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TAYLOR EXPANSION FOR MATRIX FUNCTIONS OF VECTOR VARIABLE USING THE KRONECKER PRODUCT

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Abstract. Taylor expansion is one of the many mathematical tools that is applied in Mechanics and Engineering. In this paper, using the partial derivative of a matrix with respect to a vector and the Kronecker product, the formulae of Taylor series of vector variable for scalar functions, vector functions and matrix functions are built and demonstrated. An example regarding the linearization of the differential equations of an elastic manipulator is presented using Taylor expansion.

Keywords: Taylor expansion, Kronecker product, the partial derivative of a matrix with respect to a vector, elastic manipulator, linearization.

1. INTRODUCTION

Taylor expansion is one of the many mathematical tools that is applied in Mechanics and Engineering [1–4]. Taylor series for multivariate scalar functions has been well documented in mathematics textbooks [5]. Recently, the partial derivatives with respect to a vector variable of vector functions and matrix functions using the Kronecker product have been studied [6,7]. This type of derivative has been used in dynamics of multi-body systems [8–11].

In the field of dynamics of many elastic objects, equations of motion have very complex forms. The simplification of these complex equations is essential. On the other hand, it is also desirable to get the solutions quickly and handily for the applications in optimum design, real-time control or optimal control. Therefore, in many cases the solutions of nonlinear partial differential equations are not desired directly; instead, appropriate techniques are used to convert them into more suitable forms which not only still adequately describe the important characteristics of the true systems but also are easier to deal with. One of these techniques is the Taylor expansion. However, applying common Taylor expansion formula for scalar functions of one or many scalar variables to the problems of multi-body systems where matrix functions of vector variables are widely used, one witnesses the inconvenience of cumbersome formulations.

In this paper, using the definition on partial derivative of a matrix with respect to a vector and Kronecker product [7–9], the formulae of Taylor expansion according to a vector **x** for scalar functions, vector functions and matrix functions will be built and demonstrated. An applied example regarding the linearization of the differential equations of an elastic manipulator will be presented.

2. SOME DEFINITIONS AND PROPERTIES: A REVIEW

2.1. Single variable Taylor series

Let $f(x)$ be an infinitely differentiable function in some open interval around $x = x_0$. Then the Taylor expansion of $f(x)$ at x_0 is [5]

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + O\left[(x - x_0)^{n+1}\right], \tag{1}$$

where $O\left[(x - x_0)^{n+1}\right]$ is the remainder.

2.2. The Kronecker product and the Kronecker exponentiation

Definition 1. Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}, \mathbf{B} = [b_{ij}] \in \mathbb{R}^{r\times s}$. Then, the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is defined as the matrix [6]

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{mr\times ns}. \tag{2}$$

Some properties of Kronecker products [6–10]

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}), \tag{3}$$

$$(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T, \tag{4}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}), \tag{5}$$

$$(\mathbf{E}_p \otimes \mathbf{x}_{n\times 1})\mathbf{A}_{p\times m}\mathbf{d}_{m\times 1} = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{x}), \tag{6}$$

$$\mathbf{d}_{p\times 1} \otimes \mathbf{x}_{n\times 1} = (\mathbf{d} \otimes \mathbf{E}_n)\mathbf{x}. \tag{7}$$

From Eqs. (6) and (7), we have

$$(\mathbf{E}_n \otimes \mathbf{a}_{m\times 1})\,\mathbf{b}_{n\times 1} = (\mathbf{b}_{n\times 1} \otimes \mathbf{E}_m)\,\mathbf{a}_{m\times 1}. \tag{8}$$

It is possible to prove

$$\mathbf{a}_{m\times 1}\mathbf{b}^T_{n\times 1} = (\mathbf{b}^T \otimes \mathbf{E}_m)(\mathbf{E}_n \otimes \mathbf{a}). \tag{9}$$

Indeed, using Eq. (5), we have

$$\begin{aligned}(\mathbf{b}^T \otimes \mathbf{E}_m)(\mathbf{E}_n \otimes \mathbf{a}) &= (\mathbf{b}^T\mathbf{E}_n) \otimes (\mathbf{E}_m\mathbf{a}) \\ &= \mathbf{b}^T \otimes \mathbf{a} = \begin{bmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_n \\ a_2b_1 & a_2b_2 & \dots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_mb_1 & a_mb_2 & \dots & a_mb_n \end{bmatrix} = \mathbf{a}\mathbf{b}^T. \end{aligned} \tag{10}$$

In the above formulae, $\mathbf{E}_m$ denotes the $m \times m$ identity matrix.

Definition 2. The k^{th}-Kronecker power of the matrix $\mathbf{A}$ (k is an integer larger than 1) is defined as follows

$$\mathbf{A}^{\otimes k} = \mathbf{A} \otimes (\mathbf{A}^{\otimes k-1}) = \underbrace{\mathbf{A} \otimes \mathbf{A} \otimes \ldots \otimes \mathbf{A}}_{k \text{ copies}}. \tag{11}$$

If $k = 1$, we have

$$\mathbf{A}^{\otimes 1} = \mathbf{A}. \tag{12}$$

2.3. The partial derivative of a matrix with respect to a vector

Let scalar $\alpha(\mathbf{x})$, vector $\mathbf{a}(\mathbf{x}) \in \mathbb{R}^m$ and matrix $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^{m\times p}$ be functions of vector variable $\mathbf{x} \in \mathbb{R}^n$.

Definition 3. The first order partial derivatives with respect to vector $\mathbf{x}$ of scalar $\alpha(\mathbf{x})$, vector $\mathbf{a}(\mathbf{x})$ and matrix $\mathbf{A}(\mathbf{x})$ are respectively defined by [7,8]

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} & \cdots & \frac{\partial \alpha}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n, \tag{13}$$

$$\frac{\partial \mathbf{a}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial a_1}{\partial \mathbf{x}} \\ \frac{\partial a_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial a_m}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \cdots & \frac{\partial a_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_m}{\partial x_1} & \frac{\partial a_m}{\partial x_2} & \cdots & \frac{\partial a_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times n}, \tag{14}$$

$$\frac{\partial \mathbf{A}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{a}_1}{\partial \mathbf{x}} & \cdots & \frac{\partial \mathbf{a}_p}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_{11}}{\partial x_1} & \cdots & \frac{\partial a_{11}}{\partial x_n} & & \frac{\partial a_{1p}}{\partial x_1} & \cdots & \frac{\partial a_{1p}}{\partial x_n} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \frac{\partial a_{m1}}{\partial x_1} & \cdots & \frac{\partial a_{m1}}{\partial x_n} & & \frac{\partial a_{mp}}{\partial x_1} & \cdots & \frac{\partial a_{mp}}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times np}. \tag{15}$$

Definition 4. The k^{th}-order partial derivatives with respect to vector $\mathbf{x}$ of scalar $\alpha(\mathbf{x})$, vector $\mathbf{a}(\mathbf{x})$ and matrix $\mathbf{A}(\mathbf{x})$ are respectively defined as follows ($k > 2$)

$$\frac{\partial^{(k)}\alpha}{\partial \mathbf{x}^{(k)}} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial^{(k-1)}\alpha}{\partial \mathbf{x}^{(k-1)}}\right) = \frac{\partial^{(k-1)}}{\partial \mathbf{x}^{(k-1)}}\left(\frac{\partial \alpha}{\partial \mathbf{x}}\right) \in \mathbb{R}^{n^k}, \tag{16}$$

$$\frac{\partial^{(k)}\mathbf{a}}{\partial\mathbf{x}^{(k)}} = \frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial^{(k-1)}\mathbf{a}}{\partial\mathbf{x}^{(k-1)}}\right) = \begin{bmatrix} \dfrac{\partial^k a_1}{\partial\mathbf{x}^k} \\ \dfrac{\partial^k a_2}{\partial\mathbf{x}^k} \\ \vdots \\ \dfrac{\partial^k a_m}{\partial\mathbf{x}^k} \end{bmatrix} \in \mathbb{R}^{m\times n^k}, \tag{17}$$

$$\frac{\partial^k\mathbf{A}}{\partial\mathbf{x}^k} = \frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial^{(k-1)}\mathbf{A}}{\partial\mathbf{x}^{(k-1)}}\right) = \begin{bmatrix} \overbrace{\dfrac{\partial^k a_{11}}{\partial\mathbf{x}^k}}^{1\times n^k} & \overbrace{\dfrac{\partial^k a_{12}}{\partial\mathbf{x}^k}}^{1\times n^k} & \dots & \overbrace{\dfrac{\partial^k a_{1p}}{\partial\mathbf{x}^k}}^{1\times n^k} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{}_{} \overbrace{\dfrac{\partial^k a_{m1}}{\partial\mathbf{x}^k}}^{1\times n^k} & \overbrace{\dfrac{\partial^k a_{m2}}{\partial\mathbf{x}^k}}^{1\times n^k} & \dots & \overbrace{\dfrac{\partial^k a_{mp}}{\partial\mathbf{x}^k}}^{1\times n^k} \end{bmatrix} \in \mathbb{R}^{m\times pn^k}. \tag{18}$$

Property 1. For the product of two matrices $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^{m\times p}$ and $\mathbf{B}(\mathbf{x}) \in \mathbb{R}^{p\times s}$, we have the following property [8]

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x})) = \frac{\partial\mathbf{A}}{\partial\mathbf{x}}(\mathbf{B}\otimes\mathbf{E}_n) + \mathbf{A}\frac{\partial\mathbf{B}}{\partial\mathbf{x}}. \tag{19}$$

Corollary. Using Eq. (19) for matrix $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^{m\times p}$ and matrix of constants $\mathbf{C} \in \mathbb{R}^{p\times s}$, we have

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{A}(\mathbf{x})\mathbf{C}) = \frac{\partial\mathbf{A}}{\partial\mathbf{x}}(\mathbf{C}\otimes\mathbf{E}_n). \tag{20}$$

Deriving the above expression with respect to the vector $\mathbf{x}$ successively, we get

$$\frac{\partial^k}{\partial\mathbf{x}^k}(\mathbf{A}(\mathbf{x})\mathbf{C}) = \frac{\partial^{k-1}}{\partial\mathbf{x}^{k-1}}\left(\frac{\partial\mathbf{A}}{\partial\mathbf{x}}(\mathbf{C}\otimes\mathbf{E}_n)\right) = \frac{\partial^{k-2}}{\partial\mathbf{x}^{k-2}}\left(\frac{\partial^2\mathbf{A}}{\partial\mathbf{x}^2}(\mathbf{C}\otimes\mathbf{E}_n^{\otimes 2})\right) = \frac{\partial^k\mathbf{A}}{\partial\mathbf{x}^k}(\mathbf{C}\otimes\mathbf{E}_n^{\otimes k}). \tag{21}$$

Property 2. Taking k^{th}-order derivative of the identity

$$\mathbf{a}(\mathbf{x}) = \sum_{i=1}^{m}\mathbf{e}_i a_i, \tag{22}$$

where $\mathbf{e}_i$ is the i^{th} column of the unit matrix with an appropriate size, one obtains

$$\frac{\partial^k}{\partial\mathbf{x}^k}\mathbf{a}(\mathbf{x}) = \sum_{i=1}^{m}\frac{\partial^k}{\partial\mathbf{x}^k}(\mathbf{e}_i a_i) = \sum_{i=1}^{m}\mathbf{e}_i\frac{\partial^k}{\partial\mathbf{x}^k}a_i. \tag{23}$$

Property 3. Taking k^{th}-order derivative of the identity

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{p}\mathbf{a}_i\mathbf{e}_i^T, \tag{24}$$

and noting (21) yield

$$\frac{\partial^k}{\partial \mathbf{x}^k}\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{p}\frac{\partial^k}{\partial \mathbf{x}^k}(\mathbf{a}_i\mathbf{e}_i^T) = \sum_{i=1}^{p}\frac{\partial^k}{\partial \mathbf{x}^k}\mathbf{a}_i(\mathbf{e}_i^T \otimes \mathbf{E}_n^{\otimes k}) = \sum_{i=1}^{p}\frac{\partial^k}{\partial \mathbf{x}^k}\mathbf{a}_i(\mathbf{e}_i^T \otimes \mathbf{E}_{n^k}). \tag{25}$$

3. TAYLOR EXPANSION FOR MATRIX FUNCTIONS OF VECTOR VARIABLE

3.1. Taylor series for scalar functions of vector variable

Let scalar $\alpha(\mathbf{x})$ be a function of vector variable $\mathbf{x} \in \mathbb{R}^n$, namely

$$\alpha = \alpha(\mathbf{x}) = \alpha(x_1, x_2, \ldots, x_n).$$

The Taylor expansion with respect to vector $\mathbf{x}$ for $\alpha(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{x}_0$ is defined as follows

$$\begin{aligned}
\alpha(\mathbf{x}) \approx \alpha(\mathbf{x}_0) &+ \frac{1}{1!}\sum_{i_1=1}^{n}\frac{\partial}{\partial x_{i_1}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0}) \\
&+ \frac{1}{2!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\frac{\partial^2}{\partial x_{i_1}\partial x_{i_2}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0})(x_{i_2} - x_{i_2 0}) \\
&+ \frac{1}{3!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\frac{\partial^3}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0})(x_{i_2} - x_{i_2 0})(x_{i_3} - x_{i_3 0}) + \ldots \\
&+ \frac{1}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\frac{\partial^k}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0})(x_{i_2} - x_{i_2 0})\ldots(x_{i_k} - x_{i_k 0}),
\end{aligned} \tag{26}$$

where

$$\mathbf{x}_0 = \begin{bmatrix} x_{10} & x_{20} & \ldots & x_{n0} \end{bmatrix}^T.$$

Lemma 1. *The following expression holds*

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\frac{\partial^k}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0})(x_{i_2} - x_{i_2 0})\ldots(x_{i_k} - x_{i_k 0}) = \frac{\partial^k}{\partial \mathbf{x}^k}\alpha(\mathbf{x}_0)\Delta^{\otimes k}, \tag{27}$$

where

$$\Delta = \mathbf{x} - \mathbf{x}_0. \tag{28}$$

Proof. It can easily be shown that Eq. (27) is true when $k = 1$

$$\begin{aligned}
\sum_{i_1=1}^{n}\frac{\partial}{\partial x_{i_1}}\alpha(\mathbf{x}_0)(x_{i_1} - x_{i_1 0}) &= \begin{bmatrix} \frac{\partial}{\partial x_1}\alpha(\mathbf{x}_0) & \frac{\partial}{\partial x_2}\alpha(\mathbf{x}_0) & \ldots & \frac{\partial}{\partial x_n}\alpha(\mathbf{x}_0) \end{bmatrix}\begin{bmatrix} (x_1 - x_{10}) \\ (x_2 - x_{20}) \\ \vdots \\ (x_n - x_{n0}) \end{bmatrix} \\
&= \frac{\partial}{\partial \mathbf{x}}\alpha(\mathbf{x}_0)\Delta^{\otimes 1}.
\end{aligned} \tag{29}$$

Assuming that Eq. (27) is correct with $k = j$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\dots\sum_{i_j=1}^{n}\frac{\partial^j}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_j}}\alpha\left(\mathbf{x}_0\right)\left(x_{i_1}-x_{i_10}\right)\left(x_{i_2}-x_{i_20}\right)\dots\left(x_{i_j}-x_{i_j0}\right)=\frac{\partial^j}{\partial\mathbf{x}^j}\alpha\left(\mathbf{x}_0\right)\Delta^{\otimes j},\quad(30)$$

we just need to prove that Eq. (27) is true with $k = j + 1$.

$$\begin{aligned}\sum_{i_1=1}^{n}\dots\sum_{i_j=1}^{n}\sum_{i_{j+1}=1}^{n}\frac{\partial^{j+1}}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_{j+1}}}\alpha\left(\mathbf{x}_0\right)\left(x_{i_1}-x_{i_10}\right)\left(x_{i_2}-x_{i_20}\right)\dots\left(x_{i_j}-x_{i_j0}\right)\left(x_{i_{j+1}}-x_{i_{j+1}0}\right)\\=\frac{\partial^{j+1}}{\partial\mathbf{x}^{j+1}}\alpha\left(\mathbf{x}_0\right)\Delta^{\otimes j+1}.\end{aligned}\quad(31)$$

Consider an integer value i_{j+1} such that $1 \le i_{j+1} \le n$. From (30) replacing α by $\dfrac{\partial\alpha}{\partial x_{i_{j+1}}}$, we have

$$\sum_{i_1=1}^{n}\dots\sum_{i_j=1}^{n}\frac{\partial^j}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_j}}\left(\frac{\partial\alpha}{\partial x_{i_{j+1}}}\right)\bigg|_{\mathbf{x}_0}\left(x_{i_1}-x_{i_10}\right)\left(x_{i_2}-x_{i_20}\right)\dots\left(x_{i_j}-x_{i_j0}\right)=\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_{i_{j+1}}}\right)\bigg|_{\mathbf{x}_0}\Delta^{\otimes j}.\quad(32)$$

Multiplying both sides of the above equation with $\left(x_{i_{j+1}}-x_{i_{j+1}0}\right)$, we get

$$\begin{aligned}\sum_{i_1=1}^{n}\dots\sum_{i_j=1}^{n}\frac{\partial^j}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_j}}\left(\frac{\partial\alpha}{\partial x_{i_{j+1}}}\right)\bigg|_{\mathbf{x}_0}\left(x_{i_1}-x_{i_10}\right)\left(x_{i_2}-x_{i_20}\right)\dots\left(x_{i_j}-x_{i_j0}\right)\left(x_{i_{j+1}}-x_{i_{j+1}0}\right)\\=\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_{i_{j+1}}}\right)\bigg|_{\mathbf{x}_0}\Delta^{\otimes j}\left(x_{i_{j+1}}-x_{i_{j+1}0}\right).\end{aligned}\quad(33)$$

Assigning values from 1 to n to i_{j+1} and adding all expressions (33), we have another representation of the left side of (31) as follows

$$\begin{aligned}&\sum_{i_{j+1}}^{n}\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_{i_{j+1}}}\right)\bigg|_{\mathbf{x}_0}\left(x_{i_{j+1}}-x_{i_{j+1}0}\right)\Delta^{\otimes j}\\&=\begin{bmatrix}\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_1}\right)\big|_{\mathbf{x}_0}\left(x_1-x_{10}\right) & \dots & \frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_n}\right)\big|_{\mathbf{x}_0}\left(x_n-x_{n0}\right)\end{bmatrix}\Delta^{\otimes j}\\&=\begin{bmatrix}\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_1}\right)\big|_{\mathbf{x}_0}\left(\left(x_1-x_{10}\right)\mathbf{E}_n\right) & \dots & \frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_n}\right)\big|_{\mathbf{x}_0}\left(\left(x_n-x_{n0}\right)\mathbf{E}_n\right)\end{bmatrix}\Delta^{\otimes j}\\&=\begin{bmatrix}\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_1}\right)\big|_{\mathbf{x}_0} & \dots & \frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial x_n}\right)\big|_{(\mathbf{x}_0)}\end{bmatrix}\begin{bmatrix}\left(x_1-x_{1\,0}\right)\mathbf{E}_n\\ \vdots\\ \left(x_n-x_{n\,0}\right)\mathbf{E}_n\end{bmatrix}\Delta^{\otimes j}\\&=\frac{\partial^j}{\partial\mathbf{x}^j}\left(\frac{\partial\alpha}{\partial\mathbf{x}}\right)\bigg|_{\mathbf{x}_0}(\Delta\otimes\mathbf{E}_n)\Delta^{\otimes j}.\end{aligned}\quad(34)$$

Using Eq. (7) and Eq. (16), Eq. (34) can be rewritten as

$$\sum_{i_{j+1}}^{n} \frac{\partial}{\partial x_{i_{j+1}}} \left(\frac{\partial^j \alpha}{\partial \mathbf{x}^j} \right) \Bigg|_{\mathbf{x}_0} \left(x_{i_{j+1}} - x_{i_{j+1}0} \right) \Delta^{\otimes j} = \frac{\partial^{j+1}}{\partial \mathbf{x}^{j+1}} \alpha(\mathbf{x}_0) \Delta^{\otimes j+1}. \tag{35}$$

Thus, (31) holds and therefore (27) is true.
Substituting Eq. (27) into Eq. (26), we get a compact formula as follows

$$\alpha(\mathbf{x}) \approx \alpha(\mathbf{x}_0) + \sum_{i=1}^{k} \frac{1}{i!} \frac{\partial^i}{\partial \mathbf{x}^i} \alpha(\mathbf{x}_0) \Delta^{\otimes i} \tag{36}$$

□

3.2. Taylor series for vector functions of vector variable

Consider a vector function of vector $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} a_1(\mathbf{x}) & a_2(\mathbf{x}) & \ldots & a_m(\mathbf{x}) \end{bmatrix}^T = \sum_{i=1}^{m} \mathbf{e}_i a_i, \quad \mathbf{a}(\mathbf{x}) \in \mathbb{R}^m. \tag{37}$$

Using Eq. (36), we have the Taylor expansion with respect to vector $\mathbf{x}$ for scalar function $a_i(\mathbf{x})$in the neighborhood of $\mathbf{x} = \mathbf{x}_0$

$$a_i(\mathbf{x}) \approx a_i(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} a_i(\mathbf{x}_0) \Delta^{\otimes j}, i = \overline{1, m}, \tag{38}$$

which leads to

$$\begin{aligned} \mathbf{a}(\mathbf{x}) &\approx \sum_{i=1}^{m} \mathbf{e}_i \left(\alpha_i(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \alpha_i(\mathbf{x}_0) \Delta^{\otimes j} \right) \\ &= \sum_{i=1}^{m} \mathbf{e}_i \alpha_i(\mathbf{x}_0) + \sum_{i=1}^{m} \mathbf{e}_i \left(\sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \alpha_i(\mathbf{x}_0) \Delta^{\otimes j} \right) \\ &= \mathbf{a}(\mathbf{x}_0) + \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{1}{j!} \mathbf{e}_i \frac{\partial^j}{\partial \mathbf{x}^j} \alpha_i(\mathbf{x}_0) \Delta^{\otimes j} \\ &= \mathbf{a}(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \left(\sum_{i=1}^{m} \mathbf{e}_i \frac{\partial^j}{\partial \mathbf{x}^j} \alpha_i(\mathbf{x}_0) \right) \Delta^{\otimes j}. \end{aligned} \tag{39}$$

Applying (23), we have

$$\sum_{i=1}^{m} \mathbf{e}_i \frac{\partial^j}{\partial \mathbf{x}^j} \alpha_i(\mathbf{x}_0) = \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}(\mathbf{x}_0), j = \overline{1, k}. \tag{40}$$

Substituting Eq. (40) into Eq. (39), we get

$$\mathbf{a}(\mathbf{x}) \approx \mathbf{a}(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}(\mathbf{x}_0) \Delta^{\otimes j}. \tag{41}$$

Eq. (41) is the Taylor series for vector function $\mathbf{a}(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{x}_0$.

3.3. Taylor series for matrix functions of vector variable

Consider a matrix function of vector $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} a_{11}(\mathbf{x}) & a_{12}(\mathbf{x}) & \dots & a_{1p}(\mathbf{x}) \\ a_{21}(\mathbf{x}) & a_{22}(\mathbf{x}) & \dots & a_{2p}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(\mathbf{x}) & a_{m2}(\mathbf{x}) & \dots & a_{mp}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1(\mathbf{x}) & \mathbf{a}_2(\mathbf{x}) & \dots & \mathbf{a}_p(\mathbf{x}) \end{bmatrix} = \sum_{i=1}^{p} \mathbf{a}_i \mathbf{e}_i^T. \quad (42)$$

Using Eq. (41), Taylor series for column vector $\mathbf{a}_i \in \mathbf{A}$ in a neighborhood of $\mathbf{x} = \mathbf{x}_0$ has the following form

$$\mathbf{a}_i(\mathbf{x}) \approx \mathbf{a}_i(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0)\, \Delta^{\otimes j}, i = \overline{1,p}. \quad (43)$$

Substituting Eq. (43) into Eq. (42), we have

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &\approx \sum_{i=1}^{p} \left(\mathbf{a}_i(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0)\, \Delta^{\otimes j} \right) \mathbf{e}_i^T \\ &= \sum_{i=1}^{p} \mathbf{a}_i(\mathbf{x}_0)\, \mathbf{e}_i^T + \sum_{i=1}^{p} \left(\sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0)\, \Delta^{\otimes j} \right) \mathbf{e}_i^T \\ &= \mathbf{A}(\mathbf{x}_0) + \sum_{j=1}^{k} \sum_{i=1}^{p} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0)\, \Delta^{\otimes j} \mathbf{e}_i^T. \end{aligned} \quad (44)$$

Applying Eq. (9), we have

$$\Delta^{\otimes j} \mathbf{e}_i^T = \left(\mathbf{e}_i^T \otimes \mathbf{E}_{n^j} \right) \left(\mathbf{E}_p \otimes \Delta^{\otimes j} \right), i = \overline{1,p}, j = \overline{1,k}. \quad (45)$$

Eq. (44) can be written in the following form

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &\approx \mathbf{A}(\mathbf{x}_0) + \sum_{j=1}^{k} \sum_{i=1}^{p} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0) \left(\mathbf{e}_i^T \otimes \mathbf{E}_{n^j} \right) \left(\mathbf{E}_p \otimes \Delta^{\otimes j} \right) \\ &= \mathbf{A}(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \left(\sum_{i=1}^{p} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{a}_i(\mathbf{x}_0) \left(\mathbf{e}_i^T \otimes \mathbf{E}_{n^j} \right) \right) \left(\mathbf{E}_p \otimes \Delta^{\otimes j} \right). \end{aligned} \quad (46)$$

Using Eq. (25), we have the Taylor expansion with respect to vector $\mathbf{x}$ for matrix function $\mathbf{A}(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{x}_0$

$$\mathbf{A}(\mathbf{x}) \approx \mathbf{A}(\mathbf{x}_0) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{x}^j} \mathbf{A}(\mathbf{x}_0) \left(\mathbf{E}_p \otimes \Delta^{\otimes j} \right). \quad (47)$$

3.4. Linearization of the matrix function of vector variables

If the quadratic or higher terms in the Taylor series (47) are negligibly small, we have the linearization formula

$$\mathbf{A}\left(\mathbf{x}\right) \approx \mathbf{A}\left(\mathbf{x}_0\right) + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0\right)\left(\mathbf{E}_p \otimes \Delta\right). \tag{48}$$

For matrix functions with two vector variables $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, we can apply (48) twice in succession as follows

$$\begin{aligned}\mathbf{A}\left(\mathbf{x},\mathbf{y}\right) &\approx \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}\right) + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right)\\ &\approx \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right) + \frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right) +\\ &\quad + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right) +\\ &\quad + \frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right).\end{aligned} \tag{49}$$

Note that the last term of (49) is a nonlinear term. The final linearization formula for a matrix function of two vector variables is

$$\begin{aligned}\mathbf{A}\left(\mathbf{x},\mathbf{y}\right) \approx \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right) &+ \frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right)\\ &+ \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right).\end{aligned} \tag{50}$$

A special case but very common in the dynamics of multi-body systems: we need to linearize the product of a matrix function and one of its vector variable

$$\begin{aligned}\mathbf{A}\left(\mathbf{x},\mathbf{y}\right)\mathbf{y} &\approx \Big[\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right) + \frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right) +\\ &\quad + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right)\Big]\left(\mathbf{y}_0 + \left(\mathbf{y}-\mathbf{y}_0\right)\right)\\ &\approx \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\mathbf{y}_0 + \frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right)\mathbf{y}_0 +\\ &\quad + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right)\mathbf{y}_0 + \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right) +\\ &\quad + \Big[\frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{y}-\mathbf{y}_0\right)\right) +\\ &\quad + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{E}_p \otimes \left(\mathbf{x}-\mathbf{x}_0\right)\right)\Big]\left(\mathbf{y}_0 + \left(\mathbf{y}-\mathbf{y}_0\right)\right).\end{aligned} \tag{51}$$

Ignoring the nonlinear terms and using Eq. (8), we have

$$\mathbf{A}\left(\mathbf{x},\mathbf{y}\right)\mathbf{y} \approx \mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\mathbf{y}_0 + \left(\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right) + \frac{\partial}{\partial \mathbf{y}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{y}_0 \otimes \mathbf{E}_n\right)\right)\left(\mathbf{y}-\mathbf{y}_0\right) + \\ + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(\mathbf{y}_0 \otimes \mathbf{E}_n\right)\left(\mathbf{x}-\mathbf{x}_0\right). \tag{52}$$

As a corollary, one can write

$$\mathbf{A}\left(\mathbf{x}\right)\mathbf{y} \approx \mathbf{A}\left(\mathbf{x}_0\right)\mathbf{y}_0 + \mathbf{A}\left(\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right) + \frac{\partial}{\partial \mathbf{x}}\mathbf{A}\left(\mathbf{x}_0\right)\left(\mathbf{y}_0 \otimes \mathbf{E}_n\right)\left(\mathbf{x}-\mathbf{x}_0\right). \tag{53}$$

4. LINEARIZATION OF THE MOTION EQUATIONS OF AN ELASTIC MANIPULATOR

In this section, we apply the Taylor expansion for matrix functions to linearize the motion equations of a flexible manipulator moving and vibrating only in a vertical plane as shown in Fig. 1. Flexible link OE is assumed to be long and slender enough for the Euler-Bernoulli beam theory to be applied. The stationary frame is denoted as Ox_0y_0. If the elastic vibration is ignored, the link move exactly the same as the moving frame denoted Oxy. The link has Young modulus E, second moment of area$I$, volumetric mass density ρ, length l, and cross-sectional area A. $\tau(t)$ is the driving torque.

Using Lagrange's equations of second kind and the Ritz-Galerkin method, we can establish the system of differential equations of motion of the system considering only the first mode shape, assuming that the other mode shapes are negligible.

In this example, the general coordinates of the manipulator are selected as follows

$$\mathbf{s} = \begin{bmatrix} q_a \\ w \end{bmatrix}, \tag{54}$$

where q_a is the rotation angle and w is the elastic displacement of the link.

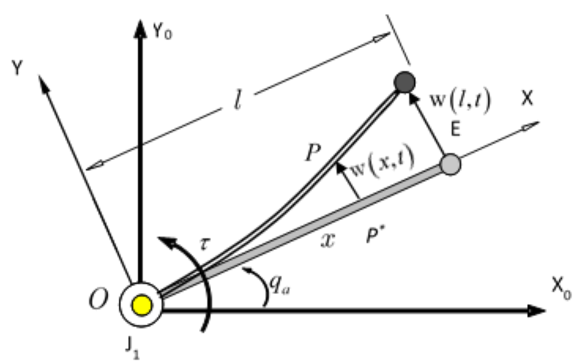

Fig. 1. A single–link flexible manipulator

The motion equations of the flexible manipulator have the following form [11]

$$\mathbf{M}\left(\mathbf{s}\right)\ddot{\mathbf{s}} + \mathbf{C}\left(\mathbf{s},\dot{\mathbf{s}}\right)\dot{\mathbf{s}} + \mathbf{g}\left(\mathbf{s}\right) = \tau\left(t\right). \tag{55}$$

Here we use the following notations

$$\mathbf{s}(t) = \mathbf{s}^{\mathbf{R}}(t) + \Delta\mathbf{s}(t) = \mathbf{s}^{\mathbf{R}}(t) + \mathbf{y}(t), \tag{56}$$

$$\dot{\mathbf{s}}(t) = \dot{\mathbf{s}}^{\mathbf{R}}(t) + \Delta\dot{\mathbf{s}}(t) = \dot{\mathbf{s}}^{\mathbf{R}}(t) + \dot{\mathbf{y}}(t), \tag{57}$$

$$\ddot{\mathbf{s}}(t) = \ddot{\mathbf{s}}^{\mathbf{R}}(t) + \Delta\ddot{\mathbf{s}}(t) = \ddot{\mathbf{s}}^{\mathbf{R}}(t) + \ddot{\mathbf{y}}(t), \tag{58}$$

where superscript **R** denotes the basic motion – the motion of the manipulator if the link is rigid

$$\mathbf{s}^{\mathbf{R}}(t) = \begin{bmatrix} q_a^{\mathbf{R}}(t) \\ q_e^{\mathbf{R}}(t) \end{bmatrix} = \begin{bmatrix} q_a^{\mathbf{R}}(t) \\ 0 \end{bmatrix}, \quad \dot{\mathbf{s}}^{\mathbf{R}}(t) = \begin{bmatrix} \dot{q}_a^{\mathbf{R}}(t) \\ 0 \end{bmatrix}, \quad \ddot{\mathbf{s}}^{\mathbf{R}}(t) = \begin{bmatrix} \ddot{q}_a^{\mathbf{R}}(t) \\ 0 \end{bmatrix}. \tag{59}$$

Applying (53) to the first term of the left-hand side of (55) results in

$$\mathbf{M}(\mathbf{s})\ddot{\mathbf{s}} \approx \mathbf{M}\left(\mathbf{s}^{\mathbf{R}}\right)\ddot{\mathbf{s}}^{\mathbf{R}} + \mathbf{M}\left(\mathbf{s}^{\mathbf{R}}\right)\ddot{\mathbf{y}} + \left.\frac{\partial \mathbf{M}}{\partial \mathbf{s}}\right|_{\mathbf{R}} (\ddot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E}_m)\mathbf{y}. \tag{60}$$

Using (52) for the second term of the left-hand side of (55) yields

$$\mathbf{C}(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}} \approx \mathbf{C}\left(\mathbf{s}^{\mathbf{R}},\dot{\mathbf{s}}^{\mathbf{R}}\right)\dot{\mathbf{s}}^{\mathbf{R}} + \left.\frac{\partial \mathbf{C}}{\partial \mathbf{s}}\right|_{\mathbf{R}} (\dot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E})\mathbf{y} + \left(\mathbf{C}\left(\mathbf{s}^{\mathbf{R}},\dot{\mathbf{s}}^{\mathbf{R}}\right) + \left.\frac{\partial \mathbf{C}}{\partial \dot{\mathbf{s}}}\right|_{\mathbf{R}} (\dot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E})\right)\dot{\mathbf{y}}. \tag{61}$$

Finally, noting (43), we have

$$\mathbf{g}(\mathbf{s}) \approx \mathbf{g}\left(\mathbf{s}^{\mathbf{R}}\right) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right|_{\mathbf{R}} \mathbf{y}. \tag{62}$$

Substituting Eqs. (60), (61) and (62) into Eq. (55) and ignoring the quadratic small quantities, we have

$$\mathbf{M}_{\mathbf{L}}(t)\ddot{\mathbf{y}} + \mathbf{D}_{\mathbf{L}}(t)\dot{\mathbf{y}} + \mathbf{K}_{\mathbf{L}}(t)\mathbf{y} = \mathbf{h}_{\mathbf{L}}(t), \tag{63}$$

where

$$\mathbf{h}_{\mathbf{L}}(t) = \tau(t) - \left[\mathbf{g}\left(\mathbf{s}^{\mathbf{R}}\right) + \mathbf{M}\left(\mathbf{s}^{\mathbf{R}}\right)\ddot{\mathbf{s}}^{\mathbf{R}} + \mathbf{C}\left(\mathbf{s}^{\mathbf{R}},\dot{\mathbf{s}}^{\mathbf{R}}\right)\dot{\mathbf{s}}^{\mathbf{R}}\right], \tag{64}$$

$$\mathbf{M}_{\mathbf{L}}(t) = \mathbf{M}\left(\mathbf{s}^{\mathbf{R}}\right), \tag{65}$$

$$\mathbf{D}_{\mathbf{L}}(t) = \mathbf{C}\left(\mathbf{s}^{\mathbf{R}},\dot{\mathbf{s}}^{\mathbf{R}}\right) + \left.\frac{\partial \mathbf{C}}{\partial \dot{\mathbf{s}}}\right|_{\mathbf{R}} (\dot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E}), \tag{66}$$

$$\mathbf{K}_{\mathbf{L}}(t) = \left.\frac{\partial \mathbf{M}}{\partial \mathbf{s}}\right|_{\mathbf{R}} (\ddot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E}_m) + \left.\frac{\partial \mathbf{C}}{\partial \mathbf{s}}\right|_{\mathbf{R}} (\dot{\mathbf{s}}^{\mathbf{R}} \otimes \mathbf{E}) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right|_{\mathbf{R}}, \tag{67}$$

or, more specifically,

$$\mathbf{M}_L(t) = \begin{bmatrix} J_1 + m_E l^2 + \frac{1}{3} m_{OE} l^2 & \rho A D_1 + m_E l X_1(l) \\ m_E l X_1 + \rho A D_1 & m_E X_1^2(l) + \rho A m_{11} \end{bmatrix}, \tag{68}$$

$$\mathbf{K}_L(t) = \begin{bmatrix} -m_E g l \sin q_a^{\mathbf{R}} - \dfrac{m_{OE} g l \sin q_a^{\mathbf{R}}}{2} & -m_E g X_1(l)\sin q_a^{\mathbf{R}} - \mu g \sin q_a^{\mathbf{R}} C_1 \\ -m_E g X_1(l)\sin q_a^{\mathbf{R}} - \mu g \sin q_a^{\mathbf{R}} C_1 & -m_E \left(\dot{q}_a^{\mathbf{R}}\right)^2 X_1^2(l) - \rho A \left(\dot{q}_a^{\mathbf{R}}\right)^2 m_{11} + E I k_{11} \end{bmatrix}, \tag{69}$$

$$\mathbf{D}_L(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{70}$$

$$\mathbf{h}_L\left(t\right)=\left[\begin{array}{c}\tau-\left(J_1+m_El^2+\dfrac{1}{3}m_{OE}l^2\right)\ddot{q}_a^{\mathbf{R}}-m_Egl\cos q_a^{\mathbf{R}}-\dfrac{m_{OE}gl\cos q_a^{\mathbf{R}}}{2}\\-\left(m_ElX_1+\rho AD_1\right)\ddot{q}_a^{\mathbf{R}}-m_EgX_1\left(l\right)\cos q_a^{\mathbf{R}}-\mu g\cos q_a^{\mathbf{R}}C_1\end{array}\right]. \quad (71)$$

The appearance of $q_a^{\mathbf{R}}$ in the above expressions implies that, if the link exhibits periodic basic motion, (63) is a linear system of ODE with periodic coefficients.

5. CONCLUSIONS

By using the Kronecker product of two matrices and the derivatives of matrices with respect to vector variable, the paper proposes the Taylor expansion formula for the matrix of vector variable in a general way. This formula is expected to ease the programming process for the equations establishment of many mechanical problems. In the applied example, the Taylor expansion was used to linearize the differential equations of motion of a flexible manipulator around the basic motion and led to a linear system of ODE with periodic coefficients. Compared with the known linearization method, the proposed method gives similar results with a shorter computer time.

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