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PERFORMANCE ANALYSIS OF GLOBAL-LOCAL MEAN SQUARE ERROR CRITERION OF STOCHASTIC LINEARIZATION FOR NONLINEAR OSCILLATORS

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Abstract. The paper presents a performance analysis of global-local mean square error criterion of stochastic linearization for some nonlinear oscillators. This criterion of stochastic linearization for nonlinear oscillators bases on dual conception to the local mean square error criterion (LOMSEC). The algorithm is generally built to multi-degree of freedom (MDOF) nonlinear oscillators. Then, the performance analysis is carried out for two applications which comprise a rolling ship oscillation and two-degree of freedom one. The improvement on accuracy of the proposed criterion has been shown in comparison with the conventional Gaussian equivalent linearization (GEL).

Keywords: probability; random; frequency response function; iteration method; mean square.

1. INTRODUCTION

One popular class of methods for approximate solutions of nonlinear systems under random excitations is GEL techniques, which are most used in structural dynamics and in the engineering mechanics applications. This is partially due to its simplicity and applicability to systems with MDOF, and ones under various types of random excitations. The key idea of GEL is to replace the nonlinear system by a linear one such that the behavior of the equivalent linear system approximates that of the original nonlinear oscillator. The standard way is that the coefficients of linearization are to be found by the classical mean square error criterion [1,2]. Although the method is very efficient, but its accuracy decreases as the nonlinearity increases and in many cases it gives very larger errors due to the non-Gaussian property of the response. That is reason why many researches have been done in recent decades on improving GEL, for example [3–11]. One among them is LOMSEC that was first proposed by N. D. Anh and Di Paola [10], and then further developed by N. D. Anh and L. X. Hung [11]. The basic difference of LOMSEC from the classical GEL is that the integration domain for mean square of response taken over finite one (local one) instead of $(-\infty, \infty)$ in the classical GEL. As LOMSEC can give a good improvement on accuracy, however, the local integration domain in question was unknown and it has resulted in the main disadvantage of LOMSEC. Recently a dual conception was proposed in the study of responses to nonlinear systems [12,13]. One remarkable advantage of the dual conception is its consideration of two different aspects of a problem in question allows the investigation to be more appropriate. Applying the dual approach to LOMSEC, a new criterion namely global-local mean square error criterion (GLOMSEC) has been recently proposed L. X. Hung et al. [14, 15]for nonlinear systems under white noise excitation, in which new values of linearization coefficients are obtained as global averaged values of all local linearization coefficients. This paper is an additional research to aim at evaluating the improved performance of the proposed criterion; herein we analyse two more applications, which are a rolling ship oscillation and two-degree-of-freedom one. The results show a significant improvement on accuracy of solutions by the new criterion compared to the ones by the classical GEL.

2. FORMULATION

Consider a MDOF nonlinear stochastic oscillator described by the following equation

$$M\ddot{q} + C\dot{q} + Kq + \Phi(q, \dot{q}) = Q(t), \tag{1}$$

where $M = \left[m_{ij}\right]_{n\times n}$, $C = \left[c_{ij}\right]_{n\times n}$, $K = \left[k_{ij}\right]_{n\times n}$ are $n \times n$ constant matrices, defined as the inertia, damping and stiffness matrices, respectively. $\Phi\left(q, \dot{q}\right) = \left[\Phi_1, \Phi_2, \ldots, \Phi_n\right]^T$ is a nonlinear n-vector function of the generalized coordinate vector $q = \left[q_1, q_2, \ldots, q_n\right]^T$ and its derivative $\dot{q} = \left[\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\right]^T$. The symbol (T) denotes the transpose of a matrix. The excitation $Q(t)$ is a zero mean stationary Gaussian random vector process with the spectral density matrix $S_Q(\omega) = \left[S_{ij}\left(\omega\right)\right]_{n\times n}$ where $S_{ij}\left(\omega\right)$ is the spectral density function of elements Q_i and Q_j.

An equivalent linear system to the original nonlinear system (1) can be defined as

$$M\ddot{q} + \left(C + C^e\right)\dot{q} + \left(K + K^e\right)q = Q(t), \tag{2}$$

where $C^e = \left[c^e_{ij}\right]_{n\times n}$, $K^e = \left[k^e_{ij}\right]_{n\times n}$ are deterministic matrices. They are to be determined so that the n-vector difference $\varepsilon = \left[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\right]^T$ between the original and the equivalent system is minimum. In the classical GEL shown in [16] by Roberts and Spanos, the matrices C^e, K^e are determined by the following criterion

$$E\left\{\varepsilon^T\varepsilon\right\} \to \min\nolimits_{c^e_{ij}, k^e_{ij}} \quad (i, j = 1, 2, \ldots, n), \tag{3}$$

where $E\{.\}$ denotes the mathematical expectation operation and c^e_{ij}, k^e_{ij} are the (i, j) elements of the matrices C^e, K^e and

$$\varepsilon = \Phi(q, \dot{q}) - C_e\dot{q} - K_e q. \tag{4}$$

Using the linearity property of the expectation operator $E\{.\}$, criterion (3) can be written as

$$E\left\{\varepsilon_\alpha^2\right\} \to \min_{c_{ij}^e, k_{ij}^e} \quad (\alpha = 1, 2, \ldots, n). \tag{5}$$

The necessary conditions for the criterion (5) to be true are

$$\frac{\partial}{\partial c_{ij}^e} E\left\{\varepsilon_\alpha^2\right\} = 0, \quad \frac{\partial}{\partial k_{ij}^e} E\left\{\varepsilon_\alpha^2\right\} = 0, \quad (i, j = 1, 2, \ldots, n). \tag{6}$$

Combine (4) and (6), after some algebraic procedures, one gets the equivalent linearization coefficients as follows

$$c_{ij}^e = E\left\{\frac{\partial \Phi_i}{\partial \dot{q}_j}\right\}, \quad k_{ij}^e = E\left\{\frac{\partial \Phi_i}{\partial q_j}\right\}, \tag{7}$$

where Φ_i is the (i) element of $\Phi(q, \dot{q})$. The spectral density matrix of the response process $q(t)$ is of the form

$$S_q(\omega) = [S_{q_i q_j}(\omega)], \quad (i, j = 1, 2, \ldots, n), \tag{8}$$

where $S_{q_i q_j}(\omega)$ is the (i, j) element of $S_q(\omega)$.

Using the matrix spectral input-output relationship to linear system (2), one gets

$$S_q(\omega) = \alpha(\omega) S_Q(\omega) \alpha^T(\omega), \tag{9}$$

where $\alpha(\omega)$ is the matrix of frequency response functions. It is known as

$$\alpha(\omega) = \left[-\omega^2 M + i\omega(C + C^e) + (K + K^e)\right]^{-1}. \tag{10}$$

The mean values of the response can be calculated by the following equations

$$\begin{aligned} E\left\{q_i q_j\right\} &= \int_{-\infty}^{\infty} S_{q_i q_j}(\omega) \mathrm{d}\omega, \quad E\left\{q q^T\right\} = \int_{-\infty}^{\infty} \alpha(-\omega) S_Q(\omega) \alpha^T(\omega) \mathrm{d}\omega, \\ E\left\{\dot{q} \dot{q}^T\right\} &= \int_{-\infty}^{\infty} \omega^2 \alpha(-\omega) S_Q(\omega) \alpha^T(\omega) \mathrm{d}\omega \end{aligned} \tag{11}$$

A set of nonlinear algebraic equations (2), (7), (9)–(11) allows to find the mean values of response. Denote $p(q)$ the stationary joint probability density function (PDF) of the vector $q = [q_1, q_2, \ldots, q_n]^T$. The criterion (5) can be written in the following form

$$E\left\{\varepsilon_\alpha^2\right\} = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \varepsilon_\alpha^2 p(q) \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n \to \min_{c_{ij}^e, k_{ij}^e} \quad (\alpha, i, j = 1, 2, \ldots, n). \tag{12}$$

As the above-mentioned that the basic difference of LOMSEC from the classical GEL is that the integration domain for mean squares of response are taken over finite one (local one). Thus, LOMSEC requires

$$E\left[\varepsilon_\alpha^2\right] = \int_{-q_{01}}^{+q_{01}} \ldots \int_{-q_{0n}}^{+q_{0n}} \varepsilon_\alpha^2 p(q) \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n \to \min_{c_{ij}^e, k_{ij}^e} \quad (\alpha, i, j = 1, 2, \ldots, n), \tag{13}$$

where $q_{01}, q_{02}, \ldots, q_{0n}$ are given positive values. The expected integrations in (13) can be transformed to non-dimensional variables by $q_{01} = r\sigma_{q1}, q_{02} = r\sigma_{q2}, \ldots, q_{0n} = r\sigma_{qn}$ with r a given positive value; $\sigma_{q1}, \sigma_{q2}, \ldots, \sigma_{qn}$ are the normal deviations of random variables of $q_1, q_2, \ldots, q_n$, respectively. Thus, criterion (13) become

$$E\left[\varepsilon_\alpha^2\right] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} \ldots \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} \varepsilon_\alpha^2 p(q) \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n \rightarrow \min_{c_{ij}^e, k_{ij}^e} \quad (\alpha, i, j = 1, 2, \ldots, n), \tag{14}$$

where $E[.]$ denotes the local mean values by LOMSEC. These values of random variables are taken as follows

$$E\left[.\right] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} \ldots \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} (.) p(q) \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n \underset{\text{For example}}{\rightarrow} E\left[q_i q_j\right] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} \ldots \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} q_i q_j p(q) \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n. \tag{15}$$

For zero-mean stationary Gaussian random variables, The classical GEL indecates that all odd-order means are null, all higher even-order means can be expressed in terms of second-order mean of the respective variable. These characteristics are also kept in LOMSEC and presented in the appendix.

In GEL, the values $\sigma_{q1}, \sigma_{q2}, \ldots, \sigma_{qn}$ are considered to be independent from c_{ij}^e, k_{ij}^e in the process of minimizing (14). Criterion (14) results in conditions for determining c_{ij}^e, k_{ij}^e as follows

$$\frac{\partial}{\partial c_{ij}^e} E\left[\varepsilon_\alpha^2\right] = 0, \quad \frac{\partial}{\partial k_{ij}^e} E\left[\varepsilon_\alpha^2\right] = 0, \quad (\alpha, i, j = 1, 2, \ldots, n). \tag{16}$$

It is seen from (14) to (16) that the elements of c_{ij}^e, k_{ij}^e are functions depending on the local mean values of random variables and also depending on r (i.e. $c_{ij}^e = c_{ij}^e(r), k_{ij}^e = k_{ij}^e(r)$), which is not explicitly expressed here. Eqs. (2), (15) and (16) allow to determine the unknowns $c_{ij}^e(r), k_{ij}^e(r)$ and the vector $q(t)$ when r is given. However, is that the local domain of integration, namely in our case the value of r, is unknown and the open question is how to find it. Using the dual approach to LOMSEC, it is suggested that instead of finding a special value of r one may consider its variation in the entire global domain of integration. Thus, the linearization coefficients $c_{ij}^e(r), k_{ij}^e(r)$ can be suggested as global mean values of all local linearization coefficients as follows

$$c_{ij}^e = \left\langle c_{ij}^e(r) \right\rangle = \lim_{s\rightarrow\infty} \frac{1}{s} \int_0^s c_{ij}^e(r) \mathrm{d}r, \quad k_{ij}^e = \left\langle k_{ij}^e(r) \right\rangle = \lim_{s\rightarrow\infty} \frac{1}{s} \int_0^s k_{ij}^e(r) \mathrm{d}r \tag{17}$$

where $\langle . \rangle$ denotes conventionally the average of operators of deterministic functions. Obviously, Eqs. (2), (15), (16), (17) allow to determine the unknowns without specifying any value of r and the new criterion may be called global–local mean square error criterion (GLOMSEC).

3. APPLICATIONS

3.1. Rolling ship oscillation

The rolling motion of a ship in random waves has been considered by Roberts [17], Roberts and Dacunha [18], David et al. [19]. The governing equation of motion, for example in [19], is

$$\ddot{\varphi} + \beta\dot{\varphi} + \alpha\dot{\varphi}\,|\dot{\varphi}| + \omega^2\varphi + \delta\varphi^3 = \sqrt{2D}w(t), \tag{18}$$

where $\varphi \leq 35^\circ$ is the roll angle from the vertical, ω is the undamped natural frequency of roll. The parameters β, α, δ are constant. The random waves is described as zero mean Gaussian white noise excitation, which is denoted by $w(t)$, and $\sqrt{2D}$ is the intensity of the white noise excitation. Note that equation (18) is only valid for $\varphi \leq 35^\circ$. This, in turn, requires that δ and D are small such that the probability for the response trajectories to depart from the region of stability in the phase plane is extremely small. Under such conditions, for practical purpose, then it is reasonable to assume the existence of stationary random rolling motion.

In order to obtain some simple analytical results, consider case with $\beta = \delta = 0$ so that the rolling ship oscillator reduces to a quadratically damped linear stiffness oscillator as follows

$$\ddot{\varphi} + \alpha\dot{\varphi}\,|\dot{\varphi}| + \omega^2\varphi = \sqrt{2D}w(t). \tag{19}$$

The exact solution of the system (19) does not exist; however, an approximate probability density function obtained by equivalent non-linearization (ENL) method following [19] or [20].

$$P(\varphi, \dot{\varphi}) = \frac{3}{2\pi\Gamma\left(\dfrac{2}{3}\right)}\left(\frac{8\alpha}{9\pi D}\right)^{\frac{2}{3}} e^{-\frac{8\alpha}{9\pi D}\left(\omega^2\varphi^2 + \dot{\varphi}^2\right)^{\frac{3}{2}}}, \tag{20}$$

where $\Gamma\left(.\right)$ is the Gamma function.

Generally, ENL gives solutions with rather high accuracy and in many cases it agrees with Monte Carlo simulation (MCS) [20]. Thus, the solutions given by ENL can be used for evaluation of accuracy of ones obtained by other approximate methods, for example GEL.

Consider the system (19) with $\omega = 1$. Denote $E\left\{\varphi^2\right\}_{NL}, E\left\{\dot{\varphi}^2\right\}_{NL}$ the square mean responses of displacement and velocity determined from the probability density function (20), respectively. Additionally, when $\omega = 1$, we have $E\left\{\varphi^2\right\}_{NL} = E\left\{\dot{\varphi}^2\right\}_{NL}$. Thus, the results are

$$E\left\{\varphi^2\right\}_{NL} = E\left\{\dot{\varphi}^2\right\}_{NL} = 0.765\left(\frac{D}{\alpha}\right)^{\dfrac{2}{3}}. \tag{21}$$

For GEL, the nonlinear system (19) is replaced by a linear one as follows

$$\ddot{\varphi} + c^e\dot{\varphi} + \varphi = \sqrt{2D}w(t), \tag{22}$$

where c^e is the linearization coefficient, for LOMSEC $c^e = c^e(r)$ as known by (16) as follows

$$\frac{\partial}{\partial c^e} E\left[\varepsilon^2\right] = \frac{\partial}{\partial c^e} E\left[\left(\alpha\dot{\varphi}\left|\dot{\varphi}\right| - c^e\dot{\varphi}\right)^2\right] = 0. \tag{23}$$

Expand (23) and utilize (A.8)–(A.9), one gets

$$c^e(r) = \alpha\sqrt{E\left\{\dot{\varphi}^2\right\}}\frac{T_{t^3,r}}{T_{1,r}}, \quad \left(T_{t^3,r} = \int_0^r t^3\eta(t)\mathrm{d}t, T_{1,r} = \int_0^r t^2\eta(t)\mathrm{d}t\right). \tag{24}$$

For the linear system (22), the mean square responses by LOMSEC are

$$E\left\{\varphi^2\right\}_L = E\left\{\dot{\varphi}^2\right\}_L = \frac{\left(\sqrt{2D}\right)^2}{2c^e(r)} = \frac{D}{c^e(r)} = \frac{D}{\alpha\sqrt{E\left\{\dot{\varphi}^2\right\}}\dfrac{T_{t^3,r}}{T_{1,r}}}. \tag{25}$$

With $r \to \infty$, (25) gives the solutions by the classical GEL as follows

$$E\left\{\varphi^2\right\}_C = E\left\{\dot{\varphi}^2\right\}_C = 0.732\left(\frac{D}{\alpha}\right)^{\frac{2}{3}}. \tag{26}$$

Apply (17) for (24), one gets the linearization coefficient by GLOMSEC as follows

$$c^e = \left\langle c^e(r)\right\rangle = \lim_{s\to\infty}\left(\frac{1}{s}\int_0^s c^e(r)\mathrm{d}r\right) = \alpha E\{\dot{\varphi}^2\}^{1/2}\lim_{s\to\infty}\left(\frac{1}{s}\int_0^s \frac{T_{t^3,r}}{T_{1,r}}\mathrm{d}r\right) \approx 1.49705\alpha E\{\dot{\varphi}^2\}^{1/2}. \tag{27}$$

The limitation element in (27) can be approximately computed to be 1.49705. The solutions obtained by GLOMSEC are

$$E\left\{\varphi^2\right\}_{GL} = E\left\{\dot{\varphi}^2\right\}_{GL} = \frac{D}{c^e} = \frac{D}{1.49705\alpha E\{\dot{\varphi}^2\}^{1/2}} = 0.76415\left(\frac{D}{\alpha}\right)^{2/3}. \tag{28}$$

Denote $Err_{(C)}, Err_{(GL)}$ the relative errors of (26) and (28) to (21) respectively, one gets

$$\begin{aligned} Err_{(C)} &= \left|\frac{E\left\{\varphi^2\right\}_C - E\left\{\varphi^2\right\}_{NL}}{E\left\{\varphi^2\right\}_{NL}}\right| * 100\% = \left|\frac{0.732 - 0.765}{0.765}\right| * 100\% = 4.314\% \\ Err_{(GL)} &= \left|\frac{E\left\{\varphi^2\right\}_{GL} - E\left\{\varphi^2\right\}_{NL}}{E\left\{\varphi^2\right\}_{NL}}\right| * 100\% = \left|\frac{0.764 - 0.765}{0.765}\right| * 100\% = 0.130\% \end{aligned} \tag{29}$$

Note that since (21), (26) and (28) all contain the same factor $\left(\dfrac{D}{\alpha}\right)^{2/3}$, so this factor is reduced in the expression (29).

The result in (29) shows that the solution by GLOMSEC agree with the one by ENL because of negligible differences between these solutions. In addition, these solutions contain the similar factor in their formulas. This means that GLOMSEC gives a significant improvement on accuracy of solution in comparison with the classical GEL.

3.2. Two-degree-of-freedom nonlinear oscillator

Consider a two-degree- of-freedom nonlinear oscillator governed by the equation [20]

$$\begin{aligned} &\ddot{x}_1 - \left(\lambda_1 - \alpha_1 \dot{x}_1^2\right)\dot{x}_1 + \omega_1^2 x_1 + a x_2 + b\left(x_1 - x_2\right)^3 = w_1(t), \\ &\ddot{x}_2 - \left(\lambda_1 - \lambda_2 - \alpha_2 \dot{x}_2^2\right)\dot{x}_2 + \omega_2^2 x_2 + a x_1 + b\left(x_2 - x_1\right)^3 = w_2(t), \end{aligned} \tag{30}$$

where $\alpha_i, a, b, \lambda_i, \omega_i (i = 1,2)$ are constant. $w_1(t), w_2(t)$ are zero mean Gaussian white noise and $E\left\{w_i(t)w_i(t+\tau)\right\} = 2\pi S_i \delta(\tau)$ $(i = 1,2)$ where $\delta(\tau)$ is Delta Dirac function, S_1, S_2 are constant values of the spectral density of $w_1(t), w_2(t)$, respectively. The equation (30) can be rewritten as follows

$$\begin{aligned} &\ddot{x}_1 - \lambda_1 \dot{x}_1 + \omega_1^2 x_1 + a x_2 + \alpha_1 \dot{x}_1^3 + b\left(x_1 - x_2\right)^3 = w_1(t), \\ &\ddot{x}_2 - \left(\lambda_1 - \lambda_2\right)\dot{x}_2 + a x_1 + \omega_2^2 x_2 + \alpha_2 \dot{x}_2^3 + b\left(x_2 - x_1\right)^3 = w_2(t). \end{aligned} \tag{31}$$

Eq. (31) can be expressed in matrix form as follows

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_1 + \lambda_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & a \\ a & \omega_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 \dot{x}_1^3 + b\left(x_1 - x_2\right)^3 \\ \alpha_2 \dot{x}_2^3 + b\left(x_2 - x_1\right)^3 \end{bmatrix} = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}. \tag{32}$$

Following Eq. (1), denote

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_1 + \lambda_2 \end{bmatrix}; K = \begin{bmatrix} \omega_1^2 & a \\ a & \omega_2^2 \end{bmatrix}; \Phi = \begin{bmatrix} \alpha_1 \dot{x}_1^3 + b\left(x_1 - x_2\right)^3 \\ \alpha_2 \dot{x}_2^3 + b\left(x_2 - x_1\right)^3 \end{bmatrix}; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{33}$$

The linear equation to (32) is taken in the form of (2) as follows

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} -\lambda_1 + c_{11}^e & c_{12}^e \\ c_{21}^e & -\lambda_1 + \lambda_2 + c_{22}^e \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 + k_{11}^e & a + k_{12}^e \\ a + k_{21}^e & \omega_2^2 + k_{22}^e \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \tag{34}$$

where $c_{ij}^e, k_{ij}^e (i,j = 1,2)$ are the linearization coefficients.

According to (4), the difference between (32) and (34) is

$$\varepsilon = \Phi(x, \dot{x}) - C^e \dot{X} - K^e X \tag{35}$$

$$\Phi(x,\dot{x}) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \dot{x}_1^3 + b\left(x_1 - x_2\right)^3 \\ \alpha_2 \dot{x}_2^3 + b\left(x_2 - x_1\right)^3 \end{bmatrix}, C^e = \begin{bmatrix} c_{11}^e & c_{12}^e \\ c_{21}^e & c_{22}^e \end{bmatrix}; \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}; K^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \dot{x}_1^3 + b\left(x_1 - x_2\right)^3 - c_{11}^e \dot{x}_1 - c_{12}^e \dot{x}_2 - k_{11}^e x_1 - k_{12}^e x_2 \\ \alpha_2 \dot{x}_2^3 + b\left(x_2 - x_1\right)^3 - c_{21}^e \dot{x}_1 - c_{22}^e \dot{x}_2 - k_{21}^e x_1 - k_{22}^e x_2 \end{bmatrix}.$$

Use (16) for determining $c_{ij}^e(r), k_{ij}^e(r) (i,j = 1,2)$

$$\frac{\partial E\left[\varepsilon_1^2\right]}{\partial c_{11}^e} = 2c_{11}^e E\left[\dot{x}_1^2\right] - 2\left\{\begin{array}{l} \alpha_1 E\left[\dot{x}_1^4\right] + b(E\left[x_1^3 \dot{x}_1\right] + 3E\left[x_1 x_2^2 \dot{x}_1\right] - 3E\left[x_1^2 x_2 \dot{x}_1\right] \\ -E\left[x_2^3 \dot{x}_1\right]) - c_{12}^e E\left[\dot{x}_1 \dot{x}_2\right] - k_{11}^e E\left[x_1 \dot{x}_1\right] - k_{12}^e E\left[x_2 \dot{x}_1\right] \end{array}\right\} = 0,$$

$$\frac{\partial E\left[\varepsilon_1^2\right]}{\partial c_{12}^e} = 2c_{12}^e E\left[\dot{x}_2^2\right] - 2\left\{\begin{array}{l} \alpha_1 E\left[\dot{x}_1^3 \dot{x}_2\right] + b(E\left[x_1^3 \dot{x}_2\right] + 3E\left[x_1 x_2^2 \dot{x}_2\right] - 3E\left[x_1^2 x_2 \dot{x}_2\right] \\ -E\left[x_2^3 \dot{x}_2\right]) - c_{11}^e E\left[\dot{x}_1 \dot{x}_2\right] - k_{11}^e E\left[x_1 \dot{x}_2\right] - k_{12}^e E\left[x_2 \dot{x}_2\right] \end{array}\right\} = 0,$$

$$
\begin{aligned}
&\frac{\partial E\left[\varepsilon_2^2\right]}{\partial c_{21}^e}=2c_{21}^e E\left[\dot{x}_1^2\right]-2\left\{\begin{array}{l}\alpha_2 E\left[\dot{x}_2^3\dot{x}_1\right]+b(E\left[x_2^3\dot{x}_1\right]+3E\left[x_1^2x_2\dot{x}_1\right]-3E\left[x_1x_2^2\dot{x}_1\right]\\ -E\left[x_1^3\dot{x}_1\right])-c_{22}^e E\left[\dot{x}_1\dot{x}_2\right]-k_{21}^e E\left[x_1\dot{x}_1\right]-k_{22}^e E\left[x_2\dot{x}_1\right]\end{array}\right\}=0,\\
&\frac{\partial E\left[\varepsilon_2^2\right]}{\partial c_{22}^e}=2c_{22}^e E\left[\dot{x}_2^2\right]-2\left\{\begin{array}{l}\alpha_2 E\left[\dot{x}_2^4\right]+b(E\left[x_2^3\dot{x}_2\right]+3E\left[x_1^2x_2\dot{x}_2\right]-3E\left[x_1x_2^2\dot{x}_2\right]\\ -E\left[x_1^3\dot{x}_2\right])-c_{21}^e E\left[\dot{x}_1\dot{x}_2\right]-k_{21}^e E\left[x_1\dot{x}_2\right]-k_{22}^e E\left[x_2\dot{x}_2\right]\end{array}\right\}=0,\\
&\frac{\partial E\left[\varepsilon_1^2\right]}{\partial k_{11}^e}=2k_{11}^e E\left[x_1^2\right]-2\left\{\begin{array}{l}\alpha_1 E\left[x_1\dot{x}_1^3\right]+b(E\left[x_1^4\right]+3E\left[x_1^2x_2^2\right]-3E\left[x_1^3x_2\right]\\ -E\left[x_1x_2^3\right])-c_{11}^e E\left[x_1\dot{x}_1\right]-c_{12}^e E\left[x_1\dot{x}_2\right]-k_{12}^e E\left[x_1x_2\right]\end{array}\right\}=0,\\
&\frac{\partial E\left[\varepsilon_1^2\right]}{\partial k_{12}^e}=2k_{12}^e E\left[x_2^2\right]-2\left\{\begin{array}{l}\alpha_1 E\left[\dot{x}_1^3x_2\right]+b(E\left[x_1^3x_2\right]+3E\left[x_1x_2^3\right]-3E\left[x_1^2x_2^2\right]\\ -E\left[x_2^4\right])-c_{11}^e E\left[\dot{x}_1x_2\right]-c_{12}^e E\left[x_2\dot{x}_2\right]-k_{11}^e E\left[x_1x_2\right]\end{array}\right\}=0,\\
&\frac{\partial E\left[\varepsilon_2^2\right]}{\partial k_{21}^e}=2k_{21}^e E\left[x_1^2\right]-2\left\{\begin{array}{l}\alpha_2 E\left[x_1\dot{x}_2^3\right]+b(E\left[x_2^3x_1\right]+3E\left[x_1^3x_2\right]-3E\left[x_1^2x_2^2\right]\\ -E\left[x_1^4\right])-c_{21}^e E\left[x_1\dot{x}_1\right]-c_{22}^e E\left[x_1\dot{x}_2\right]-k_{22}^e E\left[x_1x_2\right]\end{array}\right\}=0,\\
&\frac{\partial E\left[\varepsilon_2^2\right]}{\partial k_{22}^e}=2k_{22}^e E\left[x_2^2\right]-2\left\{\begin{array}{l}\alpha_2 E\left[x_2\dot{x}_2^3\right]+b(E\left[x_2^4\right]+3E\left[x_1^2x_2^2\right]-3E\left[x_1x_2^3\right]\\ -E\left[x_1^3x_2\right])-c_{21}^e E\left[\dot{x}_1x_2\right]-c_{22}^e E\left[x_2\dot{x}_2\right]-k_{21}^e E\left[x_1x_2\right]\end{array}\right\}=0.
\end{aligned}
\tag{36}
$$

In order to simplify the calculation, assume that x_1, x_2 are independent from each other. As known that if is a stationary Gaussian random process with zero mean, so is $\dot{x}(t)$. Besides, a stationary random process is orthogonal to its derivative, so x_1, x_2 are independent from $\dot{x}_1, \dot{x}_2$, respectively. Use (A.3), (A.6) and (A.8) in the appendix to determine the local means in (36) and note that $E\left[x_i^{2n+1}x_j^{2m+1}\right]=0\ (i\neq j)$. Thus, (36) gives the following result

$$
\begin{aligned}
&c_{11}^e(r)=\alpha_1\frac{E\left[\dot{x}_1^4\right]}{E\left[\dot{x}_1^2\right]}=\alpha_1 E\left\{\dot{x}_1^2\right\}\frac{T_{2,r}}{T_{1,r}},\\
&c_{12}^e(r)=c_{21}^e(r)=0,\\
&c_{22}^e(r)=\alpha_2\frac{E\left[\dot{x}_2^4\right]}{E\left[\dot{x}_2^2\right]}=\alpha_2 E\left\{\dot{x}_2^2\right\}\frac{T_{2,r}}{T_{1,r}},\\
&k_{11}^e(r)=b\frac{E\left[x_1^4\right]+3E\left[x_1^2\right]E\left[x_2^2\right]}{E\left[x_1^2\right]}=b\left(E\left\{x_1^2\right\}\frac{T_{2,r}}{T_{1,r}}+3E\left\{x_2^2\right\}\frac{T_{1,r}}{T_{0,r}}\right),\\
&k_{12}^e(r)=b\frac{-E\left[x_2^4\right]-3E\left[x_1^2\right]E\left[x_2^2\right]}{E\left[x_2^2\right]}=b\left(-E\left\{x_2^2\right\}\frac{T_{2,r}}{T_{1,r}}-3E\left\{x_1^2\right\}\frac{T_{1,r}}{T_{0,r}}\right),\\
&k_{21}^e(r)=b\frac{-E\left[x_1^4\right]-3E\left[x_1^2\right]E\left[x_2^2\right]}{E\left[x_1^2\right]}=b\left(-E\left\{x_1^2\right\}\frac{T_{2,r}}{T_{1,r}}-3E\left\{x_2^2\right\}\frac{T_{1,r}}{T_{0,r}}\right),\\
&k_{22}^e(r)=b\frac{E\left[x_2^4\right]+3E\left[x_1^2\right]E\left[x_2^2\right]}{E\left[x_2^2\right]}=b\left(E\left\{x_2^2\right\}\frac{T_{2,r}}{T_{1,r}}+3E\left\{x_1^2\right\}\frac{T_{1,r}}{T_{0,r}}\right).
\end{aligned}
\tag{37}
$$

In (37), let $r \to \infty$, it gives the linearization coefficients by the classical GEL as follows

$$\begin{aligned} &c^e_{11} = 3\alpha_1 E\left\{\dot{x}_1^2\right\}, \quad c^e_{12} = c^e_{21} = 0, \quad c^e_{22} = 3\alpha_2 E\left\{\dot{x}_2^2\right\}, \\ &k^e_{11} = k^e_{22} = 3b\left(E\left\{x_1^2\right\} + E\left\{x_2^2\right\}\right), \quad k^e_{12} = k^e_{21} = -3b\left(E\left\{x_1^2\right\} + 3bE\left\{x_2^2\right\}\right). \end{aligned} \tag{38}$$

The following factors are defined and replaced in (38)

$$\frac{T_{2,\infty}}{T_{1,\infty}} = \frac{\int\limits_0^\infty t^4\eta(t)\mathrm{d}t}{\int\limits_0^\infty t^2\eta(t)\mathrm{d}t} = 3, \quad \frac{T_{1,\infty}}{T_{0,\infty}} = \frac{\int\limits_0^\infty t^2\eta(t)\mathrm{d}t}{\int\limits_0^\infty \eta(t)\mathrm{d}t} = 1, \quad \eta(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}.$$

Apply (17) to (37), one obtains the linearization coefficients by GLOMSEC as follows

$$\begin{aligned}
c^e_{11} &= \langle c^e_{11}(r)\rangle = \alpha_1 E\left\{\dot{x}_1^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right), \\
c^e_{22} &= \langle c^e_{22}(r)\rangle = \alpha_2 E\left\{\dot{x}_2^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right), c^e_{12} = c^e_{21} = 0, \\
k^e_{11} &= \langle k^e_{11}(r)\rangle = b\left(E\left\{x_1^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right) + 3E\left\{x_2^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{1,r}}{T_{0,r}}\mathrm{d}r\right)\right), \\
k^e_{12} &= \langle k^e_{12}(r)\rangle = -b\left(E\left\{x_2^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right) + 3E\left\{x_1^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{1,r}}{T_{0,r}}\mathrm{d}r\right)\right), \\
k^e_{21} &= \langle k^e_{21}(r)\rangle = -b\left(E\left\{x_1^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right) + 3E\left\{x_2^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{1,r}}{T_{0,r}}\mathrm{d}r\right)\right), \\
k^e_{22} &= \langle k^e_{22}(r)\rangle = b\left(E\left\{x_2^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right) + 3E\left\{x_1^2\right\} \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{1,r}}{T_{0,r}}\mathrm{d}r\right)\right),
\end{aligned} \tag{39}$$

where the limitation factors can be approximately computed to be

$$\lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r\right) \approx 2.41189, \quad \lim_{s\to\infty}\left(\frac{1}{s}\int\limits_0^s \frac{T_{1,r}}{T_{0r}}\mathrm{d}r\right) \approx 0.83706.$$

Consider the white noise spectral densities of $w_1(t), w_2(t)$ respectively are $S_1 = S_2 = S_0$, the spectral density matrix $S_w(\omega)$ of $w(t)$ is defined by

$$S_w(\omega) = \begin{bmatrix} S_0 & 0 \\ 0 & S_0 \end{bmatrix}. \tag{40}$$

The frequency response function to linear system (34) is

$$\alpha(\omega) = \left[-\omega^2 M + i\omega(C + C^e) + (K + K^e)\right]^{-1}. \tag{41}$$

The matrices in (41) ware defined in (33) and (35) to be

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_1 + \lambda_2 \end{bmatrix}, K = \begin{bmatrix} \omega_1^2 & a \\ a & \omega_2^2 \end{bmatrix}, C^e = \begin{bmatrix} c_{11}^e & c_{12}^e \\ c_{21}^e & c_{22}^e \end{bmatrix}, K^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix}.$$

After some matrix operations, the frequency response function (41) is defined as follows

$$\alpha(\omega) = \begin{bmatrix} -\omega^2 + i\omega(-\lambda_1 + c_{11}^e) + \omega_1^2 + k_{11}^e & i\omega c_{12}^e + a + k_{12}^e \\ i\omega c_{21}^e + a + k_{21}^e & -\omega^2 + i\omega(-\lambda_1 + \lambda_2 + c_{22}^e) + \omega_2^2 + k_{22}^e \end{bmatrix}^{-1}. \tag{42}$$

In order to have a close equation system determining the unknowns, all the $E\left\{x_i^2\right\}$, $E\left\{\dot{x}_i^2\right\}, (i = 1, 2)$ must be defined. Use (11), (40) and after some matrix operations one gets

$$\begin{aligned}
E\left\{xx^T\right\} &= S_0 \int_{-\infty}^{+\infty} \begin{bmatrix} \alpha_{11}(\omega)\alpha_{11}(-\omega) + \alpha_{12}(\omega)\alpha_{12}(-\omega) & \alpha_{11}(-\omega)\alpha_{21}(\omega) + \alpha_{12}(-\omega)\alpha_{22}(\omega) \\ \alpha_{11}(\omega)\alpha_{21}(-\omega) + \alpha_{12}(\omega)\alpha_{22}(-\omega) & \alpha_{21}(\omega)\alpha_{21}(-\omega) + \alpha_{22}(\omega)\alpha_{22}(-\omega) \end{bmatrix} \mathrm{d}\omega, \\
E\left\{x_1^2\right\} &= S_0 \int_{-\infty}^{+\infty} \left(|\alpha_{11}(\omega)|^2 + |\alpha_{12}(\omega)|^2\right) \mathrm{d}\omega, \quad E\left\{x_2^2\right\} = S_0 \int_{-\infty}^{+\infty} \left(|\alpha_{21}(\omega)|^2 + |\alpha_{22}(\omega)|^2\right) \mathrm{d}\omega, \\
E\left\{\dot{x}\dot{x}^T\right\} &= S_0 \int_{-\infty}^{+\infty} \omega^2 \begin{bmatrix} \alpha_{11}(\omega)\alpha_{11}(-\omega) + \alpha_{12}(\omega)\alpha_{12}(-\omega) & \alpha_{11}(-\omega)\alpha_{21}(\omega) + \alpha_{12}(-\omega)\alpha_{22}(\omega) \\ \alpha_{11}(\omega)\alpha_{21}(-\omega) + \alpha_{12}(\omega)\alpha_{22}(-\omega) & \alpha_{21}(\omega)\alpha_{21}(-\omega) + \alpha_{22}(\omega)\alpha_{22}(-\omega) \end{bmatrix} \mathrm{d}\omega, \\
E\left\{\dot{x}_1^2\right\} &= S_0 \int_{-\infty}^{+\infty} \omega^2 \left(|\alpha_{11}(\omega)|^2 + |\alpha_{12}(\omega)|^2\right) \mathrm{d}\omega, \quad E\left\{\dot{x}_2^2\right\} = S_0 \int_{-\infty}^{+\infty} \omega^2 \left(|\alpha_{21}(\omega)|^2 + |\alpha_{22}(\omega)|^2\right) \mathrm{d}\omega,
\end{aligned} \tag{43}$$

where the elements α_{ij} are defined from (42). Eq. (43) is solved either together with (38) or (39) to define the unknowns by the classical GEL or by GLOMSEC, respectively. In order to solve the above equations, it is needed to utilize computationally approximate methods, for example, an iteration method is applied as follows: (i) Assign an initial value to the mean square responses of (43); (ii) Use (38) or (39) to determine the instantaneous linearization coefficients by the classical GEL or GLOMSEC, respectively; (iii) Use (42) and (43) to determine new instantaneous value of the responses; (iv) Repeat steps (ii) and (iii) until results from cycle to cycle have a difference to be less than 10^{-4}.

For purpose of evaluating the accuracy of solutions while the original nonlinear system (30) does not have the exact solution, one can use an approximate probability density function given by ENL method that was reported in [20] as follows.

$$p(x_1, \dot{x}_1, x_2, \dot{x}_2) = Ce^{-\left(\frac{1}{\pi S_i}\right)\left[\frac{9}{32}(\alpha_1 + \alpha_2)\left(\frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + U\right)^2 + \left(\frac{1}{2}\lambda_2 - \lambda_1\right)\left(\frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + U\right)\right]}, \tag{44}$$

where $U(x_1, x_2)$ is the potential energy of the system.

$$U(x_1, x_2) = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + a x_1 x_2 + \frac{b}{4}(x_1 - x_2)^4, \tag{45}$$

and C is the normalization constant defined by

$$C = \left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\left(\frac{1}{\pi S_i}\right)\left[\frac{9}{32}(\alpha_1+\alpha_2)\left(\frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2+U\right)^2+\left(\frac{1}{2}\lambda_2-\lambda_1\right)\left(\frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2+U\right)\right]} \prod_{i=1}^{2} \mathrm{d}x_i \mathrm{d}\dot{x}_i\right]^{-1}. \tag{46}$$

The mean square responses $E\left\{x_i^2\right\}_{NL}$ obtained by ENL are

$$E\left\{x_i^2\right\}_{NL} = C\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_i^2 e^{-\left(\frac{1}{\pi S_i}\right)\left[\frac{9}{32}(\alpha_1+\alpha_2)\left(\frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2+U\right)^2+\left(\frac{1}{2}\lambda_2-\lambda_1\right)\left(\frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2+U\right)\right]} \prod_{i=1}^{2} \mathrm{d}x_i \mathrm{d}\dot{x}_i. \tag{47}$$

Consider two cases of the given parameters. Tabs. 1 and 2 show the mean square responses of x_1, x_2 as well as their relative errors to solutions by ENL method (see also Figs. 1 and 2).

Table 1. The mean squares of x_1, x_2 versus α ($\alpha_1 = \alpha_2 = \alpha$) while $\lambda_1 = \lambda_2 = \omega_1 = \omega_2 = a = b = S_0 = 1$

α_1, α_2	$E\left\{x_1^2\right\}_{NL}$	$E\left\{x_1^2\right\}_{C}$	Err_C $\lvert\%\rvert$	$E\left\{x_1^2\right\}_{GL}$	Err_{GL} $\lvert\%\rvert$	$E\left\{x_2^2\right\}_{NL}$	$E\left\{x_2^2\right\}_{C}$	Err_C $\lvert\%\rvert$	$E\left\{x_2^2\right\}_{GL}$	Err_{GL} $\lvert\%\rvert$
0.1	1.57273	1.21597	22.684	1.40692	10.543	1.57273	1.15079	26.829	1.32675	15.640
1	0.49622	0.42145	15.068	0.48835	1.586	0.49622	0.36966	25.505	0.41930	15.501
5	0.25327	0.21986	13.191	0.25395	0.268	0.25327	0.20466	19.193	0.23409	7.573
10	0.19437	0.17091	12.070	0.19735	1.533	0.19437	0.16233	16.484	0.18625	4.178

Table 2. The mean squares of x_1, x_2 versus b while $\lambda_1 = \lambda_2 = \omega_1 = \omega_2 = a = \alpha_1 = \alpha_2 = S_0 = 1$

b	$E\left\{x_1^2\right\}_{NL}$	$E\left\{x_1^2\right\}_{C}$	Err_C $\lvert\%\rvert$	$E\left\{x_1^2\right\}_{GL}$	Err_{GL} $\lvert\%\rvert$	$E\left\{x_2^2\right\}_{NL}$	$E\left\{x_2^2\right\}_{C}$	Err_C $\lvert\%\rvert$	$E\left\{x_2^2\right\}_{GL}$	Err_{GL} $\lvert\%\rvert$
1	0.49622	0.42145	15.068	0.48835	1.586	0.49622	0.36966	25.505	0.41928	15.505
10	0.36492	0.29566	18.980	0.33460	8.309	0.36492	0.29040	20.421	0.32769	10.202
50	0.33076	0.28086	15.086	0.31644	4.329	0.33076	0.28048	15.201	0.31597	4.472
100	0.32340	0.27930	13.636	0.31453	2.743	0.32340	0.27920	13.667	0.31440	2.783

From the relative errors of the approximate solutions with respect to the ones by ENL, it can be seen that GLOMSEC gives a significant improvement on accuracy of solution in comparison with the classical GEL, especially when the nonlinearity is strong.

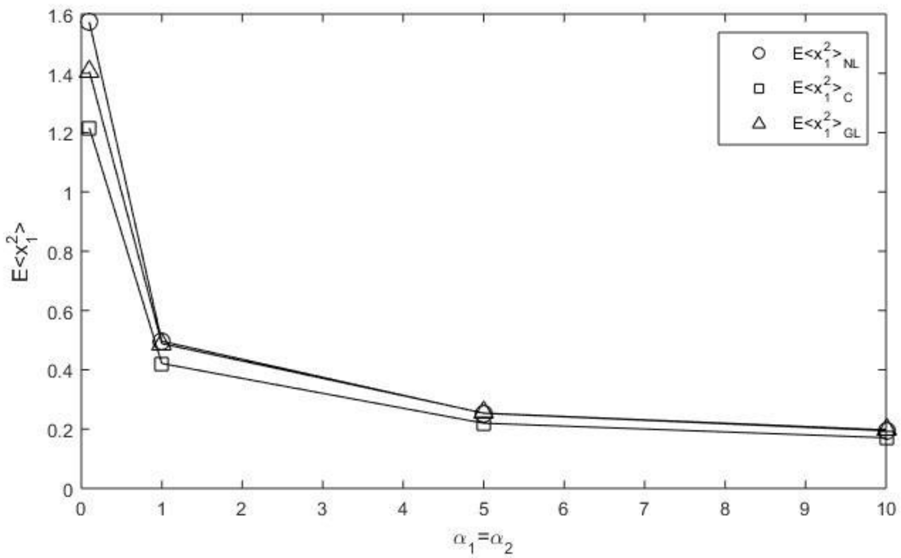

Fig. 1. The mean square of x_1 versus α

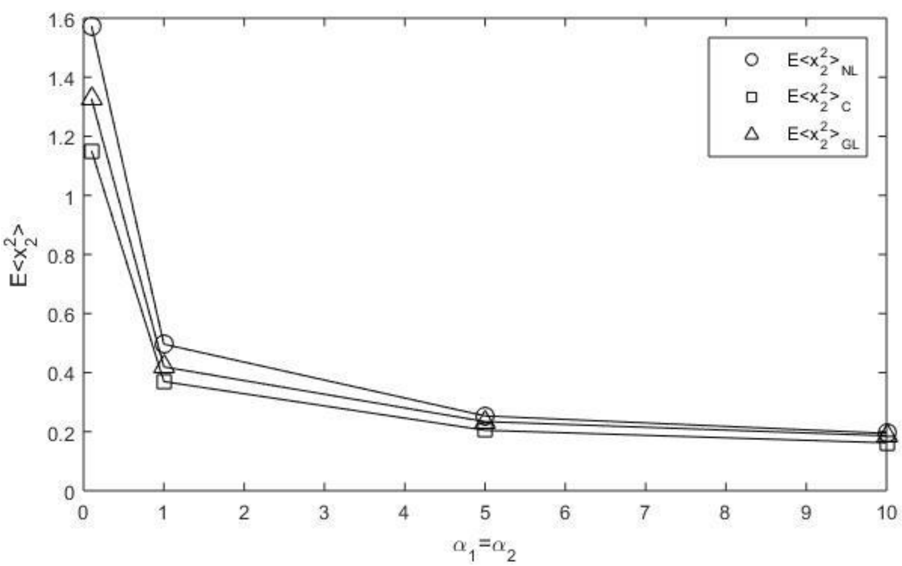

Fig. 2. The mean square of x_2 versus α

4. CONCLUSION

This paper presents the proposed criterion with its algorithm built to MDOF nonlinear oscillators under Gaussian white noise excitation. The mode of formulating algorithm is also mainly based on the classical GEL. However, a key problem is to determine the matrix of equivalent linearization coefficients in which the constant linearization coefficients are defined as global mean values of all local linearization coefficients. The paper is an additional research to our previous ones [14, 15] to aim at evaluating the improved performance of the proposed criterion; herein we analyse two applications, which are a rolling ship oscillation and two-degree-of-freedom one. The results show a significant improvement on accuracy of solutions by GLOMSEC in comparison with the ones by the classical GEL.

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APPENDIX

Suppose that the components of the vector $x = (x_1, x_2, \ldots, x_n)^T$ are zero-mean stationary Gaussian random variables. Denote $E\{.\}$ global mean values of random variables taken as follows

$$E\{.\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (.)\, p(x) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n, \tag{A.1}$$

where $p(x)$ is the stationary joint probability density function. For the Gaussian random processes with zero mean, one has the following general expressions for expectations [2]

$$E\{x_1 x_2 \ldots x_{2n+1}\} = 0, \quad E\{x_1 x_2 \ldots x_{2n}\} = \sum_{\text{all dependent pairs}} \left(\prod_{i \neq j} E\{x_i x_j\} \right), \tag{A.2}$$

where the number of independent pair is equal to $(2n)!/(2^n n!)$ For example,

$$\begin{aligned} &E\{x_1 x_2 x_3\} = 0, \\ &E\{x_1 x_2 x_3 x_4\} = E\{x_1 x_2\} E\{x_3 x_4\} + E\{x_2 x_3\} E\{x_1 x_4\} + E\{x_1 x_3\} E\{x_2 x_4\}, \\ &E\{x_1 x_2 x_3 x_4 x_5\} = 0. \end{aligned} \tag{A.3}$$

If x_i and x_j $(i \neq j)$ are uncorrelated, i.e. independent, then $E\{x_i x_j\} = 0$, and $E\{x_i^{2n+1} x_j^{2m+1}\} = 0$. Besides, formula (A.2) results in the following consequences

$$E\left\{x_i^{2n} x_j^{2m}\right\} = E\left\{x_i^{2n}\right\} E\left\{x_j^{2m}\right\} = (2n-1)!!\left(E\left\{x_i^2\right\}\right)^n (2m-1)!!\left(E\left\{x_j^2\right\}\right)^m, \tag{A.4}$$

where n and m are natural numbers. Denote $[.]$ the local mean values of random variables taken as follows.

$$E[.] = \int_{-r\sigma_{x1}}^{+r\sigma_{x1}} \cdots \int_{-r\sigma_{xn}}^{+r\sigma_{xn}} (.) p(x) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n, \tag{A.5}$$

where $\sigma_{x1}, \sigma_{x2} \ldots, \sigma_{xn}$ are the normal deviations of random variables, respectively, and r is a given positive value. Due to the symmetry of the expected integrations in (A.5), hereby (A.2) are also applied to the local mean values. If x_i and x_j $(i \neq j)$ are uncorrelated, i.e. independent, then $E\left[x_i x_j\right] = 0$, and $E\left[x_i^{2n+1} x_j^{2m+1}\right] = 0$. All higher even-order local means $E\left[x_i^{2n} x_j^{2m}\right]$ can be expressed in terms of second order global means $E\left\{x_i^2\right\}$ and $E\left\{x_j^2\right\}$ as follows [16].

$$E\left[x_i^{2n} x_j^{2m}\right] = E\left[x_i^{2n}\right] E\left[x_j^{2m}\right] = 2T_{n,r}\left(E\left\{x_i^2\right\}\right)^n 2T_{m,r}\left(E\left\{x_j^2\right\}\right)^m, \tag{A.6}$$

where

$$T_{n,r} = \int_0^r t^{2n}\eta(t)\mathrm{d}t, \quad T_{m,r} = \int_0^r t^{2m}\eta(t)\mathrm{d}t, \quad \eta(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}. \tag{A.7}$$

If $n = 0, m \neq 0$ or $n \neq 0, m = 0$, then (A.6) leads to the following results, respectively

$$E\left[x_i^0 x_j^{2m}\right] = 2T_{0,r} 2T_{m,r}\left(E\left\{x_j^2\right\}\right)^m, \; E\left[x_i^{2n} x_j^0\right] = 2T_{n,r}\left(E\left\{x_i^2\right\}\right)^n 2T_{0,r} \text{ with } T_{0,r} = \int_0^r \eta(t)\mathrm{d}t. \tag{A.8}$$

If $r \to \infty$, (A.5) and (A.7) will give the same result as (A.4) of the classical case.

A local mean of $x_i^2 |x_i|$ that arises in an application of the paper was presented in [15], the obtained result as follows

$$\begin{aligned} E\left[x_i^2 |x_i|\right] &= \int_{-r\sigma_{xi}}^{+r\sigma_{xi}} x_i^2 |x_i| \, p(x_i)\mathrm{d}x_i = 2\int_0^{+r\sigma_{xi}} x_i^3 p(x_i)\mathrm{d}x_i \\ &= 2\int_0^r t^3\sigma_{xi}^3 \frac{1}{\sqrt{2\pi}\sigma_{xi}} e^{-t^2\sigma_{xi}^2/2\sigma_{xi}^2}\sigma_{xi}\mathrm{d}t = 2\sigma_{xi}^3\int_0^r t^3\eta(t)\mathrm{d}t, \end{aligned} \tag{A.9}$$

$$E\left[x_i^2 |x_i|\right] = 2T_{t^3,r}\left(E\left\{x_i^2\right\}\right)^{3/2}, \quad T_{t^3,r} = \int_0^r t^3\eta(t)\mathrm{d}t.$$

If $x = (x_1, x_2, \ldots, x_n)^T$ is the displacement vector, then $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)^T$ is the velocity vector and we also obtain the same formulas, respectively, for the random variables of velocity.