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RESONANT AND ANTIRESONANT FREQUENCIES OF MULTIPLE CRACKED BAR

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Abstract. The natural frequencies or related resonant frequencies have been widely used for crack detection in structures by the vibration-based technique. However, antiresonant frequencies, the zeros of frequency response function, are less involved to use for the problem because they have not been thoroughly studied. The present paper addresses analysis of antiresonant frequencies of multiple cracked bar in comparison with the resonant ones. First, exact characteristic equations for the resonant and antiresonant frequencies of bar with arbitrary number of cracks are conducted in a new form that is explicitly expressed in term of crack severities. Then, the conducted equations are employed for analysis of variation of resonant and antiresonant frequencies versus crack position and depth. Numerical results show that antiresonant frequencies are indeed useful indicators for crack detection in bar mutually with the resonant ones.

Keywords: multi-cracked bar; longitudinal vibration; frequency equation; antiresonant frequency.

1. INTRODUCTION

Natural frequencies of a structure are an important dynamical characteristic that is usually computed by solving the so-called characteristic or frequency equation of the structure. Establishing the frequency equation for a structure gets to be crucial for both the analysis and identification of the structure. Adams et al. [1] are the first authors who established exact frequency equation for bar with single crack adopted by the spring model. Narkis [2] and Morassi [3] first obtained closed form solution in locating a crack using frequency equation of longitudinal vibration. More comprehensive study on both the forward and inverse problems in free vibration of multiple cracked bar was accomplished in References [4–10]. However, the study showed that unique solution of the crack detection cannot be found by using only natural frequencies. Some efforts have been made to solve the problem by encompassing other vibration characteristics such mode shapes [11–13] or frequency response function [14], but it was successful when antiresonant frequencies have been employed [15–17]. Nevertheless, using additionally the antiresonant frequencies for crack detection in bar enables to obtain unique solution of the crack detection problem only for free end bar. This may be caused from that the antiresonant frequencies of cracked bar with different boundary conditions have not been exhaustively investigated.

The present paper is devoted to study systematically variation of antiresonant frequencies of bar versus crack parameters mutually with resonant frequencies. First, there is derived a new form of characteristic equations for both resonant and antiresonant frequencies of multiple cracked bars. Then, the established equations are used for investigating change in the frequencies caused by presence of cracks. Numerical results have been examined to illustration of the proposed herein theory.

2. GENERAL FREQUENCY EQUATION FOR MULTIPLE CRACKED BAR

Let's consider longitudinal vibration in a bar that is described by the equation [14]

$$\Phi''(x) + \lambda^2\Phi(x) = 0,\ x \in (0,1),\ \lambda = \omega L\sqrt{\rho/E}, \tag{1}$$

under general boundary conditions

$$\alpha_0\Phi(0) + \beta_0\Phi'(0) = 0, \quad \alpha_1\Phi(1) + \beta_1\Phi'(1) = 0, \tag{2}$$

with the material, geometry and boundary constants $E, \rho, L, \alpha_0, \beta_0, \alpha_1, \beta_1$. Suppose that the bar is damaged to crack at arbitrary number n of positions e_j: $0 \le e_1 < ... < e_n \le 1$. For cracks modeled by transitional spring of stiffness K_j, conditions at the crack positions are [18]

$$\Phi'(e_j + 0) = \Phi'(e_j - 0),\ \Phi(e_j + 0) = \Phi(e_j - 0) + \gamma_j\Phi'(e_j), \tag{3}$$

$$\gamma_j = EA/LK_j = 2(1 - \nu^2)(h/L)\theta(a_j/h), j = 1, ..., n,$$

$$\theta(z) = 0.9852z^2 + 0.2381z^3 - 1.0368z^4 + 1.2055z^5 + 0.5803z^6 - 1.0368z^7 + 0.7314z^8. \tag{4}$$

It can be shown that any solution of equation (1) satisfying the first boundary condition in (2) at $x = 0$ and conditions (3) inside the bar is expressed in the form [14]

$$\Phi(x) = CL(x, \lambda), \tag{5}$$

where C is a constant and function

$$L(\lambda x) = L_0(\lambda x) + \sum_{k=1}^{n} \mu_k K(x - e_k), \tag{6}$$

$$K(x) = \begin{cases} 0 & \text{for } x < 0 \\ \cos\lambda x & \text{for } x \ge 0 \end{cases},\ K'(x) = \begin{cases} 0 & \text{for } x < 0 \\ -\lambda\sin\lambda x & \text{for } x \ge 0 \end{cases},$$

$$L_0(\lambda x) = (\alpha_0 \sin\lambda x - \lambda\beta_0\cos\lambda x),$$

$$\mu_j = \gamma_j\left[L'_0(\lambda e_j) - \lambda\sum_{k=1}^{j-1}\mu_k \sin\lambda(e_j - e_k)\right], j = 1, ..., n. \tag{7}$$

Substituting expression (5) into the second boundary condition in (2) at $x = 1$ yields

$$C[\alpha_1 L(1, \lambda) + \beta_1 L'(1, \lambda)] = 0,$$

that would have nontrivial solution with respect to constants C under the condition

$$D(\lambda) \equiv d_0(\lambda) + \sum_{j=1}^{n} H(1-e_j)\mu_j = 0, \tag{8}$$

where $d_0(\lambda) = \alpha_1 L_0(\lambda) + \beta_1 L'_0(\lambda); H(x) = \alpha_1 \cos\lambda x - \lambda\beta_1 \sin\lambda x$. The Eq. (8) is general form of frequency equation for multiple cracked bar that in combination with Eq. (7) enables to compute eigenvalues $\lambda_1, \lambda_2, \lambda_3, ...$ dependently on crack parameters. The obtained equation is implicit regarding crack magnitudes $\gamma_1, ..., \gamma_n$, so that solving that equation with respect to the eigenvalues or natural frequencies needs to compute the so-called damage parameters $\mu_1, ..., \mu_n$ defined by Eqs. (7). It would be much simplified in solution of both the forward and inverse problems for cracked bar if an explicit expression of the characteristic equation regarding the crack magnitudes $\gamma_1, ..., \gamma_n$ is available. Indeed, the recurrent relationships (7) can be rewritten as

$$\begin{aligned}
&\mu_1 = \gamma_1 L'_0(\lambda e_1),\\
&\mu_2 = \gamma_2 L'_0(\lambda e_2) - \lambda\gamma_1\gamma_2 L'_0(\lambda e_1)\sin\lambda(e_2-e_1),\\
&\mu_3 = \gamma_3 L'_0(\lambda e_3) - \lambda\gamma_1\gamma_3 L'_0(\lambda e_1)\sin\lambda(e_3-e_1) - \lambda\gamma_2\gamma_3 L'_0(\lambda e_2)\sin\lambda(e_3-e_2)\\
&\qquad\qquad + \lambda^2\gamma_1\gamma_2\gamma_3 L'_0(\lambda e_1)\sin\lambda(e_2-e_1)\sin\lambda(e_3-e_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Substituting latter expressions into (8) one obtains

$$\begin{aligned}
D(\lambda) \equiv\ & d_0(\lambda) + \sum_{j=1}^{n}\gamma_j d_1(\lambda, e_j) - \lambda\sum_{j=2}^{n}\sum_{k=1}^{j-1} d_2(\lambda, e_j, e_k)\gamma_j\gamma_k\\
& + \lambda^2\sum_{j=3}^{n}\sum_{k=2}^{j-1}\sum_{r=1}^{k-1} d_3(\lambda, e_j, e_k, e_r)\gamma_j\gamma_k\gamma_r + ... + (-\lambda)^{n-1} d_n(\lambda, e_n, ..., e_1)\gamma_1\gamma_2...\gamma_n\\
=\ & d_0(\lambda) + \sum_{k=1}^{n}\sum_{1\le i_1<i_2<...<i_k\le n}(-\lambda)^{k-1} d_k(\lambda, e_{i_k}, e_{i_{k-1}}, ..., e_{i_1})\gamma_{i_1}\gamma_{i_2}...\gamma_{i_k} = 0,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
d_0(\lambda) &= \alpha_1 L_0(\lambda) + \beta_1 L'_0(\lambda),\\
d_1(\lambda, e_j) &= H(1-e_j)L'_0(\lambda e_j),\\
d_2(\lambda, e_j, e_k) &= H(1-e_j)\sin\lambda(e_j-e_k)L'_0(\lambda e_k),\\
d_3(\lambda, e_j, e_k, e_r) &= H(1-e_j)\sin\lambda(e_j-e_k)\sin\lambda(e_k-e_r)L'_0(\lambda e_r),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
d_n(\lambda, e_n, ..., e_1) &= H(1-e_n)\sin\lambda(e_n-e_{n-1})\sin\lambda(e_{n-1}-e_{n-2})\sin\lambda(e_2-e_1)L'_0(\lambda e_1).
\end{aligned} \tag{10}$$

The obtained equation (9) is desired explicit form of the characteristic equation that provides an efficient tool for solving not only the forward but also the inverse problem of multiple cracked bar. Note, another form of the characteristic equation for multiple cracked rod was exactly obtained by Shifrin in [7], but that equation is implicit with respect to the crack magnitudes, likely, the equation given by Khiem et al. in [14].

In the case of intact bar, Eq. (9) becomes

$$d_0(\lambda) \equiv (\alpha_0\alpha_1 + \lambda^2\beta_0\beta_1)\sin\lambda - \lambda(\alpha_1\beta_0 - \alpha_0\beta_1)\cos\lambda = 0. \tag{11}$$

Next, for bar with single, double and triple cracks exact frequency equations get respectively the forms

$$D_1(\lambda) \equiv d_0(\lambda) + \gamma_1 d_1(\lambda, e_1) = 0, \tag{12}$$

$$D_2(\lambda) \equiv d_0(\lambda) + \gamma_1 d_1(\lambda, e_1) + \gamma_2 d_1(\lambda, e_2) - \lambda\gamma_1\gamma_2 d_2(\lambda, e_2, e_1) = 0, \tag{13}$$

$$\begin{aligned} D_3(\lambda) \equiv\; & d_0(\lambda) + \gamma_1 d_1(\lambda, e_1) + \gamma_2 d_1(\lambda, e_2) + \gamma_3 d_1(\lambda, e_3) - \lambda\gamma_1\gamma_2 d_2(\lambda, e_2, e_1), \\ & - \lambda\gamma_1\gamma_3 d_2(\lambda, e_3, e_1) - \lambda\gamma_2\gamma_3 d_2(\lambda, e_3, e_2) + \lambda^2\gamma_1\gamma_2\gamma_3 d_3(\lambda, e_3, e_2, e_1) = 0. \end{aligned} \tag{14}$$

Moreover, if the cracks are small so that asymptotic approximations of first, second and third order respectively for the frequency equation are

$$d_0(\lambda) + \sum_{j=1}^{n} \gamma_j d_1(e_j) = 0, \tag{15}$$

$$d_0(\lambda) + \sum_{j=1}^{n} \gamma_j d_1(e_j) - \lambda \sum_{j=2}^{n}\sum_{k=1}^{j-1} d_2(\lambda, e_j, e_k)\gamma_j\gamma_k = 0, \tag{16}$$

$$d_0(\lambda) + \sum_{j=1}^{n} \gamma_j d_1(e_j) - \lambda \sum_{j=2}^{n}\sum_{k=1}^{j-1} d_2(\lambda, e_j, e_k)\gamma_j\gamma_k + \lambda^2 \sum_{j=3}^{n}\sum_{k=2}^{j-1}\sum_{r=1}^{k-1} d_3(\lambda, e_j, e_k, e_r)\gamma_j\gamma_k\gamma_r = 0. \tag{17}$$

Finally, it has to note that general boundary conditions (2) include all the conventional end conditions and the elastic ones in dependence on the specific combinations of parameters $(\alpha_0, \beta_0, \alpha_1, \beta_1)$. Namely, for the case of free-free ends, (a) $\Phi'(0) = \Phi'(1) = 0$; fixed ends, (b) $\Phi(0) = \Phi(1) = 0$ and fixed-free ends, (c) $\Phi(0) = \Phi'(1) = 0$, the parameters get respectively

$$\text{(a) } \alpha_0 = \alpha_1 = 0, \beta_0 = \beta_1 = 1,\ \text{(b) } \alpha_0 = \alpha_1 = 1, \beta_0 = \beta_1 = 0,\ \text{(c) } \beta_0 = \alpha_1 = 0, \alpha_0 = \beta_1 = 1. \tag{18}$$

So that, for the listed above boundary conditions one has

(a) Free-Free ends: $L_0(x) = -\lambda\cos\lambda x, L'_0(x) = \lambda^2\sin\lambda x, H_1(x) = -\lambda\sin\lambda x$, and therefore

$$\begin{aligned} d_0(\lambda) &= \lambda^2\sin\lambda, \\ d_1(\lambda, e) &= -\lambda^3\sin\lambda e\sin\lambda(1-e), \\ d_2(\lambda, e_1, e_2) &= -\lambda^3\sin\lambda e_1\sin\lambda(e_2 - e_1)\sin\lambda(1 - e_2), \end{aligned} \tag{19}$$

(b) Fixed-Fixed ends: $L_0(x) = \sin\lambda x, L'_0(x) = \lambda\cos\lambda x, H_1(x) = \cos\lambda x$ and

$$\begin{aligned} d_0(\lambda) &= \sin\lambda, \\ d_1(\lambda, e) &= \cos\lambda(1-e)\sin\lambda e, \\ d_2(\lambda, e_1, e_2) &= \lambda\cos\lambda(1-e_2)\sin\lambda(e_2-e_1)\cos\lambda e_1. \end{aligned} \tag{20}$$

(c) Fixed-Free ends: $L_0(x) = \sin\lambda x, L'_0(x) = \lambda\cos\lambda x, H_1(x) = -\lambda\sin\lambda x$ and

$$\begin{aligned} d_0(\lambda) &= \lambda\cos\lambda x, \\ d_1(\lambda, e) &= -\lambda^2\sin\lambda(1-e)\cos\lambda e, \\ d_2(\lambda, e_1, e_2) &= -\lambda^2\sin\lambda(1-e_2)\sin\lambda(e_2-e_1)\cos\lambda e_1. \end{aligned} \tag{21}$$

If both the ends of bar are supported by translational springs of stiffness S_0, S_1, the parameters $(\alpha_0, \beta_0, \alpha_1, \beta_1)$ are defined as $\alpha_0 = \alpha_1 = 1, \beta_0 = -EA/S_0, \beta_1 = EA/S_1$, so that

$$L_0(x) = \sin\lambda x + \lambda\beta_0\cos\lambda x, L'_0(x) = \lambda\cos\lambda x - \lambda^2\beta_0\sin\lambda x, H_1(x) = \cos\lambda x - \lambda\beta_1\sin\lambda x,$$

and

$$\begin{aligned} d_0(\lambda) &= \lambda(\beta_0+\beta_1)\cos\lambda + (1-\lambda^2\beta_0\beta_1)\sin\lambda, \\ d_1(\lambda, e) &= \lambda[\cos\lambda(1-e) - \lambda\beta_1\sin\lambda(1-e)](\cos\lambda e - \lambda\beta_0\sin\lambda e), \\ d_2(\lambda, e_1, e_2) &= \lambda[\cos\lambda(1-e_2) - \lambda\beta_1\sin\lambda(1-e_2)](\cos\lambda e_1 - \lambda\beta_0\sin\lambda e_1)\sin\lambda(e_2-e_1). \end{aligned} \tag{22}$$

Moreover, in the case of small cracks first order asymptotic approximations of the frequency equation for the conventional (Free-Free; Fixed-Fixed; Fixed-Free) boundary conditions are

$$\sin\lambda - \lambda\sum_{j=1}^{n}\gamma_j\sin\lambda e_j\sin\lambda(1-e_j) = 0, \tag{23}$$

$$\sin\lambda + \lambda\sum_{j=1}^{n}\gamma_j\cos\lambda e_j\cos\lambda(1-e_j) = 0, \tag{24}$$

$$\cos\lambda - \lambda\sum_{j=1}^{n}\gamma_j\cos\lambda e_j\sin\lambda(1-e_j) = 0. \tag{25}$$

Also, assuming $\lambda = \lambda_0 + \Delta\lambda$ with λ_0 being the frequency parameter of intact bar and small $\Delta\lambda$, the latter equations yield all those obtained in earlier studies, for example, [8].

3. ANTIRESONANT FREQUENCY EQUATION FOR MULTIPLE CRACKED BARS

As well known in the vibration theory, resonant frequencies of a mechanical system are poles of the system's frequency Response Function (FRF) while antiresonant ones are zeros of the FRF. In case of systems without damping, the resonant frequencies are identical to natural frequencies determined as roots of the frequency equations. As zeros of FRF, antiresonant frequencies of a multiple cracked bar are seeking as follows. General expression of FRF for multiple cracked bars has been obtained by Khiem et al. [12,13] in the form

$$FRF(\omega, x_0, x) = \frac{L}{\lambda EF}\left\{\sin\lambda(x-x_0) - \frac{g(1-x_0)[\alpha_0\sin\lambda x - \beta_0\lambda\cos\lambda x + \sum_{j=1}^{n}\mu_j S(x-e_j)]}{d_0(\lambda) + \sum_{j=1}^{n}\mu_j[\alpha_1\cos\lambda(1-e_j) - \beta_1\lambda\sin\lambda(1-e_j)]}\right\}, \tag{26}$$

where $g(x) = \alpha_1\sin\lambda x + \beta_1\lambda\cos\lambda x$ and function $S(x) = \{0 \text{ if } x < 0; \cos\lambda x \text{ if } x \geq 0\}$. Letting $x = x_0 = 1$, that implies the FRF determined with both the input and output applied at the right end of bar, Eq. (26) is simplified to

$$FRF(\omega, 1, 1) = \frac{-\beta_1 L}{EF}\frac{[\alpha_0\sin\lambda - \beta_0\lambda\cos\lambda + \sum_{j=1}^{n}\mu_j\cos\lambda(1-e_j)]}{d_0(\lambda) + \sum_{j=1}^{n}\mu_j[\alpha_1\cos\lambda(1-e_j) - \beta_1\lambda\sin\lambda(1-e_j)]}. \tag{27}$$

Therefore, antiresonant frequencies can be sought by solving the equation

$$\alpha_0\sin\bar{\lambda} - \beta_0\bar{\lambda}\cos\bar{\lambda} + \sum_{j=1}^{n}\mu_j\cos\bar{\lambda}(1-e_j) = 0,$$

with respect to $\bar{\lambda}$ or

$$\bar{d}_0(\bar{\lambda}) + \sum_{j=1}^{n}\bar{H}(1-e_j)\mu_j = 0, \tag{28}$$

where $\bar{d}_0(\bar{\lambda}) = \alpha_0\sin\bar{\lambda} - \beta_0\bar{\lambda}\cos\bar{\lambda}$; $\bar{H}(x) = \cos\bar{\lambda}x$. Eq. (28) has the same form as Eq. (8) where the functions $d_0(\lambda), H(x)$ are replaced by $\bar{d}_0(\lambda), \bar{H}(x)$ and in both the equations the parameters μ_j are expressed by the same equations (7). Thus, equation for antiresonant frequencies (called herein antiresonant frequency equation) can be derived as

$$\bar{d}_0(\bar{\lambda}) + \sum_{k=1}^{n}\sum_{1\leq i_1<i_2<\ldots<i_k\leq n}(-\bar{\lambda})^{k-1}\bar{d}_k(\bar{\lambda}, e_{i_1}, e_{i_2}, \ldots, e_{i_k})\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k} = 0, \tag{29}$$

where

$$\begin{aligned}
\bar{d}_1(\lambda, e_j) &= \bar{H}(1-e_j)L'_0(\bar{\lambda}e_j),\\
\bar{d}_2(\bar{\lambda}, e_j, e_k) &= \bar{H}(1-e_j)\sin\bar{\lambda}(e_j-e_k)L'_0(\bar{\lambda}e_k),\\
\bar{d}_3(\lambda, e_j, e_k, e_r) &= \bar{H}(1-e_j)\sin\bar{\lambda}(e_j-e_k)\sin\bar{\lambda}(e_k-e_r)L'_0(\bar{\lambda}e_r),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\bar{d}_n(\bar{\lambda}, e_n, \ldots, e_1) &= \bar{H}(1-e_n)\sin\bar{\lambda}(e_n-e_{n-1})\sin\bar{\lambda}(e_{n-1}-e_{n-2})\sin\bar{\lambda}(e_2-e_1)L'_0(\bar{\lambda}e_1).
\end{aligned} \tag{30}$$

Similarly, the first, second and third order asymptotic approximations of the antiresonant frequency equation can be obtained respectively as

$$\bar{d}_0(\bar{\lambda}) + \sum_{j=1}^{n} \gamma_j \bar{d}_1(\bar{\lambda}, e_j) = 0, \tag{31}$$

$$\bar{d}_0(\bar{\lambda}) + \sum_{j=1}^{n} \gamma_j \bar{d}_1(\bar{\lambda}, e_j) - \bar{\lambda} \sum_{j=2}^{n} \sum_{k=1}^{j-1} \bar{d}_2(\bar{\lambda}, e_j, e_k) \gamma_j \gamma_k = 0, \tag{32}$$

$$\bar{d}_0(\bar{\lambda}) + \sum_{j=1}^{n} \gamma_j \bar{d}_1(\bar{\lambda}, e_j) - \bar{\lambda} \sum_{j=2}^{n} \sum_{k=1}^{j-1} \bar{d}_2(\bar{\lambda}, e_j, e_k) \gamma_j \gamma_k + \bar{\lambda}^2 \sum_{j=3}^{n} \sum_{k=2}^{j-1} \sum_{r=1}^{k-1} \bar{d}_3(\bar{\lambda}, e_j, e_k, e_r) \gamma_j \gamma_k \gamma_r = 0. \tag{33}$$

Since the FRF (26) is meaningless at the fixed ends of bar, the antiresonant frequency equations (29) and (31)–(33) are applied only for free end bar and cantilever bar. In the latter cases of boundary conditions, the first order approximate antiresonant frequency equation are

$$\cos \bar{\lambda} - \bar{\lambda} \sum_{j=1}^{n} \gamma_j \cos\bar{\lambda}(1 - e_j) \sin \bar{\lambda} e_j = 0, \tag{34}$$

$$\sin \bar{\lambda} + \bar{\lambda} \sum_{j=1}^{n} \gamma_j \cos \bar{\lambda}(1 - e_j) \cos \bar{\lambda} e_j = 0. \tag{35}$$

Eq. (35) shows that antiresonant frequencies of fixed-free bar are resonant frequencies of fixed end bar (see Eq. (24), so they are the same for symmetric cracks. Nevertheless, antiresonant frequencies of free-free end bar, likely resonant frequencies of cantilever bar, have not the symmetric effect. The latter fact has been employed by Rubio et al. [17] to obtain unique solution in localization of single and double crack in free-free end rod from given resonant and antiresonant frequencies. However, as shown below, the result cannot be extended for other cases of boundary conditions, even if other pair of resonant and antiresonant frequencies are used.

To validate the proposed theoretical development, antiresonant frequencies of the free end bar that was experimentally examined by the authors of Ref. [15] are computed and compared to the measured ones (see Tab. 1).

Obviously, calculated and measured antiresonant frequencies are excellently agreed (descrepancy between them is less than 1%. However, the descrepancy increases with severity of damage, esspecially, for higher frequencies. Note, deviation between calculated and measured first antiresonant frequency is of the same order 7% for both the cases of damage severity D_1 and D_2. This is perhaps caused by inaccuracy of the crack model used for representing the saw cut in the experimentation.

4. CRACK-INDUCED CHANGE IN RESONANT AND ANTIRESONANT FREQUENCIES (NUMERICAL RESULTS)

The problem of single crack detection in free end bar has been thoroughly studied by Morassi and his coworkers. However, it is necessary to note that the unique solution

Table 1. Antiresonant frequencies of intact and cracked bar compared to the measured ones

Mode No	Intact bar		Damage senario D_1		Damage senario D_2	
	Exp. [15]	Present (deviation, %)	Exp. [15]	Present (deviation, %)	Exp. [15]	Present (deviation, %)
1	468.6	470.6 (0.42)	439.5	470.3 (7.0)	432.9	465.3 (7.48)
2	1411.7	1411.7 (0)	1409.3	1406.4 (0.2)	1365.6	1301.7 (4.67)
3	2328.4	2352.8 (1.05)	2337.0	2339.6 (0.1)	2324.4	2132.9 (8.23)
4	3265.8	3294.0 (0.86)	-	3282.1 (-)	3102.5	3134.1 (1.01)
5	4216.6	4235.1 (0.43)	-	4232.9 (-)	3722.1	4200.8 (12.86)
6	5145.1	5176.3 (0.67)	-	5173.3 (-)	4866.6	5098.5 (4.76)

Damage scenarios D_1, D_2 correspond to different depth (6 and 15 mm) of crack at position e = 0.55/2.747

in locating single crack was attained in [17] because only first resonant and antiresonant frequencies have been used. The unique solution could not be obtained by using a pair of second or higher resonant and antiresonant frequencies. Obviously, the resonant and antiresonant frequencies used for obtaining unique solution have no critical point, crack occurred at which do not change the frequencies. Consequently, it can be expected that existence of the critical points for resonant and antiresonant frequencies destroys the uniqueness of solution in crack detection problem by using the frequencies. Therefore, knowing the critical points, that are called hereby nodes of resonant and antiresonant frequencies, is important in solving the crack detection problem.

The above equations show that nodes of resonant frequencies can be sought from equation $d_1(\lambda_0, x) = 0$, where λ_0 is solution of frequency equation in case of uncracked bar. Nodes of five lowest resonant frequencies are given in Tab. 2 for the cases of classical boundary conditions.

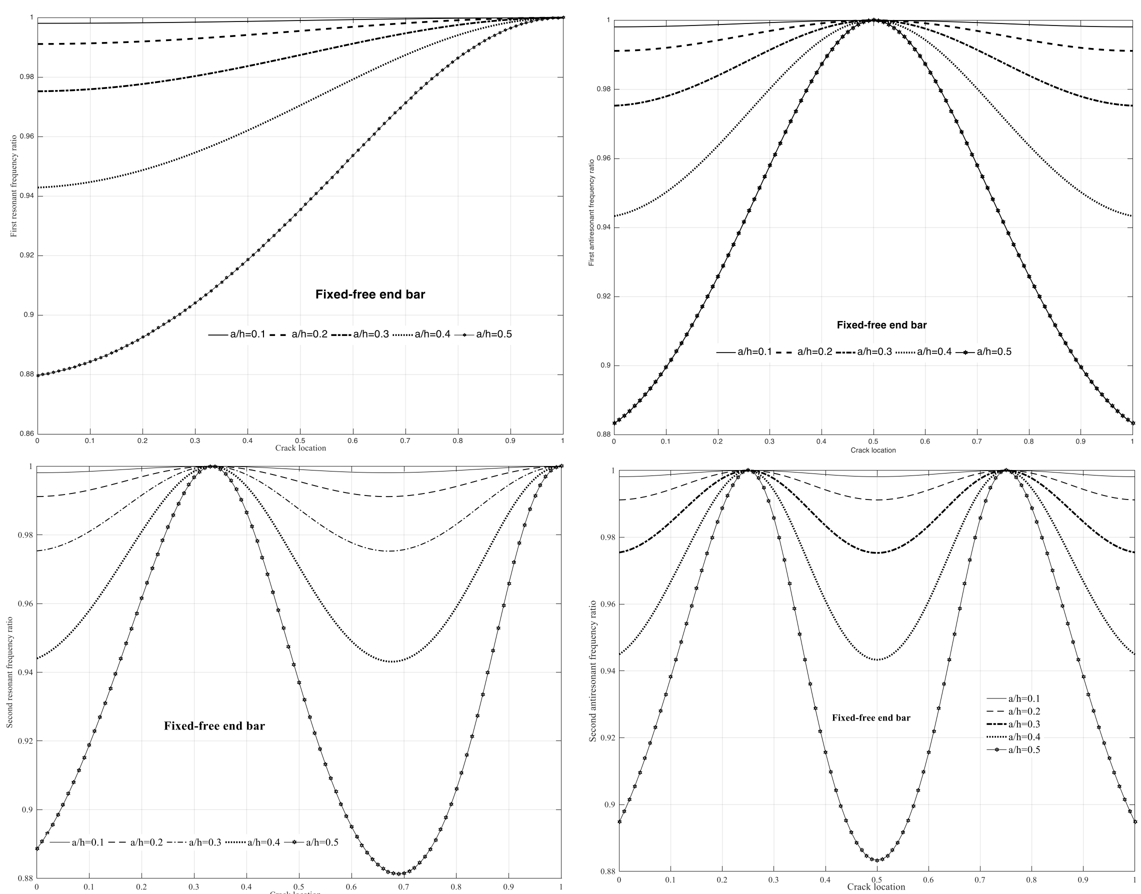
Table 2. Nodes of resonant frequencies for bar with classical boundary conditions

Mode No	Fixed end bar	Free-free end bar	Fixed-free end bar
1	1/2	not available	not available
2	1/4 3/4	0.5	1/3
3	1/6 1/2 5/6	1/3 2/3	0.2 0.6
4	1/8 3/8 5/8 7/8	0.25 0.5 0.75	1/7 3/7 5/7
5	1/10 3/10 01/2 7/10 9/10	0.2 0.4 0.6 0.8	1/9 3/9 5/9 7/9

For finding nodes of antiresonant frequencies of free-free bar and fixed-free bar one has the following equations $\cos\lambda_0(1-x)\sin\lambda_0 x = 0$ and $\cos\lambda_0(1-x)\cos\lambda_0 x = 0$, respectively. Solutions of the equations for five modes are given in Tab. 3. Evidently, node of resonant frequencies in fixed end bar exactly coincide with nodes of antiresonant frequencies in fixed-free end bar. All the calculated nodes of resonant and antiresonant frequencies of the fixed-free (Fig. 1) and free-free (Fig. 2) bars can be observed in Figs. 1–2 where there are shown ratios of the frequencies to those of intact bar. The ratios (resonant on the left and antiresonant – on the right) are plotted versus crack position (from 1 to 1) in different crack depth (10%–50%).

Table 3. Nodes of antiresonant frequencies for bar with free ends and fixed-free ends

Mode No	Fixed-free end bar	Free-free bar
1	1/2	not available
2	1/4 3/4	2/3
3	1/6 1/2 5/6	2/5 4/5
4	1/8 3/8 5/8 7/8	2/7 4/7 6/7
5	1/10 3/10 1/2 7/10 8/10	2/9 4/9 6/9 8/9



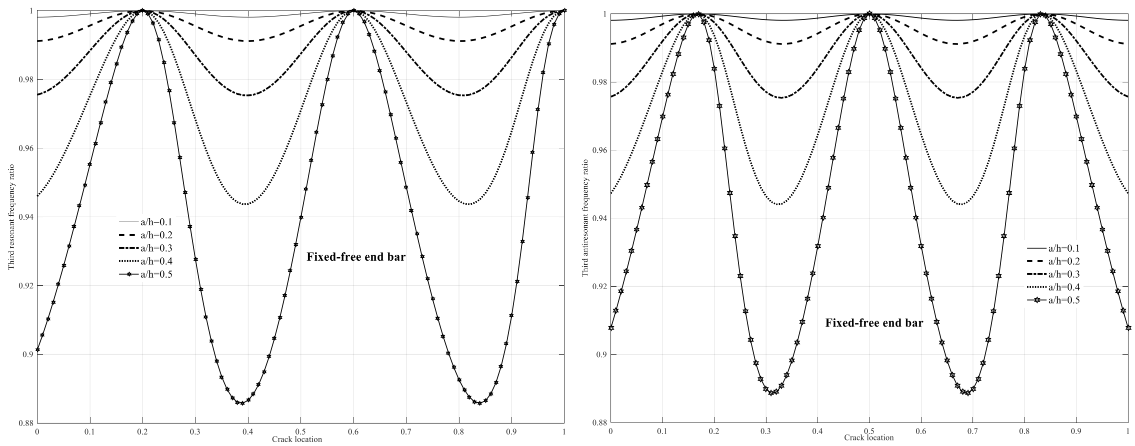
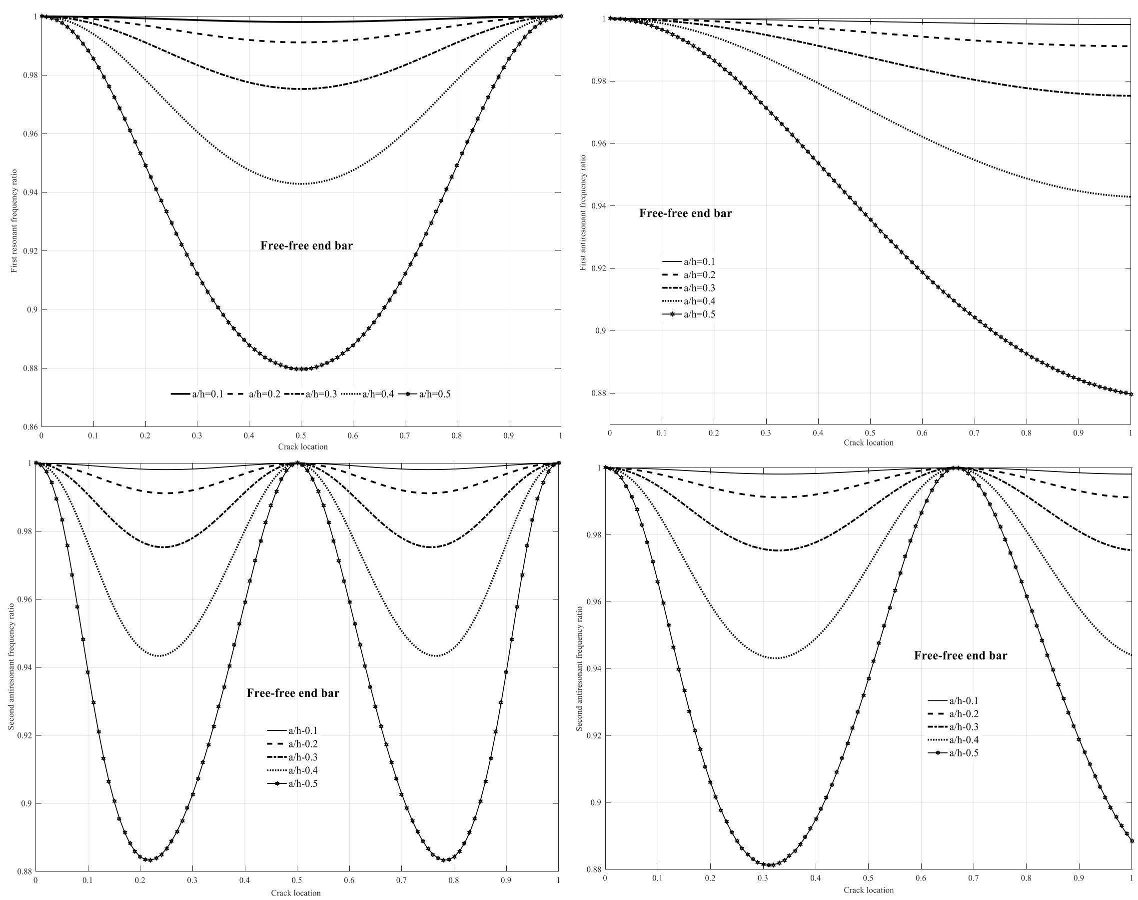


Fig. 1. Variation of three lowest resonant (left) and antiresonant (right) frequencies of fixed-free bar versus crack position with different crack depth (10%–50%)



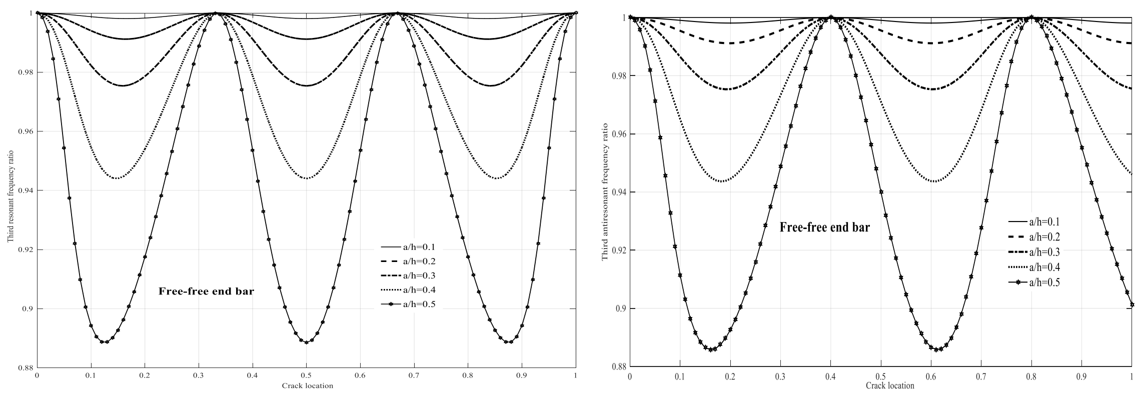
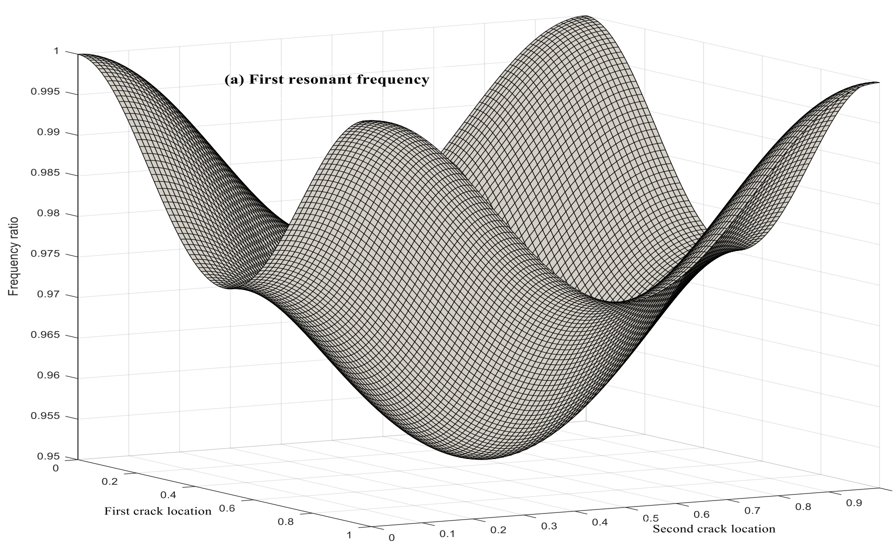


Fig. 2. Variation of three lowest resonant (left) and antiresonant (right) frequencies of free-free bar versus crack position with different crack depth (10%–50%)

Observing graphics given in the Figures demonstrates that crack at free end of bar makes no effect on the resonant frequencies, while it would do significant change in antiresonant frequencies if the frequency response function is defined at this position. Likely to the resonant frequencies, antiresonant frequencies are all monotonically reduced with increasing depth of crack except the nodes (Tab. 3) where they are unaffected by the crack presence.

The ratios of resonant and antiresonant frequencies computed for free-free end bar with two cracks are presented respectively in Figs. 3–4. Obviously, symmetric cracks make the same effect on resonant frequencies of the bar, but this is not true for antiresonant frequencies. Also, the larger number of cracks makes more reduction of antiresonant frequencies.



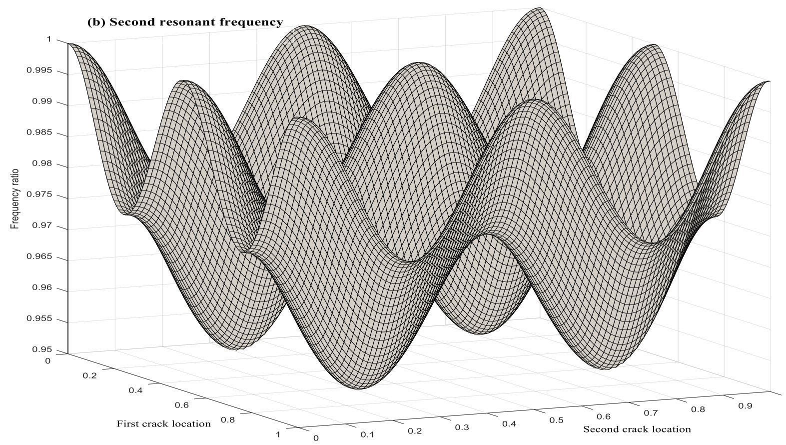


Fig. 3. Variation of first and second resonant frequencies versus position of two cracks with equal depth 30% for free end bar

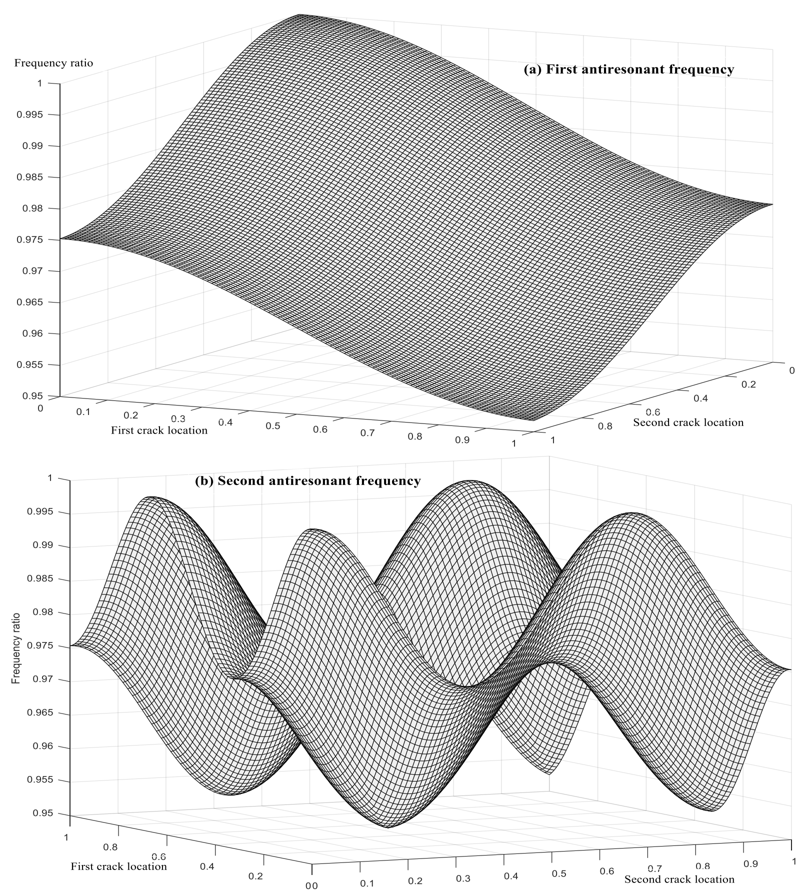


Fig. 4. Variation of first and second antiresonant frequencies versus positions of two cracks with equal depth 30% for free end bar

5. CONCLUSIONS

In the present work there has been derived a novel form of characteristic equation for resonant and antiresonant frequencies of multiple cracked bar that is explicitly expressed in terms of crack magnitudes. The conducted characteristic equations are general regarding boundary conditions and exact in comparison with the numerous approximate ones known in the literature. These characteristic equations provide a useful tool for developing crack detection procedures in bar.

The antiresonant frequencies of bar with single and double cracks have been examined versus crack position and depth mutually with the resonant ones. The obtained results show that there exist also nodes for antiresonant frequencies but they are different from those of resonant ones. Furthermore, resonant frequencies are defined independently upon where frequency response is measured, while antiresonant frequencies are strongly dependent on the FRF's measurement. Therefore, effect of crack position on an antiresonant frequency may be also different if the antiresonant frequency is extracted from different FRFs. The observed different properties of resonant and antiresonant frequencies may be helpful for detecting cracks in bar by using both of them.

The question that is open in this study is how to determine antiresonant frequencies of cracked bar with fixed-fixed ends. This problem is easily solved for uncracked bar, but it is unsolved for a bar with a crack because the points selected for measurement of FRF may disregard effect of the crack on the FRF. The problem mentioned above is subject for further study of the authors.

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