  $(\theta^\alpha, \zeta)$  to  $\tilde{\mathbf{x}}$  is one-to-one and orientation preserving if and only if  $\mu > 0$ . Following conventional usage [7], we refer to the region of space in which this condition holds as *shell space*.

Our development requires expressions for the three-dimensional deformation gradient and its through-thickness derivatives. Let  $\tilde{\chi}(\tilde{\mathbf{x}})$  be the three-dimensional deformation with gradient  $\tilde{\mathbf{F}}(\tilde{\mathbf{x}})$ . We define

$$\hat{\chi}(\theta^\alpha, \zeta) = \tilde{\chi}(x(\theta^\alpha) + \zeta N(\theta^\alpha)) \quad \text{and} \quad \hat{\mathbf{F}}(\theta^\alpha, \zeta) = \tilde{\mathbf{F}}(x(\theta^\alpha) + \zeta N(\theta^\alpha)). \tag{20}$$

Then,

$$\hat{\mathbf{F}}(\mu dx + Nd\zeta) = d\hat{\chi} = \hat{\chi}_{,\alpha} d\theta^\alpha + \hat{\chi}' d\zeta = (\nabla \hat{\chi})dx + \hat{\chi}' d\zeta, \tag{21}$$

where  $d\theta^\alpha = A^\alpha \cdot dx$ ,  $\hat{\chi}_{,\alpha} = \partial \hat{\chi} / \partial \theta^\alpha$ ,  $\hat{\chi}' = \partial \hat{\chi} / \partial \zeta$  and

$$\nabla \hat{\chi} = \hat{\chi}_{,\alpha} \otimes A^\alpha, \tag{22}$$

is the surfacial deformation gradient. Thus,

$$\hat{\mathbf{F}}\mathbf{1}\mu = \nabla \hat{\chi} \quad \text{and} \quad \hat{\mathbf{F}}\mathbf{N} = \hat{\chi}', \tag{23}$$

in which the orthogonal decomposition  $\hat{\mathbf{F}} = \hat{\mathbf{F}}\mathbf{1} + \hat{\mathbf{F}}\mathbf{N} \otimes \mathbf{N}$  has been used. If the configuration  $\kappa$  is contained in shell space, as we assume, then  $\mu^{-1}$  exists and

$$\hat{\mathbf{F}} = (\nabla \hat{\chi})\mu^{-1} + \hat{\chi}' \otimes \mathbf{N}. \tag{24}$$

The interfacial value of  $\hat{\mathbf{F}}$ , and those of its through-thickness derivatives  $\hat{\mathbf{F}}'$  and  $\hat{\mathbf{F}}''$ , are needed in Section 4. These, and other variables defined on the interface, are identified by the absence of superposed carets. The required expressions are [6]

$$\mathbf{F} = \nabla \mathbf{r} + \mathbf{d} \otimes \mathbf{N}, \quad \mathbf{F}' = \nabla \mathbf{d} + (\nabla \mathbf{r})\mathbf{B} + \mathbf{g} \otimes \mathbf{N}, \tag{25}$$

and

$$\mathbf{F}'' = \nabla \mathbf{g} + 2(\nabla \mathbf{d})\mathbf{B} + 2(\nabla \mathbf{r})\mathbf{B}^2 + \mathbf{h} \otimes \mathbf{N}, \tag{26}$$

where

$$\mathbf{r} = \chi, \quad \mathbf{d} = \chi', \quad \mathbf{g} = \chi'' \quad \text{and} \quad \mathbf{h} = \chi''', \tag{27}$$

are mutually *independent* functions of  $\theta^\alpha$  and the operation  $\nabla(\cdot)$  is defined by (22). Together they furnish the coefficient vectors in the expansion

$$\hat{\chi} = \mathbf{r} + \zeta \mathbf{d} + \frac{1}{2} \zeta^2 \mathbf{g} + \frac{1}{6} \zeta^3 \mathbf{h} + \dots \tag{28}$$

of the three-dimensional deformation. Evidently  $\mathbf{r}(\theta^\alpha) = \hat{\chi}(x(\theta^\alpha))$  is the position of a material point on the deformed image  $\omega$  of the interface  $\Omega$ ; its gradient

$$\nabla \mathbf{r} = \mathbf{a}_\alpha \otimes A^\alpha, \quad \text{where} \quad \mathbf{a}_\alpha = \mathbf{r}_{,\alpha}, \tag{29}$$

maps  $T_{\Omega(p)}$  to the tangent plane  $T_{\omega(p)}$  to  $\omega$  at the material point  $p$ .

#### 4. EXPANSION OF THE THREE-DIMENSIONAL ENERGY

##### 4.1. Preliminaries

We are concerned with a two-ply laminate consisting of upper (+) and lower (-) laminae. Consider a function

$$F(\zeta) = \{F_+; \quad \zeta \in (0, \eta h) F_-; \quad \zeta \in (-(1 - \eta)h, 0)\}, \tag{30}$$

where  $\eta \in [0, 1]$  if fixed. Thus  $\eta h$  and  $(1 - \eta)h$  are the fractions of the total thickness apportioned to the upper and lower laminae, respectively. The integral of  $F$  over  $\kappa$  is then given by

$$\int_\kappa F dv = \int_\Omega I da, \quad \text{with} \quad I = \int_C G d\zeta, \quad G = \mu F \quad \text{and} \quad C = [-(1 - \eta)h, \eta h]. \tag{31}$$

We wish to estimate the integral  $I$  for small  $h$ . To this end we write

$$\begin{aligned} I &= I_+(h) + I_-(h), \quad \text{where} \\ I_+(h) &= \int_0^{\eta h} G_+(\zeta) d\zeta \quad \text{and} \quad I_-(h) = \int_{-(1-\eta)h}^0 G_-(\zeta) d\zeta, \quad \text{with} \quad G_{\pm} = \mu F_{\pm}. \end{aligned} \quad (32)$$

For example,

$$I_+(h) = I_+(0) + hI'_+(0) + \frac{1}{2}h^2I''_+(0) + \frac{1}{6}h^3I'''_+(0) + o(h^3), \quad (33)$$

where  $I_+(0) = 0$  and, by Leibniz' Rule,  $I'_+(h) = \eta G_+(\eta h)$ . Accordingly,  $I''_+(h) = \eta^2 G'_+(\eta h)$ ,  $I'''_+(h) = \eta^3 G''_+(\eta h)$  and

$$I_+(h) = \eta h(G_+)_{0} + \frac{1}{2}\eta^2 h^2(G'_+)_{0} + \frac{1}{6}\eta^3 h^3(G''_+)_{0} + o(h^3), \quad (34)$$

where the subscript  $_0$  refers to function values at  $\zeta = 0$ . In the same way we derive

$$I_-(h) = (1-\eta)h(G_-)_{0} - \frac{1}{2}(1-\eta)^2 h^2(G'_-)_{0} + \frac{1}{6}(1-\eta)^3 h^3(G''_-)_{0} + o(h^3), \quad (35)$$

and conclude that

$$\begin{aligned} I &= h[\eta(G_+)_{0} + (1-\eta)(G_-)_{0}] + \frac{1}{2}h^2[\eta^2(G'_+)_{0} - (1-\eta)^2(G'_-)_{0}] \\ &\quad + \frac{1}{6}h^3[\eta^3(G''_+)_{0} + (1-\eta)^3(G''_-)_{0}] + o(h^3), \end{aligned} \quad (36)$$

where, in view of (18) and (32)<sub>3</sub>,

$$(G_{\pm})_{0} = (F_{\pm})_{0}, \quad (G'_{\pm})_{0} = (F'_{\pm})_{0} - 2H(F_{\pm})_{0} \quad \text{and} \quad (G''_{\pm})_{0} = (F''_{\pm})_{0} - 4H(F'_{\pm})_{0} + 2K(F_{\pm})_{0}. \quad (37)$$

#### 4.2. Small-thickness estimate of the strain energy

The strain energy of the shell is

$$\mathcal{E} = \int_{\kappa} \tilde{\Psi} dv = \int_{\Omega} S da, \quad (38)$$

where

$$S = \int_{\mathcal{C}} \mu \tilde{\Psi}(\hat{\mathbf{F}}; \hat{\mathbf{x}}) d\zeta, \quad (39)$$

is the areal strain-energy density on  $\Omega$ . Here  $\tilde{\Psi}$  plays the role of  $F(\zeta)$  of the previous subsection, in which the functional dependence on the through-thickness coordinate  $\zeta$  arises implicitly, via  $\hat{\mathbf{F}}$ , or explicitly, via  $\hat{\mathbf{x}}$  (see (13)).

Henceforth we assume the materials of which both laminae are made to be such that the associated energy densities do not depend *explicitly* on  $\zeta$ , i.e.,  $\tilde{\Psi}(\hat{\mathbf{F}}; \hat{\mathbf{x}}) = \tilde{\Psi}(\hat{\mathbf{F}}; \mathbf{x})$ . Accordingly, for the two laminae we have

$$(F_{\pm})_{0} = \Psi_{\pm}(F_{\pm}; \mathbf{x}), \quad (F'_{\pm})_{0} = \mathbf{P}_{\pm}(F_{\pm}; \mathbf{x}) \cdot \mathbf{F}'_{\pm} \quad \text{and} \quad (F''_{\pm})_{0} = \mathbf{P}_{\pm}(F_{\pm}; \mathbf{x}) \cdot \mathbf{F}''_{\pm} + \mathbf{F}'_{\pm} \cdot \mathcal{M}_{\pm}(F_{\pm}; \mathbf{x})[\mathbf{F}'_{\pm}], \quad (40)$$

where, in each lamina,

$$\Psi(\mathbf{F}; \mathbf{x}) = \tilde{\Psi}_0, \quad \mathbf{P}(\mathbf{F}; \mathbf{x}) = \tilde{\mathbf{P}}_0 \quad \text{and} \quad \mathcal{M}(\mathbf{F}; \mathbf{x}) = (\tilde{\Psi}_{\hat{\mathbf{F}}\hat{\mathbf{F}}})_0, \quad (41)$$

with subscripts  $\pm$  appended as appropriate. Alternatively, we may use the strain energy expressed as a function of the strain to derive

$$(F_{\pm})_{0} = U_{\pm}(\mathbf{E}_{\pm}; \mathbf{x}), \quad (F'_{\pm})_{0} = \mathbf{S}_{\pm}(\mathbf{E}_{\pm}; \mathbf{x}) \cdot \mathbf{E}'_{\pm} \quad \text{and} \quad (F''_{\pm})_{0} = \mathbf{S}_{\pm}(\mathbf{E}_{\pm}; \mathbf{x}) \cdot \mathbf{E}''_{\pm} + \mathbf{E}'_{\pm} \cdot \mathcal{C}_{\pm}(\mathbf{E}_{\pm}; \mathbf{x})[\mathbf{E}'_{\pm}], \quad (42)$$

where

$$U(\mathbf{E}; \mathbf{x}) = \tilde{U}_0, \quad \mathbf{S}(\mathbf{E}; \mathbf{x}) = \tilde{\mathbf{S}}_0 \quad \text{and} \quad \mathcal{C}(\mathbf{E}; \mathbf{x}) = (\tilde{U}_{\hat{\mathbf{E}}\hat{\mathbf{E}}})_0, \quad (43)$$

and  $\mathbf{E}, \mathbf{E}'$  and  $\mathbf{E}''$  are evaluated at  $\zeta = 0$ .

We assume the laminae to be perfectly bonded, and hence to deform without sliding or delaminating. They therefore share a common interfacial deformation field  $\mathbf{r}(\mathbf{x})$ , implying (cf. (25), (26)) that

$$\mathbf{F}_{\pm} = \nabla \mathbf{r} + \mathbf{d}_{\pm} \otimes \mathbf{N}, \quad \mathbf{F}'_{\pm} = \nabla \mathbf{d}_{\pm} + (\nabla \mathbf{r})\mathbf{B} + \mathbf{g}_{\pm} \otimes \mathbf{N}, \quad (44)$$

and

$$\mathbf{F}''_{\pm} = \nabla \mathbf{g}_{\pm} + 2(\nabla d_{\pm})\mathbf{B} + 2(\nabla \mathbf{r})\mathbf{B}^2 + \mathbf{h}_{\pm} \otimes \mathbf{N}. \quad (45)$$

### 4.3. Small-strain approximation

Fiber-reinforced composites of interest in engineering applications typically withstand small elastic strains before yielding or fracturing. In view of the intended application to such materials we simplify our further development by assuming *a priori* that  $|\tilde{\mathbf{E}}|$  is small in  $\kappa$ . This implies that  $|\mathbf{E}|$  is small on  $\Omega$ , but of course this does *not* imply that  $\mathbf{E}'$  and  $\mathbf{E}''$  are small. We assume the latter to be bounded and estimate the various terms in (42) to leading order in  $|\mathbf{E}|$ . For example, with reference to (9), in each lamina we have, on suppressing the subscripts  $\pm$  and using the notation  $\simeq$  to identify expressions valid to leading order in  $|\mathbf{E}|$ ,

$$U(\mathbf{E}; \mathbf{x}) \simeq \frac{1}{2} \mathbf{E} \cdot \mathbf{C}^*(\mathbf{x})[\mathbf{E}] \quad \text{and} \quad \mathbf{S}(\mathbf{E}; \mathbf{x}) \simeq \mathbf{C}^*(\mathbf{x})[\mathbf{E}], \quad \text{where} \quad \mathbf{C}^*(\mathbf{x}) = \mathbf{C}(\mathbf{0}; \mathbf{x}). \quad (46)$$

Using (42) in (37) we then find

$$(G_{\pm})_0 \simeq \frac{1}{2} \mathbf{E}_{\pm} \cdot \mathbf{C}_{\pm}^*(\mathbf{x})[\mathbf{E}_{\pm}], \quad (G'_{\pm})_0 \simeq \mathbf{C}_{\pm}^*(\mathbf{x})[\mathbf{E}_{\pm}] \cdot \mathbf{E}'_{\pm} \quad \text{and} \quad (G''_{\pm})_0 \simeq \mathbf{E}'_{\pm} \cdot \mathbf{C}_{\pm}^*(\mathbf{x})[\mathbf{E}'_{\pm}]. \quad (47)$$

Accordingly, with the areal strain energy  $S$  playing the role of  $G$  in the expansion (36), we obtain the estimate

$$S = W + o(h^3), \quad (48)$$

where

$$\begin{aligned} W &\simeq \frac{1}{2} h \{ \eta \mathbf{E}_+ \cdot \mathbf{C}_+^*(\mathbf{x})[\mathbf{E}_+] + (1 - \eta) \mathbf{E}_- \cdot \mathbf{C}_-^*(\mathbf{x})[\mathbf{E}_-] \} \\ &\quad + \frac{1}{2} h^2 \{ \eta^2 \mathbf{C}_+^*(\mathbf{x})[\mathbf{E}_+] \cdot \mathbf{E}'_+ - (1 - \eta)^2 \mathbf{C}_-^*(\mathbf{x})[\mathbf{E}_-] \cdot \mathbf{E}'_- \} \\ &\quad + \frac{1}{6} h^3 \{ \eta^3 \mathbf{E}'_+ \cdot \mathbf{C}_+^*(\mathbf{x})[\mathbf{E}'_+] + (1 - \eta)^3 \mathbf{E}'_- \cdot \mathbf{C}_-^*(\mathbf{x})[\mathbf{E}'_-] \}, \end{aligned} \quad (49)$$

and conclude, with the aid of (2), (38) and (48), that the shell strain energy, accurate to order  $h^3$ , is given by

$$E = \int_{\Omega} W da. \quad (50)$$

In each lamina we have

$$\mathbf{E} = \frac{1}{2} \{ \mathbf{c} + (\nabla \mathbf{r})^t \mathbf{d} \otimes \mathbf{N} + \mathbf{N} \otimes (\nabla \mathbf{r})^t \mathbf{d} + (\mathbf{d} \cdot \mathbf{d}) \mathbf{N} \otimes \mathbf{N} - \mathbf{I} \}, \quad (51)$$

with  $\mathbf{d} = \mathbf{d}_{\pm}$ , as appropriate, where

$$\mathbf{c} = (\nabla \mathbf{r})^t \nabla \mathbf{r} = a_{\alpha\beta} \mathbf{A}^{\alpha} \otimes \mathbf{A}^{\beta}, \quad (52)$$

is the interfacial Cauchy–Green deformation tensor, and

$$a_{\alpha\beta} = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta}, \quad (53)$$

is the metric induced by the convected coordinates on the deformed interface.

It proves advantageous to introduce a two-vector  $\mathbf{e} \in T_{\Omega(p)}$  such that

$$\mathbf{d} = \lambda \mathbf{n} + (\nabla \mathbf{r}) \mathbf{e}, \quad (54)$$

where  $\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$  is the unit normal to  $T_{\omega(p)}$ ,  $\lambda = \mathbf{n} \cdot \mathbf{d}$  and  $(\nabla \mathbf{r}) \mathbf{e} = e^{\alpha} \mathbf{a}_{\alpha}$  with  $e^{\alpha} = \mathbf{A}^{\alpha} \cdot \mathbf{e}$ ; thus,  $e^{\alpha}$  are the contravariant components of  $\mathbf{d}$  on  $T_{\omega(p)}$ . With this we have

$$\mathbf{E} = \boldsymbol{\epsilon} + \text{Sym}(\boldsymbol{\gamma} \otimes \mathbf{N}) + \varepsilon \mathbf{N} \otimes \mathbf{N}, \quad (55)$$

where

$$\boldsymbol{\epsilon} = \frac{1}{2} (\mathbf{c} - \mathbf{1}), \quad (56)$$

is the interface strain,

$$\boldsymbol{\gamma} = \mathbf{c} \mathbf{e}, \quad (57)$$

is the transverse shear strain, and

$$\varepsilon = \frac{1}{2}(\lambda^2 + \mathbf{e} \cdot \mathbf{c}\mathbf{e} - 1), \quad (58)$$

is the normal strain.

The restriction  $\det F > 0$ , together with  $\det F = F^* N \cdot FN$ , where  $F^*$  is the cofactor of  $F$ , combined with (44)<sub>1</sub> and the Piola–Nanson formula  $F^* N = |F^* N| \mathbf{n}$  [9], implies that  $\lambda > 0$ .

Further, by differentiating (4) with respect to  $\zeta$  and evaluating the result at  $\zeta = 0$ , we obtain

$$\mathbf{E}' = \text{Sym}(\mathbf{F}' \mathbf{F}'). \quad (59)$$

Observing that  $F$  and the rotation factor  $R$  in its polar decomposition differ by terms of order  $|E|$ , we find that to the order of approximation consistent with the development leading to (49),

$$\mathbf{E}' \simeq \text{Sym}(\mathbf{R}' \mathbf{F}'). \quad (60)$$

In particular, the decomposition (cf. (17))

$$\mathbf{R} = \mathbf{R}\mathbf{1} + \mathbf{R}\mathbf{N} \otimes \mathbf{N}, \quad (61)$$

implies, with reference to (25)<sub>1</sub>, that

$$\mathbf{R}\mathbf{1} \simeq \nabla \mathbf{r} \quad \text{and} \quad \mathbf{R}\mathbf{N} \simeq \mathbf{d}, \quad (62)$$

and hence, because  $\nabla \mathbf{r}$  maps  $T_{\Omega(p)}$  to  $T_{\omega(p)}$ , that

$$\mathbf{d} \simeq \mathbf{n}, \quad (63)$$

the unit normal to  $T_{\omega(p)}$ . This, however, does *not* imply that  $\nabla \mathbf{d} \simeq \nabla \mathbf{n}$ .

## 5. REFLECTION SYMMETRY

In most applications the laminate is constructed in such a way that the material properties of the individual laminae exhibit reflection symmetry with respect to the tangent plane  $T_{\Omega(p)}$  at the material point  $p$ . That is, the strain-energy functions  $U_{\pm}$  of the laminae satisfy

$$U(\mathbf{E}; \mathbf{x}) = U(\mathbf{Q}^t \mathbf{E} \mathbf{Q}; \mathbf{x}), \quad (64)$$

where

$$\mathbf{Q} = \mathbf{I} - 2\mathbf{N} \otimes \mathbf{N}, \quad (65)$$

is the reflection through  $T_{\Omega(p)}$ . We obtain

$$\mathbf{Q}^t \mathbf{E} \mathbf{Q} = \boldsymbol{\varepsilon} - \text{Sym}(\boldsymbol{\gamma} \otimes \mathbf{N}) + \varepsilon \mathbf{N} \otimes \mathbf{N}, \quad (66)$$

and conclude, with (55) and (57), that the strain energy is then an even function of  $\mathbf{e}$ .

Let  $\Gamma(e_{\alpha}) = U(\mathbf{E}(e_{\alpha}))$ , in which all variables other than  $\mathbf{e}$  are fixed and  $e_{\alpha} = \mathbf{A}_{\alpha} \cdot \mathbf{e}$ . Reflection symmetry implies that the function  $\Gamma(e_{\alpha})$  is even:  $\Gamma(e_{\alpha}) = \Gamma(-e_{\alpha})$ . Using superposed dots to denote variations, we derive

$$\begin{aligned} (\partial \Gamma / \partial e_{\alpha}) \dot{e}_{\alpha} &= \dot{U} \\ &= \mathbf{S} \cdot \dot{\mathbf{E}} = \dot{\mathbf{e}} \cdot \{ \mathbf{c}\mathbf{1}\mathbf{S}\mathbf{N} + (\mathbf{S} \cdot \mathbf{N} \otimes \mathbf{N}) \boldsymbol{\gamma} \}, \end{aligned} \quad (67)$$

yielding

$$\partial \Gamma / \partial e_{\alpha} = \mathbf{A}^{\alpha} \cdot \{ \mathbf{c}\mathbf{1}\mathbf{S}\mathbf{N} + (\mathbf{S} \cdot \mathbf{N} \otimes \mathbf{N}) \boldsymbol{\gamma} \}, \quad (68)$$

in which we have inserted the factor  $\mathbf{1}$  to account for the fact that  $c$  maps  $T_{\Omega(p)}$  to itself. Because  $\Gamma$  is even, these derivatives vanish at  $e_{\alpha} = 0$ , and hence at  $\boldsymbol{\gamma} = \mathbf{0}$ . Thus,  $\mathbf{A}^{\alpha} \cdot \mathbf{c}\mathbf{1}\mathbf{S}\mathbf{N} = 0$ , which reduce, at leading order in strain, to

$$0 = \mathbf{A}^{\alpha} \cdot \mathbf{S}\mathbf{N} = \mathbf{A}^{\alpha} \cdot (\mathbf{C}^*[\mathbf{E}])\mathbf{N}, \quad (69)$$

whenever the strain is of the form

$$\mathbf{E} = \boldsymbol{\varepsilon} + \frac{1}{2}(\lambda^2 - 1)\mathbf{N} \otimes \mathbf{N}. \quad (70)$$

Because (69) is an identity for strains within this restricted class, we have  $A^\alpha \cdot (C^*[\dot{E}])N = 0$ , i.e.,

$$\dot{\epsilon}_{\beta\lambda} A^\alpha \cdot (C^*[A^\beta \otimes A^\lambda])N + \dot{\lambda} A^\alpha \cdot (C^*[N \otimes N])N = 0, \quad (71)$$

where we have replaced  $\lambda$  by unity to secure an expression valid to leading order in strain. This holds for all  $\dot{\lambda}$  and all (symmetric)  $\dot{\epsilon}_{\beta\lambda}$ . Accordingly, identifying  $N$  with the '3-direction', we conclude that

$$C^{*\alpha 3\beta\lambda} = 0 \quad \text{and} \quad C^{*\alpha 333} = 0, \quad (72)$$

which, because of the major and minor symmetries of  $C^*$ , implies, as is well known [10], that all moduli with an odd number of indices equal to 3 vanish. Because these moduli are fixed, independent of strain, these restrictions apply in general to materials possessing reflection symmetry, with important consequences for the further development of the model.

## 6. REDUCTION

In this work we identify equilibria of the shell with states that satisfy the virtual-power equality

$$\dot{E} = P, \quad (73)$$

where the superposed dot is again used to denote a variational derivative and  $P$  is the virtual power of the loads acting on the shell, the form of which is made explicit below. Here  $E$  is a functional of the interfacial deformation  $r$  together with the lamina fields  $\lambda_\pm$ ,  $e_\pm$  and  $g_\pm$ . We compute the variations of the energy with respect to the latter fields in the present section. These yield restrictions which are used to effect a final reduction of the energy to a functional of the interfacial deformation.

### 6.1. Variation with respect to $\lambda_\pm$ and $e_\pm$

For fixed  $r$  and  $g_\pm$  we have

$$\dot{E} = \int_{\Omega} \dot{W} da, \quad (74)$$

where, from (49),

$$\dot{W} = h\{\eta C_+^*[E_+] \cdot \dot{E}_+ + (1 - \eta) C_-^*[E_-] \cdot \dot{E}_-\} + O(h^2), \quad (75)$$

with

$$\dot{E} = \text{Sym}(\dot{\gamma} \otimes N) + \dot{\epsilon} N \otimes N; \quad \dot{\gamma} = c\dot{e}, \quad \dot{\epsilon} = \dot{\lambda}\dot{\lambda} + \gamma \cdot \dot{e}, \quad (76)$$

in each lamina, in which the variations  $\dot{\lambda}$  and  $\dot{e}$  are independent and use has been made of the major symmetry of the moduli.

We assume the virtual power  $P$  to be insensitive to  $\dot{\lambda}$  and  $\dot{e}$ ; in other words, that there are no power-conjugate actions associated with these variations. Taking  $\dot{e} = \mathbf{0}$  first, we conclude, on dividing (75) by  $h$ , that  $\dot{\lambda}\dot{\lambda} C^*[E] \cdot N \otimes N + O(h)$  vanishes, and hence, on passing to the limit, that

$$N \cdot (C^*[E])N = 0, \quad (77)$$

at leading order. Taking this into account, in the same way we find that variation with respect to  $e$  yields

$$\mathbf{1}(C^*[E])N = \mathbf{0}, \quad (78)$$

again at leading order in  $h$ . Together these furnish the restriction

$$(C^*[E])N = \mathbf{0}, \quad (79)$$

on the equilibrium lamina strain  $E$ . This implies that that the laminae are in a state of plane stress at the interface. In the case of reflection symmetry, the restrictions (72) on the moduli may be used to reduce this to

$$(C^{*33\beta\lambda} \epsilon_{\beta\lambda} + C\epsilon)N + C^{*\alpha 3\beta 3} \gamma_\beta A_\alpha = \mathbf{0}, \quad \text{where} \quad C = C^{*3333}. \quad (80)$$

In view of the strong-ellipticity condition (12), the symmetric tensor  $\mathcal{A}$ , defined by

$$\mathcal{A}v = (C^*[v \otimes N])N, \quad (81)$$

satisfies  $v \cdot \mathcal{A}v > 0$  for all non-zero  $v$  and is accordingly positive definite. Reflection symmetry reduces this to

$$\mathcal{A} = C^{*\alpha 3\beta 3} A_\alpha \otimes A_\beta + CN \otimes N, \quad (82)$$

from which we conclude that the submatrix  $(C^{*\alpha 3\beta 3})$  is positive definite and that  $C > 0$ . Then, (80) yields the restrictions

$$\gamma = \mathbf{0}, \quad \boldsymbol{\epsilon} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}}(\boldsymbol{\epsilon}) = -C^{-1}C^{*33\beta\lambda}\boldsymbol{\epsilon}_{\beta\lambda}, \quad (83)$$

and hence the strain

$$\boldsymbol{E} = \boldsymbol{\epsilon} + \bar{\boldsymbol{\varepsilon}}(\boldsymbol{\epsilon})\boldsymbol{N} \otimes \boldsymbol{N}. \quad (84)$$

It follows from (54) and (58) that

$$\boldsymbol{d} = \lambda \boldsymbol{n}, \quad \text{where} \quad \frac{1}{2}(\lambda^2 - 1) = \bar{\boldsymbol{\varepsilon}}(\boldsymbol{\epsilon}), \quad (85)$$

corresponding to classical Kirchhoff–Love kinematics with thickness distension. In contrast to conventional treatments [7], however, where such kinematics are typically assumed at the outset, here we have *derived* this result as the leading-order thin-shell approximation for laminae exhibiting reflection symmetry.

After some algebra we then find, on making use of (72) and (84), that

$$\boldsymbol{E} \cdot \boldsymbol{C}^*(x)[\boldsymbol{E}] = \boldsymbol{\epsilon} \cdot \boldsymbol{D}(x)[\boldsymbol{\epsilon}], \quad (86)$$

with

$$\boldsymbol{D} = \mathcal{D}^{\alpha\beta\lambda\mu} \boldsymbol{A}_\alpha \otimes \boldsymbol{A}_\beta \otimes \boldsymbol{A}_\lambda \otimes \boldsymbol{A}_\mu, \quad (87)$$

where

$$\mathcal{D}^{\alpha\beta\lambda\mu} = C^{*\alpha\beta\lambda\mu} - C^{-1}C^{*33\alpha\beta}C^{*33\lambda\mu}, \quad (88)$$

are the plane-stress moduli, possessing major symmetry and both minor symmetries. We note that the positive-definiteness of  $\boldsymbol{D}$  on the space of symmetric 2-tensors follows, via (86), from that of  $\boldsymbol{C}^*$  on the space of symmetric 3-tensors.

## 6.2. Variation with respect to $\boldsymbol{g}_\pm$

Using (85)<sub>1</sub>, we reduce (25)<sub>2</sub> to

$$\boldsymbol{F}' \simeq \nabla \boldsymbol{n} + (\nabla \boldsymbol{r})\boldsymbol{B} + \boldsymbol{g} \otimes \boldsymbol{N} + \boldsymbol{n} \otimes \nabla \lambda, \quad (89)$$

where we have replaced  $\lambda$  by unity, again to secure an estimate valid to consistent order in strain. With (61)–(63) we then obtain

$$\boldsymbol{R}^t \boldsymbol{F}' \simeq (\nabla \boldsymbol{r})^t \nabla \boldsymbol{n} + \boldsymbol{N} \otimes (\nabla \boldsymbol{n})^t \boldsymbol{n} + \boldsymbol{c}\boldsymbol{B} + \boldsymbol{N} \otimes \boldsymbol{B}(\nabla \boldsymbol{r})^t \boldsymbol{n} + \boldsymbol{R}^t \boldsymbol{g} \otimes \boldsymbol{N} + (\nabla \boldsymbol{r})^t \boldsymbol{n} \otimes \nabla \lambda + \boldsymbol{N} \otimes \nabla \lambda. \quad (90)$$

This cumbersome expression is substantially simplified by noting that (cf. (14))

$$(\nabla \boldsymbol{n})dx = d\boldsymbol{n} = -\boldsymbol{b}d\boldsymbol{r} = -\boldsymbol{b}(\nabla \boldsymbol{r})dx, \quad (91)$$

where  $\boldsymbol{b} = b_{\alpha\beta} \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$  - a symmetric linear map from  $T_{\omega(p)}$  to itself - is the curvature tensor on the deformed surface  $\omega$ . Thus,  $\nabla \boldsymbol{n} = -\boldsymbol{b}(\nabla \boldsymbol{r})$  and

$$(\nabla \boldsymbol{r})^t \nabla \boldsymbol{n} = \boldsymbol{\kappa}, \quad (92)$$

where

$$\boldsymbol{\kappa} = -(\nabla \boldsymbol{r})^t \boldsymbol{b}(\nabla \boldsymbol{r}) = -b_{\alpha\beta} \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta, \quad (93)$$

is the symmetric pullback of the deformed curvature, mapping  $T_{\Omega(p)}$  to itself. Further,  $(\nabla \boldsymbol{n})^t \boldsymbol{n} = -(\nabla \boldsymbol{r})^t \boldsymbol{b}\boldsymbol{n}$  and  $(\nabla \boldsymbol{r})^t \boldsymbol{n} = \boldsymbol{A}^\alpha (\boldsymbol{a}_\alpha \cdot \boldsymbol{n})$ , which both vanish identically, whereas  $\boldsymbol{c} = 2\boldsymbol{\epsilon} + \mathbf{1} \simeq \mathbf{1}$ . We thus obtain

$$\boldsymbol{R}^t \boldsymbol{F}' \simeq -\boldsymbol{\rho} + \boldsymbol{R}^t \boldsymbol{g} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \nabla \lambda, \quad (94)$$

where

$$\boldsymbol{\rho} = -(\boldsymbol{\kappa} + \boldsymbol{B}) = \rho_{\alpha\beta} \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta, \quad \text{with} \quad \rho_{\alpha\beta} = b_{\alpha\beta} - B_{\alpha\beta}, \quad (95)$$

is the bending strain of the interface. Accordingly (cf. (60)),

$$\boldsymbol{E}'_\pm \simeq -\boldsymbol{\rho} + \text{Sym}(\boldsymbol{\mu}_\pm \otimes \boldsymbol{N}), \quad (96)$$

in the laminae, where

$$\boldsymbol{\mu}_\pm = \boldsymbol{R}^t \boldsymbol{g}_\pm + \nabla \lambda_\pm. \quad (97)$$

Before proceeding we use (96), finding that

$$\mathcal{C}^*[E] \cdot E' = -\mathcal{C}^*[E] \cdot \rho + (\mathcal{C}^*[E])N \cdot \mu, \quad (98)$$

in each lamina, in which the second term vanishes by (79). Eqs. (72) and (88) also give

$$\mathcal{C}^*[E] \cdot \rho = \rho_{\alpha\beta}(\mathcal{C}^{*\alpha\beta\lambda\mu}\epsilon_{\lambda\mu} + \mathcal{C}^{*\alpha\beta 33}\bar{\epsilon}) = \mathcal{D}^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}\epsilon_{\lambda\mu}, \quad (99)$$

where we have used (83). Thus,

$$\mathcal{C}^*[E] \cdot E' = -\rho \cdot \mathcal{D}[\epsilon] = -\epsilon \cdot \mathcal{D}[\rho]. \quad (100)$$

Accordingly, the order -  $h^2$  coupling terms in (49) do not involve  $g_{\pm}$ ; the latter therefore occur only in the order -  $h^3$  terms. Proceeding with the variation of the energy with respect to  $g$  in each lamina separately, and assuming there are no corresponding virtual power terms, with reference to (49) we derive

$$0 = (\mathcal{C}^*[E'])N \cdot \dot{\mu} = R(\mathcal{C}^*[E'])N \cdot \dot{g}, \quad (101)$$

in either lamina, at leading order in  $h$ . This furnishes the restriction

$$(\mathcal{C}^*[E'])N = 0, \quad (102)$$

on the interfacial value of  $E'$ . This in turn implies that  $S'N$  vanishes, and thus, with  $SN = 0$  (cf. (79)), that the Piola traction  $\tilde{P}N$  on the interface vanishes through order  $O(\zeta)$ .

Substituting (96), we reduce (102) to

$$\mathcal{A}\mu = (\mathcal{C}^*[\rho])N, \quad (103)$$

where (cf. (82))

$$\mathcal{A}\mu = \mathcal{C}^{*\alpha 3\beta 3}\mu_{\beta}A_{\alpha} + C\mu_3N, \quad (104)$$

and

$$(\mathcal{C}^*[\rho])N = \mathcal{C}^{*33\alpha\beta}\rho_{\alpha\beta}N. \quad (105)$$

Consequently,  $\mu_{\beta} = 0$ , i.e.,

$$1R^t g = -\nabla\lambda, \quad (106)$$

while  $\mu_3 = N \cdot R^t g$  reduces to

$$N \cdot R^t g = C^{-1}\mathcal{C}^{*33\alpha\beta}\rho_{\alpha\beta}. \quad (107)$$

We obtain

$$E' \simeq -\rho + (N \cdot R^t g)N \otimes N, \quad (108)$$

and conclude, as in the derivation of (86), that

$$E' \cdot \mathcal{C}^*[E'] = \rho \cdot \mathcal{D}[\rho], \quad (109)$$

in each lamina.

### 6.3. The energy as a functional of the interfacial deformation

We are finally in a position to express the strain energy as a functional of the interfacial deformation alone. Collecting (49), (86), (100) and (109), we have

$$\begin{aligned} W &= \frac{1}{2}h\epsilon \cdot \{\eta\mathcal{D}_+(x) + (1-\eta)\mathcal{D}_-(x)\}[\epsilon] \\ &\quad - \frac{1}{2}h^2\epsilon \cdot \{\eta^2\mathcal{D}_+(x) - (1-\eta)^2\mathcal{D}_-(x)\}[\rho] \\ &\quad + \frac{1}{6}h^3\rho \cdot \{\eta^3\mathcal{D}_+(x) + (1-\eta)^3\mathcal{D}_-(x)\}[\rho]. \end{aligned} \quad (110)$$

In the case of pure bending, defined by the condition  $\epsilon \equiv 0$  on  $\Omega$ , the various small-strain estimates adopted in the foregoing become exact, and (110) reduces to

$$W = \frac{1}{6}h^3\rho \cdot \{\eta^3\mathcal{D}_+(x) + (1-\eta)^3\mathcal{D}_-(x)\}[\rho]. \quad (111)$$

The two- and three-dimensional energies,  $E$  and  $\mathcal{E}$  respectively, are then related by

$$\mathcal{E}/h^3 = E/h^3 + o(h^3)/h^3, \quad (112)$$

and passage to the limit  $h \rightarrow 0$  yields  $E$  as the rigorous leading-order energy for small  $h$ .

However, because of Gauss' *Theorema Egregium* [8], the constraint on  $\epsilon$  limits the applicability of this model to deformations for which the Gaussian curvature of  $\omega$  coincides pointwise with that of  $\Omega$ , as in the bending of a plane to a cylinder, for example. Despite its status as a rigorous limit, the model for pure bending is therefore of limited utility and we do not consider it further.

In the case of laminae with equal moduli ( $\mathcal{D}_\pm = \mathcal{D}$ ) having an interface located at the midsurface ( $\eta = \frac{1}{2}$ ), (110) reduces to Koiter's decoupled energy [4]

$$W = \frac{1}{2}h\epsilon \cdot \mathcal{D}(x)[\epsilon] + \frac{1}{24}h^3\rho \cdot \mathcal{D}(x)[\rho], \quad (113)$$

which has been justified by careful asymptotic analysis of the three-dimensional theory for small  $h$  (see [3]; Sec. 4.1, and [6]).

## 7. EQUILIBRIUM

Our derivation of the equilibrium and boundary conditions proceeds, as is customary [4], from the virtual power statement (73). We use this statement to extract expressions for the actions acting on the shell that contribute to the virtual power  $P$ , i.e., that are compatible with the expressions (50) and (110) for the strain energy. Naturally we could proceed in an alternative fashion to specify these actions in the three-dimensional setting and derive the virtual power in the two-dimensional theory via asymptotic analysis, as we have done for the strain energy [6]. In practice, however, this procedure serves merely to provide interpretations of the actions occurring in the two-dimensional theory in terms of their three-dimensional antecedents, while playing no substantive role in the specification of two-dimensional boundary-value problems.

### 7.1. Variational derivatives and response functions

As the shell deforms the strain-energy function (110) evolves in response to variations  $\dot{a}_{\alpha\beta}$  and  $\dot{b}_{\alpha\beta}$  of the metric and curvature of the deforming surface  $\omega$ . This follows from the structure of (110) and the definition (56) (with (52)) of the surface strain  $\epsilon$ , and the definition (95) of the bending strain  $\rho$ . Thus we have (74), where  $\dot{W}$  is given by

$$\dot{W} = \frac{1}{2}N^{\alpha\beta}\dot{a}_{\alpha\beta} + M^{\alpha\beta}\dot{b}_{\alpha\beta}, \quad (114)$$

where, by abuse of notation,

$$N^{\alpha\beta} = \frac{\partial W}{\partial a_{\alpha\beta}} + \frac{\partial W}{\partial a_{\beta\alpha}}, \quad M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial W}{\partial b_{\alpha\beta}} + \frac{\partial W}{\partial b_{\beta\alpha}}\right), \quad (115)$$

in which we have exploited the symmetries  $\dot{a}_{\alpha\beta} = \dot{a}_{\beta\alpha}$  and  $\dot{b}_{\alpha\beta} = \dot{b}_{\beta\alpha}$  and the factor  $\frac{1}{2}$  is included in (114) rather than (115)<sub>1</sub> for the sake of later convenience.

Explicit expressions for the response functions  $N^{\alpha\beta}$  and  $M^{\alpha\beta}$  may be derived by computing  $\dot{W}$  from (110) and comparing the result with (114). Using

$$\dot{\epsilon} = \frac{1}{2}\dot{a}_{\alpha\beta}A^\alpha \otimes A^\beta \quad \text{and} \quad \dot{\rho} = -\dot{b}_{\alpha\beta}A^\alpha \otimes A^\beta, \quad (116)$$

together with the major symmetries of  $\mathcal{D}_\pm$ , we obtain

$$N^{\alpha\beta} = h\{\eta\mathcal{D}_+^{\alpha\beta\lambda\mu}(x) + (1-\eta)\mathcal{D}_-^{\alpha\beta\lambda\mu}(x)\}\epsilon_{\lambda\mu} - \frac{1}{2}h^2\{\eta^2\mathcal{D}_+^{\alpha\beta\lambda\mu}(x) - (1-\eta)^2\mathcal{D}_-^{\alpha\beta\lambda\mu}(x)\}\rho_{\lambda\mu}, \quad (117)$$

and

$$M^{\alpha\beta} = \frac{1}{2}h^2\{\eta^2\mathcal{D}_+^{\alpha\beta\lambda\mu}(x) - (1-\eta)^2\mathcal{D}_-^{\alpha\beta\lambda\mu}(x)\}\epsilon_{\lambda\mu} - \frac{1}{3}h^3\{\eta^3\mathcal{D}_+^{\alpha\beta\lambda\mu}(x) + (1-\eta)^3\mathcal{D}_-^{\alpha\beta\lambda\mu}(x)\}\rho_{\lambda\mu}. \quad (118)$$

We proceed to express (114) in terms of the variation  $\mathbf{u}(\mathbf{x}) = \dot{\mathbf{r}}$  of the interfacial position field. To this end we use

$$\dot{a}_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{u}_{,\beta} + \mathbf{a}_\beta \cdot \mathbf{u}_{,\alpha}, \quad (119)$$

together with

$$\mathbf{u}_{;\alpha\beta} = \mathbf{u}_{,\alpha\beta} - \Gamma_{\alpha\beta}^\lambda \mathbf{u}_{,\lambda}, \quad (120)$$

in which  $\Gamma_{\alpha\beta}^\lambda$  are the Christoffel symbols on the equilibrium surface  $\omega$  and subscripts preceded by semi-colons identify covariant derivatives. By varying the Gauss equations [8]

$$\mathbf{r}_{,\alpha\beta} = \Gamma_{\alpha\beta}^\lambda \mathbf{a}_\lambda + b_{\alpha\beta} \mathbf{n}, \quad (121)$$

and using  $\dot{\mathbf{a}}_\lambda = \mathbf{u}_{,\lambda}$ , we reduce (110) to

$$\mathbf{u}_{;\alpha\beta} = \dot{\Gamma}_{\alpha\beta}^\lambda \mathbf{a}_\lambda + b_{\alpha\beta} \dot{\mathbf{n}} + \dot{b}_{\alpha\beta} \mathbf{n}, \quad (122)$$

and conclude that

$$\dot{b}_{\alpha\beta} = \mathbf{n} \cdot \mathbf{u}_{;\alpha\beta}. \quad (123)$$

However, for our present purposes it is more appropriate to express  $\dot{b}_{\alpha\beta}$  in terms of  $\mathbf{u}_{,\alpha}$  and  $\mathbf{u}_{|\alpha\beta}$ , where

$$\mathbf{u}_{|\alpha\beta} = \mathbf{u}_{,\alpha\beta} - \bar{\Gamma}_{\alpha\beta}^\lambda \mathbf{u}_{,\lambda}, \quad (124)$$

is the referential covariant derivative, in which  $\bar{\Gamma}_{\alpha\beta}^\lambda$  are the Christoffel symbols on  $\Omega$ . To achieve this we solve for  $\mathbf{u}_{,\alpha\beta}$  and substitute into (120), obtaining

$$\mathbf{u}_{;\alpha\beta} = \mathbf{u}_{|\alpha\beta} - S_{\alpha\beta}^\mu \mathbf{u}_{,\mu}, \quad (125)$$

where

$$S_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\lambda - \bar{\Gamma}_{\alpha\beta}^\lambda. \quad (126)$$

Thus,

$$\dot{b}_{\alpha\beta} = \mathbf{n} \cdot (\mathbf{u}_{|\alpha\beta} - S_{\alpha\beta}^\lambda \mathbf{u}_{,\lambda}). \quad (127)$$

It then follows from (114) that

$$\dot{W} = N^\alpha \cdot \mathbf{u}_{,\alpha} + M^{\alpha\beta} \cdot \mathbf{u}_{|\alpha\beta}, \quad (128)$$

with

$$N^\alpha = N^{\alpha\beta} \mathbf{a}_\beta + N^\alpha \mathbf{n}, \quad M^{\alpha\beta} = M^{\alpha\beta} \mathbf{n}, \quad (129)$$

in which

$$N^\alpha = -M^{\beta\lambda} S_{\beta\lambda}^\alpha. \quad (130)$$

The non-standard terms  $N^\alpha$  are due to the use of covariant derivatives on  $\Omega$ , whereas standard shell theory is typically developed in terms of covariant derivatives on  $\omega$  [3,4].

## 7.2. Equilibrium equations and boundary conditions

Proceeding with the reduction of (73), we define

$$\varphi^\alpha = \mathbf{T}^\alpha \cdot \mathbf{u} + M^{\alpha\beta} \cdot \mathbf{u}_{,\beta}, \quad (131)$$

with

$$\mathbf{T}^\alpha = N^\alpha - M_{|\beta}^{\alpha\beta}, \quad (132)$$

where

$$M_{|\beta}^{\alpha\beta} = M_{|\beta}^{\alpha\beta} \mathbf{n} - M^{\alpha\lambda} b_{\lambda}^\beta \mathbf{a}_\beta, \quad (133)$$

in which use has been made of the Weingarten equation [8]

$$\mathbf{n}_{,\beta} = -b_{\beta}^\lambda \mathbf{a}_\lambda, \quad (134)$$

where  $b_{\beta}^\lambda = a^{\lambda\alpha} b_{\alpha\beta}$  are the mixed curvature components and  $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$  is the reciprocal metric. With this we have

$$\dot{W} = \varphi_{|\alpha}^\alpha - \mathbf{u} \cdot \mathbf{T}_{|\alpha}^\alpha, \quad (135)$$

and Stokes' theorem may then be used to reduce (74) to

$$\dot{E} = \int_{\partial\Omega} \varphi^\alpha \nu_\alpha ds - \int_{\Omega} \mathbf{u} \cdot \mathbf{T}_{|\alpha}^\alpha da, \quad (136)$$

wherein  $\nu = \nu_\alpha \mathbf{A}^\alpha$  is the rightward unit normal to  $\partial\Omega$ .

The virtual power statement (73) thus accommodates a distributed load  $\mathbf{g}$ , per unit area of  $\Omega$ , which contributes  $\int_{\Omega} \mathbf{g} \cdot \mathbf{u} da$  to the virtual power. It follows immediately from the Fundamental Lemma of the Calculus of Variations that the equilibrium equation, holding in  $\Omega$ , is

$$\mathbf{T}_{|\alpha}^\alpha + \mathbf{g} = \mathbf{0}. \quad (137)$$

Turning to the boundary terms, a standard integration-by-parts procedure may be used to recast the first integral in (136) as

$$\int_{\partial\Omega} \varphi^\alpha \nu_\alpha ds = \int_{\partial\Omega} \{(\mathbf{T}^\alpha \nu_\alpha - (\mathbf{M}^{\alpha\beta} \nu_\alpha \tau_\beta)') \cdot \mathbf{u} + \mathbf{M}^{\alpha\beta} \nu_\alpha \nu_\beta \cdot \mathbf{u}_\nu\} ds - \sum [\mathbf{M}^{\alpha\beta} \nu_\alpha \tau_\beta]_i \cdot \mathbf{u}_i, \quad (138)$$

where  $\boldsymbol{\tau} = \tau_\alpha \mathbf{A}^\alpha = \mathbf{N} \times \boldsymbol{\nu}$  is the unit tangent to  $\partial\Omega$ ,  $\mathbf{u}_\nu = \nu^\alpha \mathbf{u}_{,\alpha}$  is the normal derivative of  $\mathbf{u}$ ,  $(\cdot)' = d(\cdot)/ds$  and the square bracket refers to the forward jump as a corner of the boundary is traversed. Thus,  $[\cdot] = (\cdot)_+ - (\cdot)_-$ , where the subscripts  $\pm$  identify the limits as a corner located at arclength station  $s$  is approached through larger and smaller values of arclength, respectively. The sum accounts for the contributions from all corners. Here we admit piecewise smooth boundaries having a finite number of jumps in the unit tangent  $\boldsymbol{\tau}$ .

From (73) and (136), it follows that the virtual power assumes the form

$$P = \int_{\Omega} \mathbf{g} \cdot \mathbf{u} da + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{u} ds + \int_{\partial\Omega_m} \mathbf{m} \cdot \mathbf{u}_\nu ds + \sum_* \mathbf{f}_i \cdot \mathbf{u}_i, \quad (139)$$

and the Fundamental Lemma furnishes

$$\mathbf{t} = \mathbf{T}^\alpha \nu_\alpha - (\mathbf{Q}\mathbf{n})', \quad \mathbf{m} = \mathbf{M}\mathbf{n} \quad \text{and} \quad \mathbf{f}_i = -[\mathbf{Q}\mathbf{n}]_i, \quad (140)$$

with

$$\mathbf{Q} = \mathbf{M}^{\alpha\beta} \nu_\alpha \tau_\beta \quad \text{and} \quad \mathbf{M} = \mathbf{M}^{\alpha\beta} \nu_\alpha \tau_\beta, \quad (141)$$

where  $\mathbf{t}$ ,  $\mathbf{m}$  and  $\mathbf{f}_i$  are the edge traction, edge double force density [11] and the corner force at the  $i^{\text{th}}$  corner, respectively. Here,  $\partial\Omega_t$  and  $\partial\Omega_m$  respectively are parts of  $\partial\Omega$  where  $\mathbf{r}$  and  $\mathbf{r}_\nu$  are not assigned, and the starred sum includes only the corners where position is not assigned.

We suppose that  $\mathbf{r}$  and  $\mathbf{r}_\nu$  are assigned on  $\partial\Omega \setminus \partial\Omega_t$  and  $\partial\Omega \setminus \partial\Omega_m$ , respectively, and position is assigned at corners that are not included in the starred sum.

### 7.3. Overall equilibrium

To interpret the double force in familiar terms, we consider the case in which no kinematical data are assigned anywhere on  $\partial\Omega$ , so that  $\partial\Omega_t = \partial\Omega_m = \partial\Omega$  and rigid-body variations of the deformation are kinematically admissible. The variational derivative of such a deformation is expressible as  $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{a}$ , where  $\mathbf{a}$  and  $\boldsymbol{\omega}$  are arbitrary spatially uniform vectors. Because the strain-energy function is invariant under such variations, we have  $\dot{E} = 0$  and (73) reduces to  $P = 0$ , i.e.,

$$\mathbf{a} \cdot \left( \int_{\Omega} \mathbf{g} da + \int_{\partial\Omega} \mathbf{t} ds + \sum \mathbf{f}_i \right) + \boldsymbol{\omega} \cdot \left\{ \int_{\Omega} \mathbf{r} \times \mathbf{g} da + \int_{\partial\Omega} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds + \sum \mathbf{r}_i \times \mathbf{f}_i \right\} = 0, \quad (142)$$

where

$$\mathbf{c} = \mathbf{r}_\nu \times \mathbf{m}. \quad (143)$$

The arbitrariness of  $\mathbf{a}$  and  $\boldsymbol{\omega}$  then yield the overall force and moment balances

$$\int_{\Omega} \mathbf{g} da + \int_{\partial\Omega} \mathbf{t} ds + \sum \mathbf{f}_i = \mathbf{0} \quad \text{and} \quad \int_{\Omega} \mathbf{r} \times \mathbf{g} da + \int_{\partial\Omega} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds + \sum \mathbf{r}_i \times \mathbf{f}_i = \mathbf{0}, \quad (144)$$

and hence the interpretation of  $\mathbf{c}$  as the couple traction.

With a little effort we can show that

$$\mathbf{c} = M \nu^\alpha \mathbf{a}_\alpha \times \mathbf{n} = -JM \tau_\alpha \mathbf{a}^\alpha, \quad (145)$$

where  $J = \sqrt{a/A}$ , with  $a = \det(a_{\alpha\beta})$  and  $A = \det(A_{\alpha\beta})$ . Using  $d\theta^\alpha = \tau^\alpha ds = \tau^{*\alpha} ds^*$ , where  $s^*$  measures arclength on  $\partial\omega$  with unit tangent  $\boldsymbol{\tau}^* = \tau^{*\alpha} \mathbf{a}_\alpha$ , together with  $ds \simeq ds^*$  at leading order in strain, we infer that  $\tau_\alpha = A_{\alpha\beta} \tau^\beta \simeq a_{\alpha\beta} \tau^\beta \simeq a_{\alpha\beta} \tau^{*\beta} = \tau_\alpha^*$ , and hence that

$$c \simeq -M\boldsymbol{\tau}^*, \quad (146)$$

a pure bending couple on  $\partial\omega$ .

However, from the variational point of view it is not consistent to specify the couple traction in a boundary-value problem. Rather, it is the double force that may be specified, and the couple traction can then be computed from (143), if desired, once the deformation is known.

## 8. CONCLUSIONS

In keeping with recent developments in shell theory in the mathematical community [3], in this work we have developed an asymptotic procedure to derive a small-thickness estimate of the strain energy of a laminated two-ply shell. This furnishes an alternative to conventional derivations of shell theories based on thickness-wise integration of the three-dimensional equilibrium equations, which do not yield information about their asymptotic structure. The interfacial plane-stress condition and consequent restrictions on the through-thickness variation of the deformation were derived, in contrast to conventional treatments based on *a priori* hypotheses. The model thus obtained furnishes a natural extension of Koiter's nonlinear shell energy [4] when specialized to small strains of laminae possessing reflection symmetry of the material properties with respect to the interfacial surface. This was used in the setting of a virtual-power statement to obtain the relevant equilibrium equations and boundary conditions. Suggestions for further work include extensions of the present framework to multi-layer shells and the consideration of large lamina strains, of particular relevance in biomechanics.

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