Vietnam Journal of Mechanics, VAST, Vol. 39, No. 3 (2017), pp. 275 – 301
DOI:10.15625/0866-7136/10750

A ONE DIMENSIONAL VARIATIONAL MODEL OF SUPERELASTICITY FOR SHAPE MEMORY ALLOYS

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Received February 21, 2017

Abstract. In this paper we propose a variational framework for the modeling of superelasticity in shape memory alloys with softening behavior. This model is valid for a class of standard rate-independent materials with a single internal variable. The quasi-static evolution is based on two physical principles: a stability criterion which selects the local minima of the total energy and an energy balance condition which ensures the absolute continuity of the total energy. The stability criterion allows to bypass non-uniqueness issues associated to softening behaviour while the energy balance condition accounts for brutal evolutions at the local levels. We investigate properties of homogeneous and non-homogenous solutions towards this variational evolution problem. Specifically, we show how softening behaviour can lead to instability of the homogeneous states. In this latter case, we show that a stable solution would consist in following the Maxwell line given by the softening behaviour, then resulting in a non-homogeneous evolution.

Keywords: Shape memory alloys (SMA), superelasticity, softening, one dimensional model, energetic approach, stability criterion.

1. INTRODUCTION

The property of superelasticity of shape memory alloys (SMA) wires such as NiTi is now vastly employed in biomedical industry for applications such as orthopedic wires or stents. It allows, through an austenite to martensite phase transformation, to achieve high macroscopic strain (up to 6%) with fully recoverable initial state. Due to the dissipative nature of this phase transformation, the force-displacement response of a loading cycle shows an hysteresis loop. As a result, SMA are also used as dampers to prevent seismic vibrations.

The modeling of SMA has shown consistent increased interest during the last twenty years, both from the experimental and theoretical point of view. In particular, several one dimensional models have been proposed to account for the "exotic" behaviour of SMA e.g. superelaticity or shape-memory effect. These fundamental researches on the one-dimensional case are not only essential for a better understanding of the behavior of SMA but are in fact directly related to practical applications such as stent or orthopedic wires *i.e.* an assembly of one-dimensional systems. Most of uniaxial experiments that highlight the superelastic behavior of SMA consists in quasi-static uniaxial tension of NiTi wires or thin strips at prescribed displacement [1–5]. An important feature of their results is that the martensitic transformation is inhomogeneous. For sufficiently slow deformation loading rates (10^{-4} to 10^{-5}), it usually consists in one or two nucleation martensitic sites that grow and propagates continuously with the loading. The global response given by the stress-nominal strain diagram follows a three-steps scenario: i) an elastic phase associated to the austenite phase, ii) a sudden drop of the stress to a plateau stress which corresponds to the initiation of the martensite transformation; the stress remains at this plateau until the transformation is complete, iii) an elastic phase associated to the fully transformed martensite phase.

We usually distinguish two classes of constitutive models that account for superelasticity of SMA. Firstly, we have those for which the constitutive behavior is derived from a micro mechanical approach by means of homogenization techniques [6, 7]. An advantage of these models is to give a direct access to the evolution of the microstructure during the macroscopic phase transformation and to relate the effective properties at the macroscopic scale to physical material parameters at the microscopic scale. However, these multiscale approaches require usually too much computational efforts to be used for structural engineering calculations. In these latter cases, one would rather rely on macroscopic phenomenological models. Those models incorporate internal variables that describe the macroscopic phase transformation such as the martensitic volume fraction or the martensite orientation strain tensor. Starting with one-dimensional models of SMA [1, 8], these latter have been successively extended to the three dimensional case, including the shape memory effect (temperature induced transformation) [9, 10].

When the aforementioned models are used in a structural calculation, they predict a homogeneous or diffuse phase transformation, even in the simplified example of a one dimensional bar under tension. Indeed, the material parameters of these models are calibrated in such a way that the intrinsic behavior of the material displays a hardening response which stabilize homogeneous evolutions. However, homogeneous transformations is more the exception than the rule as it only occurs experimentally for a very short nominal strain range (less than 1%) that follows the austenite elastic phase [11]. On the contrary, localized transformations constitutes the main mode of phase change. To capture these localized phase transformation that are observed experimentally, one has to introduce a softening behavior at the local level in the model as it would to loss of ellipticity and localization of deformation. This has been confirmed by recent experiments on NiTi by [12] based on an original idea of [13]. It consists in controlling the strain at every point of a softening unidimensional specimen by bonding it to a hardening material such as stainless steel with the same geometry. As a result, the global behavior of this sandwich exhibits a structural hardening behavior which is associated to a diffuse transformation. By subtracting the response of the stainless steel, one has access to the intrinsic behavior of the softening material. In fact, such idea was used in a different context by [14] to isolate the underlying softening behavior of concrete. All the experimental evidences show that softening is a key ingredient for a correct modeling of phase transformation in SMA [15].

Introducing softening behavior in SMA models leads to theoretical issues such as non-convexity and non-uniqueness of the response. Based on Ericksen's paper [16], it is widely accepted that the homogeneous response given by a softening model is non-stable and that a stable response can be obtained by following the corresponding Maxwell line (equal area rule). However, the *qualitative* model of Ericksen formulated in a one dimensional context is a non-linear elastic model and does not introduce any dissipative processes that subtend macroscopic phase transformation in SMA. Softening in SMA can either introduced phenomenologically or through a multi-scale simulation. In this latter case, the softening is directly incorporated as an ingredient in the microscale behavior [17, 18]. This allows a better characterization of the formation of the microstructure by taking into account the interface energy between austenite and martensitic variants. Due to computational effort required in the calculation of the microscale behavior and its upscaling to the macroscale, these multi-scale approaches are not always suitable for structural calculations.

In this article, we propose to introduce softening at a phenomenological level by making use of energetic approaches that have been introduced in the context of rate-independent processes [19] such as fracture [20], damage mechanics [21] as well as plasticity [22] or phase transformation [23, 24]. They consist in formulating the quasi-static evolution problem in terms of the total energy of the problem based on two physical principles: a stability criterion which consists in selecting the local (or global) minima of the total energy of the system among admissible functional spaces and an energy balance condition which enforces the (absolute) continuity of the total energy with respect to the loading parameter. These two energetic principles are fundamental in the view of dealing with softening behavior. Indeed, the classical formulations of the macroscopic phase transformation in SMA are based on a Kuhn-Tucker formulation. Although such formulation is sufficient to ensure the well-posedness and uniqueness of the evolution problem for stress hardening materials, it shows some limitations when it comes to stress softening behavior. The first issue with softening is the non-uniqueness of the response due to the non-convexity of the model which allows to construct an infinity of solutions including homogeneous as well as localized responses which satisfy the Kuhn-Tucker conditions. The introduction of a stability criterion based on minima of the total energy aims to select only the physically acceptable solutions among those ones. The second issue with softening is the temporal discontinuities that can occur during a quasi-static evolution. Indeed, experimental and numerical evidences show that the nucleation of martensite can be a *brutal* phenomenon, resulting in a non-smooth evolution of the local strain and phase transformation fields with respect to time. However, the Kunh-Tucker consistency condition which involves the rate of the phase transformation variable implies that the evolution must be sufficiently smooth in time. By introducing an energy balance condition, we aim to generalize the Kunh-Tucker consistency condition to a larger class of evolution, including non-regular solutions. These two concepts, namely the stability criterion and the energy balance condition, make an extensive use of the total energy of the system, hence requiring a rigorous definition of this latter. The *standard* structure of the model introduced by [25] is the starting point of such energetic approach: the total energy of the system is obtained as the sum of the free energy and the dissipated energy. The goal of this paper is then to propose, in a one-dimensional context, an energetic framework for the quasi-static evolution of superelastic SMA: how this variational approach compares with previous formulations and what improvements it brings when dealing with softening SMA will be emphasized throughout the article.

The paper is organized as follows. In Section 2, we derive a one dimensional standard model of SMA by means of thermodynamic work property. Definitions of hardening and softening behaviors as well as energetic interpretations of material parameters are given based on the intrinsic response of the model. Section 3 is devoted to the study of a one dimensional superelastic SMA bar under tension. The evolution problem is formulated in terms of a stability criterion based on the selection of *local* minima of the total energy and a balance of energy. Stability of homogeneous states and localized states is investigated with respect to the intrinsic behavior of the material. Finally, evolution of localized states along the Maxwell line in the global diagram is shown to be a result of the energetic formulation of the problem.

The following notations are used: the dependence on the time parameter t is indicated by a subscript whereas the dependence on the spatial coordinate x is indicated classically by parentheses, *e.g.* $x \mapsto u_t(x)$ stands for the displacement field at time t. In general, the material functions of the phase transformation variable are represented by sans serif letters, like E, G or R. The prime denotes either the derivative with respect to x or the derivative with respect to the phase transformation variable, the dot stands for the time derivative, *e.g.* $u_t'(x) = \partial u_t(x)/\partial x$, $\mathsf{E}'(\alpha) = d\mathsf{E}(\alpha)/d\alpha$ or $\dot{u}_t(x) = \partial u_t(x)/\partial t$.

2. ONE DIMENSIONAL CONSTITUTIVE MODEL OF SHAPE MEMORY ALLOY

2.1. Derivation of phase transformation laws based on a Drucker-Ilyushin work property in Marigo [26]

We will consider only quasi-static mechanical loadings and neglect all inertia and viscous effects. All the processes are assumed to be isothermal. This is only an approximation since in practice the austenite to martensite phase transformation is an exothermic process. However, for sufficiently slow deformation loading rates (10^{-4} to 10^{-5}), the isothermal assumption is acceptable since the local change of temperature can be neglected due to the temperature diffusion. The small strain formalism is adopted as it is usually sufficient, at least in a first attempt, to capture most of the features of SMA *e.g.* super elasticity, hysteresis [27–29]. We will focus on the tensile behavior of the SMA in this paper and thus we will consider only positive strain values. Our one dimensional SMA model will be built within the class of rate-independent dissipative materials with internal variables. We start from the simplest assumption by considering two phases, an austenite phase and a single martensite phase. The transformation from one phase to the other is modeled by a unique scalar internal variable, called the phase transformation variable, and which satisfies the following hypothesis:

(1) The scalar phase transformation variable α lies in the interval $[\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$ where α_{A} and α_{M} are two real values such that $\alpha_{\mathsf{A}} < \alpha_{\mathsf{M}}$. When $\alpha = \alpha_{\mathsf{A}}$ (*resp.* $\alpha = \alpha_{\mathsf{M}}$), the material is in a fully austenite (*resp.* martensite) phase.
(2) For a given $\alpha \in [\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$, admissible strain states are bound to lie in a non-empty closed and connected subset $\mathbb{E}(\alpha)$ in the space of strain $\mathbb{R}^+$, the so-called *elastic domain*. It is such that the phase transformation variable α does not evolve as long as the strain path remains in the open subset $\mathring{\mathbb{E}}(\alpha)$. An evolution of the internal variable occurs only when the strain lies on the boundary of $\partial\mathbb{E}(\alpha)$.

Remark 1. *Strictly intermediate values of α between α_{A} and α_{M} describes a fine mixture of martensite and austenite phases at a level below the scale of macroscopic observation. As a result, α does not corresponds necessarily in this model to the volume fraction of one of the micro-constituent. A direct link between α and the underlying microstructure* e.g. *volume fraction of austenite or geometric distribution of phases, is only possible if α is upscaled from a homogenization process. Here, we rather choose to adopt from the start a purely macroscopic point of view and directly postulate the existence of α as well as its interval of definition $[\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$.*

[26] derived the constitutive behavior for such class of rate-independent materials based a thermodynamic principle called Drucker-Ilyushin work property. This thermodynamic principle is more restrictive than first and second thermodynamic principles as it allows to derive more information on the model. For instance, the author showed by means of the work property that the evolution of elastoplastic materials obeys to the maximal dissipation principle while the evolution of damage materials is ruled by the energy release rate. Therefore, the *standard*[1] evolution laws of the internal variables become a *consequence* of the work property and not a *postulate*, hence giving a physical justification to the convenient energetic frameworks. We recall here the definition of the work property and their consequences on the formulation of a SMA model. We invite the reader to refer to [26] for detailed proofs of the main results.

Let us call $\mathcal{C}$ a strain cycle performed during the time interval $t \in [0, T]$. The initial strain and phase transformation variable state at $t = 0$ reads $(\varepsilon(0), \alpha(0))$ while the final state at $t = T$ reads $(\varepsilon(1), \alpha(1))$, with the condition $\varepsilon(0) = \varepsilon(1)$ for cycle's closure. At any time t, the strain $\varepsilon(t)$ must be admissible *i.e.* $\varepsilon(t)$ must belong to the elastic domain $\mathbb{E}(\alpha(t))$ at any t. Then the work property can be formulated as follows

Definition 2.1. *The material satisfies the* work property *if and only if for any strain cycle $\mathcal{C}$ as described above, the total strain work in this cycle remains positive:*

$$\forall \mathcal{C}, \quad \int_{\mathcal{C}} \sigma \dot{\varepsilon} dt \geq 0. \tag{1}$$

Equipped with the work property, two fundamental properties are obtained. First, by considering strain cycles that lie in $\mathring{\mathbb{E}}(\alpha)$ *i.e.* reversible cycles, one obtains [30]

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[1] in the sense of [25]

Proposition 2.2. *There exists a potential* $(\varepsilon, \alpha) \mapsto \phi(\varepsilon, \alpha)$ *defined over* $\mathbb{R}^+ \times [\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$ *such that the stress-strain relation derives from this potential*

$$\sigma = \frac{\partial \phi}{\partial \varepsilon}(\varepsilon, \alpha). \tag{2}$$

The evolution laws of the internal variable are obtained by considering dissipative strain cycles for which the internal variable and the elastic domain evolve. The central result of [26] is that the elastic domain is characterized by an energy release rate inequality. Specifically we have

Proposition 2.3. *There exists a differentiable positive scalar function* $\alpha \mapsto \mathsf{R}(\alpha)$ *with* $\mathsf{R}'(\alpha) \geq 0$ *defined over* $[\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$ *such that the elastic domain* $\mathbb{E}(\alpha)$ *in the space of strain is given by*

$$\mathbb{E}(\alpha) = \begin{cases} \left\{\varepsilon \in \mathbb{R}^+ : -\frac{\partial \phi}{\partial \alpha}(\varepsilon, \alpha) \leq \mathsf{R}'(\alpha)\right\} & \text{if} \quad \alpha = \alpha_{\mathsf{A}}, \\ \left\{\varepsilon \in \mathbb{R}^+ : -\mathsf{R}'(\alpha) \leq -\frac{\partial \phi}{\partial \alpha}(\varepsilon, \alpha) \leq \mathsf{R}'(\alpha)\right\} & \text{if} \quad \alpha_{\mathsf{A}} < \alpha < \alpha_{\mathsf{M}}, \\ \left\{\varepsilon \in \mathbb{R}^+ : -\mathsf{R}'(\alpha) \leq -\frac{\partial \phi}{\partial \alpha}(\varepsilon, \alpha)\right\} & \text{if} \quad \alpha = \alpha_{\mathsf{M}}. \end{cases} \tag{3}$$

Moreover the evolution of the internal variable satisfies the following consistency equation

$$\left(-\frac{\partial \phi}{\partial \alpha}\right)\dot{\alpha} = \mathsf{R}'(\alpha)|\dot{\alpha}| \quad \Leftrightarrow \quad \begin{cases} \dot{\alpha} > 0 \Rightarrow -\frac{\partial \phi}{\partial \alpha}(\varepsilon, \alpha) = \mathsf{R}'(\alpha), \\ \dot{\alpha} < 0 \Rightarrow \frac{\partial \phi}{\partial \alpha}(\varepsilon, \alpha) = \mathsf{R}'(\alpha). \end{cases} \tag{4}$$

where $\dot{\alpha}$ *stands as the right derivative in time of* α*.*

Note that the second thermodynamics is automatically satisfied with this class of model as the intrinsic dissipation rate $\dot{\mathcal{D}}$, given by

$$\dot{\mathcal{D}} = \sigma\dot{\varepsilon} - \dot{\phi} = -\frac{\partial \phi}{\partial \alpha}\dot{\alpha} = \mathsf{R}'(\alpha)|\dot{\alpha}|, \tag{5}$$

is positive. To go further in the characterization of the free potential ϕ and in accordance with the small deformation assumption, let us perform a Taylor expansion of $\phi(\cdot, \alpha)$ in ε near 0. Up to the second order ε, the Taylor expansion can be always cast into the following form

$$\phi(\varepsilon, \alpha) = \frac{1}{2}\mathsf{E}(\alpha)(\varepsilon - \mathsf{p}(\alpha))^2 + \mathsf{G}(\alpha), \tag{6}$$

where E, p and G are material functions which only depend on α. Given the expansion (6), the stress-strain relation reads

$$\sigma = \mathsf{E}(\alpha)(\varepsilon - \mathsf{p}(\alpha)). \tag{7}$$

The parameters E and p are depicted on Fig. 1 where is represented a typical stress-strain response of a volume element of shape memory alloys. For a given value α of the phase transformation variable, the parameter $\mathsf{E}(\alpha)$ corresponds to the Young modulus of the transformed material while $\mathsf{p}(\alpha)$ is the transformation strain associated to the macroscopic orientation of the austenite-martensite mixture. As the the austenite phase is stiffer than the martensite phase, we have $\mathsf{E}(\alpha_{\mathsf{A}}) = \mathsf{E}_{\mathsf{A}} > \mathsf{E}(\alpha_{\mathsf{M}}) = \mathsf{E}_{\mathsf{M}} > 0$. Moreover, since the austenite phase is free of any transformation strain, then $\mathsf{p}(\alpha_{\mathsf{A}}) = 0$. Finally, as ϕ is determined up to a constant, we will assume that $\mathsf{G}(\alpha_{\mathsf{A}}) = 0$.

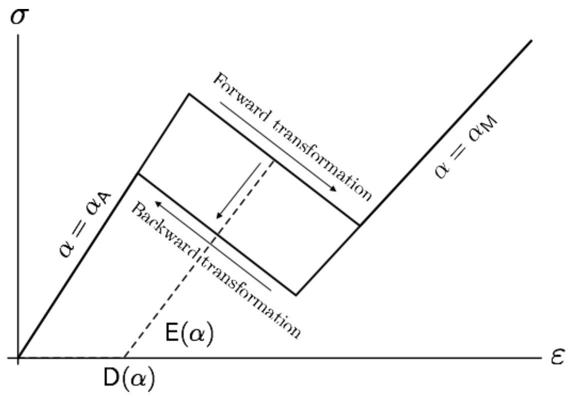

Fig. 1. Stress-strain response of a material point

In accordance with the superelastic behavior of SMA, let us now make extra hypothesis on the material functions E, p, G and R for intermediate values between α_{A} and α_{M}:

Hypothesis 2.4. *We assume that all the material functions are twice differentiable. In addition, we suppose* $\alpha \mapsto \mathsf{E}(\alpha)$ *to be a decreasing function while* $\alpha \mapsto \mathsf{p}(\alpha)$*,* $\alpha \mapsto \mathsf{R}(\alpha)$ *and* $\alpha \mapsto \mathsf{G}(\alpha) - \mathsf{R}(\alpha)$ *to be increasing functions over* $[\alpha_{\mathsf{A}}, \alpha_{\mathsf{M}}]$ i.e.

$$\mathsf{E}' < 0, \quad \mathsf{p}' > 0, \quad \mathsf{R}' > 0, \quad \mathsf{G}' - \mathsf{R}' > 0. \tag{8}$$

The positivity of E' means that the Young's modulus of a microstructural mixture of austenite and martensite is decreasing from E_{A} to E_{M} as the the material is progressively transformed. The negativity of p' signifies that the phase transformation strain is increasing during the transformation from 0 to $\mathsf{p}(1)$. To explain the assumptions on the monotonicity of R and $\mathsf{G} - \mathsf{R}$, let us analyze the stress-strain diagram in which those quantities have an energetic interpretation. First, let us compute the work of strain throughout a strain cycle $t \mapsto \varepsilon_t^*$. The phase transformation variable evolution during this cycle is denoted $t \mapsto \alpha_t^*$. We suppose that the material point is initially at rest and the state of strain and phase transformation is $(0, \alpha_{\mathsf{A}})$. We then apply a monotonically increasing strain loading up to the state (ε, α). Finally, we close the cycle by unloading the applied strain until we reach the initial state $(0, \alpha_{\mathsf{A}})$. For this cycle, we suppose that the phase transformation variable is increasing during the loading stage and decreasing during the unloading stage[2]. The total strain work performed during this cycle $\mathcal{C}$ reads

$$\int_{\mathcal{C}} \sigma_t \dot{\varepsilon}_t^* dt = \int_{\mathcal{C}} \frac{\partial \phi}{\partial \varepsilon}(\varepsilon_t^*, \alpha_t^*) \dot{\varepsilon}_t^* dt = \underbrace{\int_{\mathcal{C}} \frac{d}{dt}(\phi(\varepsilon_t^*, \alpha_t^*)) dt}_{=0 \text{ (cycle)}} - \int_{\mathcal{C}} \frac{\partial \phi}{\partial \alpha}(\varepsilon_t^*, \alpha_t^*) \dot{\alpha}_t^* dt \tag{9}$$

$$= \int_{\mathcal{C}} \mathsf{R}'(\alpha_t^*) |\dot{\alpha}_t^*| dt = \underbrace{(\mathsf{R}(\alpha) - \mathsf{R}(\alpha_{\mathsf{A}}))}_{\dot{\alpha}>0 \text{ (loading)}} - \underbrace{(\mathsf{R}(\alpha_{\mathsf{A}}) - \mathsf{R}(\alpha))}_{\dot{\alpha}<0 \text{ (unloading)}} = 2\mathsf{R}(\alpha).$$

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[2] This assumption will be enforced later through the Hypothesis 2.7 on by imposing an strain-hardening behavior to the SMA.

Therefore, $2\mathsf{R}(\alpha)$ represents the total energy dissipated by the phase transformation process during a cycle where the phase transformation variable reaches the value α. As a result, it makes physically sense to make the growth assumption on R'. To give more insight now on $\mathsf{G}-\mathsf{R}$, let us compute the total strain work during the loading stage only *i.e.* from the initial state $(0,\alpha_\mathsf{A})$ to the intermediary state (ε,α)

$$\begin{aligned}\int_{(0,0)}^{(\varepsilon,\alpha)} \sigma_t \dot{\varepsilon}_t^* dt &= \int_{(0,0)}^{(\varepsilon,\alpha)} \frac{d}{dt}(\phi(\varepsilon_t^*,\alpha_t^*))dt - \int_{(0,0)}^{(\varepsilon,\alpha)} \frac{\partial\phi}{\partial\alpha}(\varepsilon_t^*,\alpha_t^*)\dot{\alpha}^* dt \\ &= (\phi(\varepsilon,\alpha)-\phi(0,\alpha_\mathsf{A})) + (\mathsf{R}(\alpha)-\mathsf{R}(\alpha_\mathsf{A})) \\ &= \frac{1}{2}\mathsf{E}(\alpha)(\varepsilon-\mathsf{p}(\alpha))^2 + \mathsf{G}(\alpha) + \mathsf{R}(\alpha).\end{aligned} \tag{10}$$

Knowing that $\frac{1}{2}\mathsf{E}(\alpha)(\varepsilon-\mathsf{p}(\alpha))^2$ is the elastic energy stored by the material point in the state (ε,α), the functions $\mathsf{G}-\mathsf{R}$ can be directly seen in the stress-strain diagram in terms of surface area, see Fig. 2.

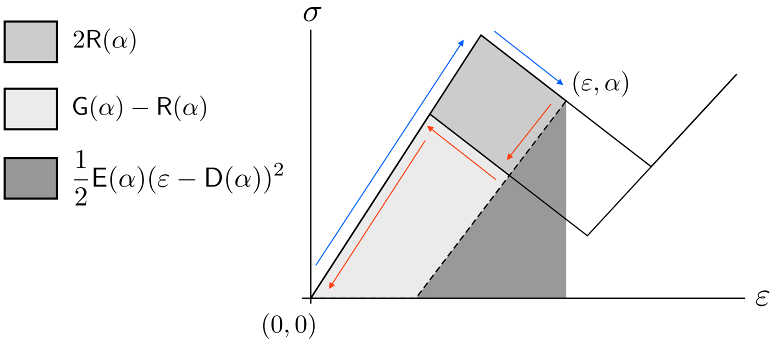

Fig. 2. Energetic interpretation of the material functions from a partial phase transformation in the intrinsic stress-strain diagram

Therefore, while 2R was linked to the dissipated energy by the phase transformation, $\mathsf{G}-\mathsf{R}$ represents the total latent heat due to the exothermic forward phase transformation and which is fully recoverable upon an endothermic reverse phase transformation. As a result, the monotonicity assumption on $\mathsf{G}-\mathsf{R}$ is physically acceptable with respect to the energetic characteristics of SMA.

2.2. Choice of the internal variable

Up to now, the one-dimensional model of SMA is fully characterized by the knowledge of α_A and α_M as well as the four material functions $\mathsf{E}(\alpha)$, $\mathsf{p}(\alpha)$, $\mathsf{G}(\alpha)$ and $\mathsf{R}(\alpha)$. We now illustrate how a change of variable on the phase transformation variable α can lead to further simplifications by reducing the number of material parameters to be identified. Let us consider two examples of change of variable.

Dissipated energy as the phase transformation variable A first possible choice is to choose the dissipated energy $\alpha \mapsto 2\mathsf{R}(\alpha)$ as the new phase transformation variable. This change of variable is legit since, in virtue of the Hypothesis **2.4**, we have $\mathsf{R}' > 0$. Let us introduce

$2\check{\mathsf{R}}_{\mathsf{M}} = 2\mathsf{R}(\alpha_{\mathsf{M}})$ the total energy dissipated by a full phase transformation cycle. Providing a renormalization by $2\check{\mathsf{R}}_{\mathsf{M}}$, the new phase transformation variable d reads

$$d : \alpha \mapsto \frac{\mathsf{R}(\alpha)}{\check{\mathsf{R}}_{\mathsf{M}}}, \quad d(\alpha_{\mathsf{A}}) = 0, \quad d(\alpha_{\mathsf{M}}) = 1. \tag{11}$$

Based on the new phase transformation variable d, the model is now defined by the free potential $\check{\varphi}$ based on the three material functions $\check{\mathsf{E}}$, $\check{\mathsf{p}}$ and $\check{\mathsf{G}}$ and the yield material parameter $\check{\mathsf{R}}_{\mathsf{M}}$

$$\check{\varphi}(\varepsilon, d) = \frac{1}{2}\check{\mathsf{E}}(d)(\varepsilon - \check{\mathsf{p}}(d))^2 + \check{\mathsf{G}}(d), \qquad \check{\mathsf{R}}(d) = \check{\mathsf{R}}_{\mathsf{M}} d. \tag{12}$$

Transformation strain as the phase transformation variable Another convenient choice is to choose instead the transformation strain $\mathsf{p}(\alpha)$ as the new phase transformation variable. Let us call $\mathsf{p}_1 = \mathsf{p}(\alpha_{\mathsf{M}})$ the transformation strain associated to the orientation of the martensite phase when the phase transformation is completed. Since $\mathsf{p}' > 0$ in virtue of the Hypothesis 2.4, this change of variable is also possible. In this case, the new phase transformation variable z reads after renormalization by p_1

$$z : \alpha \mapsto \frac{\mathsf{p}(\alpha)}{\mathsf{p}_1}, \quad z(\alpha_{\mathsf{A}}) = 0, \quad z(\alpha_{\mathsf{M}}) = 1. \tag{13}$$

Based on the new phase transformation variable z, the model is defined by the following $\hat{\varphi}$ based on the two material functions $\hat{\mathsf{E}}$ and $\hat{\mathsf{G}}$ and the yield function $\hat{\mathsf{R}}$

$$\hat{\varphi}(\varepsilon, z) = \frac{1}{2}\hat{\mathsf{E}}(z)(\varepsilon - \mathsf{p}_1 z)^2 + \hat{\mathsf{G}}(z), \qquad \hat{\mathsf{R}}(z). \tag{14}$$

The model after this change of variable is characterized by three material functions $\hat{\mathsf{E}}$, $\hat{\mathsf{G}}$, $\hat{\mathsf{R}}$ and the yield material parameter p_1.

In the rest of the study, we will adopt this change of variable related to the transformation strain. With this change of variable, the elastic domain in the space of strain reads

$$\hat{\mathbb{E}}(z) = \begin{cases} \left\{\varepsilon \in \mathbb{R}^+ : -\frac{\partial\hat{\varphi}}{\partial\alpha}(\varepsilon, z) \leq \hat{\mathsf{R}}'(z)\right\} & \text{if} \quad z = 0, \\ \left\{\varepsilon \in \mathbb{R}^+ : \left| -\frac{\partial\hat{\varphi}}{\partial\alpha}(\varepsilon, z)\right| \leq \hat{\mathsf{R}}'(z)\right\} & \text{if} \quad 0 < z < 1, \\ \left\{\varepsilon \in \mathbb{R}^+ : -\hat{\mathsf{R}}'(z) \leq -\frac{\partial\hat{\varphi}}{\partial z}(\varepsilon, z)\right\} & \text{if} \quad z = 1. \end{cases} \tag{15}$$

while the stress-strain relation and consistency condition are given by

$$\sigma = \frac{\partial\hat{\varphi}}{\partial\varepsilon}(\varepsilon, z), \qquad \left(-\frac{\partial\hat{\varphi}}{\partial z}\right)\dot{z} = \hat{\mathsf{R}}'(z)|\dot{z}|. \tag{16}$$

2.3. Intrinsic response, strain hardening and stress softening properties

We establish in this section the intrinsic response of the material from which we will define the hardening and softening properties of the model. Within a strain cycle the response of the material point during the loading/unloading phase is given by

Proposition 2.5. *Let us introduce the fours functions* $z \mapsto \bar{\varepsilon}_{\mathsf{AM}}(z)$, $z \mapsto \bar{\sigma}_{\mathsf{AM}}(z)$, $z \mapsto \bar{\varepsilon}_{\mathsf{MA}}(z)$ *and* $z \mapsto \bar{\sigma}_{\mathsf{MA}}(z)$ *on* $[0,1]$ *as follows*

$$\bar{\varepsilon}_{\mathsf{AM}}(z) = z\mathsf{p}_1 - \frac{\mathsf{p}_1\hat{\mathsf{E}}(z)}{\hat{\mathsf{E}}'(z)}\left(\sqrt{1+2\hat{\mathsf{S}}'(z)\frac{\hat{\mathsf{G}}'(z)+\hat{\mathsf{R}}'(z)}{\mathsf{p}_1^2}}-1\right), \quad \bar{\sigma}_{\mathsf{AM}}(z) = \hat{\mathsf{E}}(z)(\bar{\varepsilon}_{\mathsf{AM}}(z) - z\mathsf{p}_1), \tag{17}$$

$$\bar{\varepsilon}_{\mathsf{MA}}(z) = z\mathsf{p}_1 - \frac{\mathsf{p}_1\hat{\mathsf{E}}(z)}{\hat{\mathsf{E}}'(z)}\left(\sqrt{1+2\hat{\mathsf{S}}'(z)\frac{\hat{\mathsf{G}}'(z)-\hat{\mathsf{R}}'(z)}{\mathsf{p}_1^2}}-1\right), \quad \bar{\sigma}_{\mathsf{MA}}(z) = \hat{\mathsf{E}}(z)(\bar{\varepsilon}_{\mathsf{MA}}(z) - z\mathsf{p}_1). \tag{18}$$

Then the response of the material point during the loading phase (ε increasing) follows three successive stages:

(1) Austenite phase: $\varepsilon \in [0, \bar{\varepsilon}_{\mathsf{AM}}(0)]$. *At the onset of the loading, the response of the material is first elastic. The phase transformation variable is equal to 0 and the stress-strain relation reads*

$$\sigma = \mathsf{E}_{\mathsf{A}}\varepsilon. \tag{19}$$

(2) Forward phase transformation: $\varepsilon \in [\bar{\varepsilon}_{\mathsf{AM}}(0), \bar{\varepsilon}_{\mathsf{AM}}(1)]$. *The stress, the strain and the phase transformation variable are linked during the forward phase transformation through the relations*

$$\varepsilon = \bar{\varepsilon}_{\mathsf{AM}}(z_\varepsilon), \quad \sigma = \bar{\sigma}_{\mathsf{AM}}(z_\varepsilon). \tag{20}$$

(3) Martensite phase: $\varepsilon \in [\bar{\varepsilon}_{\mathsf{AM}}(1), +\infty)$. *The response becomes again elastic. The phase transformation variable is equal to 1 and the stress-strain relation reads*

$$\sigma = \mathsf{E}_{\mathsf{M}}(\varepsilon - \mathsf{p}_1). \tag{21}$$

By unloading the material point in its martensite phase (ε decreasing), the response of the material point follows also three successive stages:

(1) Martensite phase: $\varepsilon \in [\bar{\varepsilon}_{\mathsf{MA}}(1), +\infty)$. *At the onset of the loading, the response of the material is first elastic. The phase transformation variable is equal to 0, while the stress is given by*

$$\sigma = \mathsf{E}_{\mathsf{M}}(\varepsilon - \mathsf{p}_1). \tag{22}$$

(2) Backward phase transformation: $\varepsilon \in [\bar{\varepsilon}_{\mathsf{MA}}(0), \bar{\varepsilon}_{\mathsf{MA}}(1)]$. *The stress, the strain and the phase transformation variable are linked during the backward phase transformation through the relations*

$$\varepsilon = \bar{\varepsilon}_{\mathsf{MA}}(z_\varepsilon), \quad \sigma = \bar{\sigma}_{\mathsf{MA}}(z_\varepsilon). \tag{23}$$

(3) Austenite phase: $\varepsilon \in [0, \bar{\varepsilon}_{\mathsf{MA}}(0)]$. *The response becomes again elastic. The phase transformation variable is equal to 1, while the stress is given by*

$$\sigma = \mathsf{E}_{\mathsf{A}}\varepsilon. \tag{24}$$

Proof. At the onset of the loading, the response of the material is elastic and no transformation occurs as long as the forward phase transformation criterion is a strict inequality. This elastic phase ends when the criterion becomes an equality *i.e.* when ε

reaches the upper bound of the elastic domain $\hat{\mathbb{E}}(0)$ given by $-\frac{\partial\hat{\phi}}{\partial\alpha}(\varepsilon,0) = \hat{\mathsf{R}}'(0)$. This leads to a second order polynomial equation $-\frac{1}{2}\hat{\mathsf{E}}'(0)\varepsilon^2 + \hat{\mathsf{E}}(0)\mathsf{p}_1\varepsilon - \hat{\mathsf{G}}'(0) = \hat{\mathsf{R}}'(0)$ to be solved. In virtue of the Hypothesis **2.4** and given the change of variable (13), we have $(\hat{\mathsf{G}} - \hat{\mathsf{R}})' > 0$ and *a fortiori* $(\hat{\mathsf{G}} + \hat{\mathsf{R}})' > 0$ since $\hat{\mathsf{R}}' > 0$. The polynomial equation has then a unique positive solution which corresponds to $\bar{\varepsilon}_{\mathsf{AM}}(0)$. For $\varepsilon > \bar{\varepsilon}_{\mathsf{AM}}(0)$, the forward phase transformation starts to occur. In virtue of the consistency condition (16), the forward phase transformation criterion is an equality during the transformation and we have $-\frac{1}{2}\hat{\mathsf{E}}'(z_\varepsilon)(\varepsilon - \mathsf{p}_1 z_\varepsilon)^2 + \hat{\mathsf{E}}(z_\varepsilon)\mathsf{p}_1(\varepsilon - \mathsf{p}_1 z_\varepsilon) - \hat{\mathsf{G}}'(z_\varepsilon) = \hat{\mathsf{R}}'(z_\varepsilon)$. Again, since $(\hat{\mathsf{G}} + \hat{\mathsf{R}})' > 0$, this second order polynomial equation in ε has a unique positive root which corresponds to $\bar{\varepsilon}_{\mathsf{AM}}(z_\varepsilon)$. When $z_\varepsilon = 1$, no constraint is imposed by the forward criterion (15) for subsequent loading and the response is again elastic.
During the unloading stage, same arguments apply to the backward phase transformation. It starts when the criterion becomes an equality *i.e.* when ε reaches the lower bound of the elastic domain $\hat{\mathbb{E}}(1)$ given by $-\frac{\partial\hat{\phi}}{\partial\alpha}(\varepsilon,1) = -\hat{\mathsf{R}}'(1)$. This leads to a second order polynomial equation $-\frac{1}{2}\hat{\mathsf{E}}'(1)\varepsilon^2 + \hat{\mathsf{E}}(1)\mathsf{p}_1\varepsilon - \hat{\mathsf{G}}'(1) = -\hat{\mathsf{R}}'(1)$. This polynomial equation has a unique positive solution in virtue of the Hypothesis **2.4** and it corresponds to $\bar{\varepsilon}_{\mathsf{MA}}(1)$. By releasing the load from $\varepsilon = \bar{\varepsilon}_{\mathsf{MA}}(1)$, the backward phase transformation occurs and in virtue of the consistency condition (16), the backward phase transformation criterion is an equality during the transformation *i.e.* $-\frac{1}{2}\hat{\mathsf{E}}'(z_\varepsilon)(\varepsilon - \mathsf{p}_1 z_\varepsilon)^2 + \hat{\mathsf{E}}(z_\varepsilon)\mathsf{p}_1(\varepsilon - \mathsf{p}_1 z_\varepsilon) - \hat{\mathsf{G}}'(z_\varepsilon) = -\hat{\mathsf{R}}'(z_\varepsilon)$. This polynomial equation in ε has a unique positive root which corresponds to $\bar{\varepsilon}_{\mathsf{MA}}(z_\varepsilon)$. When $z_\varepsilon = 0$, no constraint is imposed anymore by the backward criterion (15) for further unloading and the response becomes again elastic. ☐

Knowing the intrinsic response of the material, let us now introduce

Definition 2.6. ***Hardening and softening phase transformation behavior.*** *The forward* (resp. *backward*) *phase transformation is said to be strain-hardening if* $\bar{\varepsilon}'_{\mathsf{AM}}(z) > 0$ (resp. $\bar{\varepsilon}'_{\mathsf{MA}}(z) > 0$)*, stress-hardening if* $\bar{\sigma}'_{\mathsf{AM}}(z) > 0$ (resp. $\bar{\sigma}'_{\mathsf{MA}}(z) > 0$) *and stress-softening if* $\bar{\sigma}'_{\mathsf{AM}}(z) < 0$ (resp. $\bar{\sigma}'_{\mathsf{MA}}(z) < 0$) *for* $z \in [0,1]$.

To ensure that no snap-back occurs in the material response during the forward/backward phase transformation, let us make

Hypothesis 2.7. *The forward and backward phase transformations are supposed to be strain-hardening.*

Under this assumption, the response of a material point during a strain cycle is necessarily unique since $\bar{\varepsilon}_{\mathsf{AM}}$ and $\bar{\varepsilon}_{\mathsf{MA}}$ are invertible. In particular, the stress-strain response during the forward transformation and backward transformation are given by $\varepsilon \mapsto \bar{\sigma}_{\mathsf{AM}}((\bar{\varepsilon}_{\mathsf{AM}})^{-1}(\varepsilon))$ and $\varepsilon \mapsto \bar{\sigma}_{\mathsf{MA}}((\bar{\varepsilon}_{\mathsf{MA}})^{-1}(\varepsilon))$, respectively.

2.4. Identification of the material functions

Let us consider the following constitutive law used in the literature [10] for which the material functions are defined as follows

$$\hat{\mathsf{E}}(z) = \frac{1}{\frac{z}{\mathsf{E_M}} + \frac{1-z}{\mathsf{E_A}}}, \quad \hat{\mathsf{G}}(z) = \hat{\mathsf{G}}_0 z + \hat{\mathcal{H}}(z), \quad \hat{\mathsf{R}}(z) = \hat{\mathsf{R}}_0 z. \tag{25}$$

Specifically, the stiffness of the mixture of austenite and martensite corresponds here to a model in series, where z plays the volume fraction of martensite. The material function $\hat{\mathcal{H}}$ is usually interpreted as an *interaction potential* between phases that satisfies $\hat{\mathcal{H}}(0) = \hat{\mathcal{H}}(1) = 0$ while $\hat{\mathsf{G}}_0$ is the heat density associated to the phase change. Based on Proposition 2.5, the softening properties will depend on the sign of the second order derivative of the interaction potential $\hat{\mathcal{H}}$. In particular, the forward *and* the backward phase transformation have a softening character if and only if $\hat{\mathcal{H}}$ is concave. Therefore, the form of the *interaction potential* plays a keyrole in the hardening/softening properties of the SMA which in return will dictate the well-posedness behavior of the problem as we will see in the next section.

3. EVOLUTION PROBLEM OF A ONE DIMENSIONAL BAR

We consider a one dimensional SMA bar $(0, L)$ of length L and axial coordinate $x \in [0, L]$ whose displacements are prescribed at the extremities of the bar. The end $x = 0$ of the bar is fixed, whereas the displacement of the end $x = L$ is prescribed to a value U_t depending on an increasing parameter t which plays the role of the "time". The nominal strain is then given by $\bar{\varepsilon}_t = U_t/L$. For this one dimensional problem, the affine space of admissible displacement field $\mathcal{C}_t$ and the convex space of admissible phase transformation variable $\mathcal{Z}_0$ read

$$\mathcal{C}_t = \{u \in H^1(0, L) : u(0) = 0, u(L) = U_t\}, \quad \mathcal{Z}_0 = L^\infty((0, L), [0, 1]), \tag{26}$$

where $H^1(0, L)$ is the space of functions that are square integrable and whose first derivatives are square integrable while $L^\infty((0, L), [0, 1])$ is the space of functions which are bounded by 0 and 1 almost everywhere. For a given $z \in \mathcal{Z}_0$, let us associate the space of admissible direction test $\mathcal{Z}(z)$

$$\mathcal{Z}(z) = \{\beta \in L^\infty((0, L), \mathbb{R}) : \beta \geq 0 \text{ where } z = 0, \quad \beta \leq 0 \text{ where } z = 1\}. \tag{27}$$

We aim here to provide an energetic formulation of the evolution problem to tackle possible non-uniqueness and stability issues of the system associated to softening behavior. Let us first determine the total energy of the body when this latter is loaded from 0 to time t. Considering a material point of the bar, the total strain work to load this material point reads

$$\begin{aligned} W_t(\varepsilon_t, z_t) &= \int_0^t \sigma_\tau \dot{\varepsilon}_\tau d\tau = \int_0^t \Big(\dot{\hat{\phi}}(\varepsilon_\tau, z_\tau) - \frac{\partial \hat{\phi}}{\partial z}(\varepsilon_\tau, z_\tau) \dot{z}_\tau \Big) d\tau \\ &= \int_0^t \Big(\dot{\hat{\phi}}(\varepsilon_\tau, z_\tau) + \hat{\mathsf{R}}'(z)|\dot{z}| \Big) d\tau \\ &= \hat{\phi}(\varepsilon_t, z_t) + \int_0^t \hat{\mathsf{R}}'(z_\tau)|\dot{z}_\tau| d\tau, \end{aligned} \tag{28}$$

where the integral term $\int_0^t \hat{\mathsf{R}}'(z)|\dot{z}|d\tau$ corresponds to the total dissipated energy during the whole process. The total energy of the bar $\mathcal{P}_t(u_t, z_t)$ in a displacement configuration $u_t \in \mathcal{C}_t$ and phase transformation variable $z_t \in \mathcal{Z}_0$ is then defined as the integral over the domain of the density of strain work

$$\mathcal{P}_t(u_t, z_t) = \int_0^L \mathsf{W}_t(u_t', z_t)dx = \mathcal{E}_t(u_t, z_t) + \int_0^L \int_0^t \hat{\mathsf{R}}'(z_\tau)|\dot{z}_\tau| d\tau dx, \tag{29}$$

where

$$\mathcal{E}_t(u_t, z_t) = \int_0^L \hat{\varphi}(u_t', z_t)dx, \tag{30}$$

corresponds to the total free energy of the system. Note that expression (29) is valid only when the evolution is "sufficiently" smooth in time as it involves the derivative in time of the phase transformation variable $\dot{z}_t$. However, smoothness of the evolution does not always hold as time discontinuities might occur in a quasi-static setting. A more general expression of the total dissipated energy that encompasses those possible time discontinuities consists in rewriting the total dissipated energy of the system as

$$\mathsf{Diss}(z_t, [0,t]) = \sup_{(\tau_i)} \sum_{i=1}^{M} \int_0^L |\hat{\mathsf{R}}(z_{\tau_i}) - \hat{\mathsf{R}}(z_{\tau_{i-1}})|dx, \tag{31}$$

where the supremum is taken over all $M \in \mathbb{N}$ and all discretizations $0 = \tau_1 < \ldots < \tau_M = t$. In particular, $\mathsf{Diss}(z, [0,t])$ coincides precisely with $\int_0^t \int_0^L \hat{\mathsf{R}}'(z_\tau)|\dot{z}_\tau| dx d\tau$ for smooth evolutions while keeping a meaning for non-smooth evolutions. Then the final expression of the total energy of the state (u_t, z_t) reads

$$\mathcal{P}_t(u_t, z_t) = \mathcal{E}_t(u_t, z_t) + \mathsf{Diss}(z_t, [0,t]). \tag{32}$$

3.1. The stability criterion and energy balance condition

An admissible state (u_t, z_t) will be called a solution of the problem if and only if it satisfies the two following conditions for any t:

$$\textbf{(S)} : \quad \begin{cases} \forall(v,\beta) \in \mathcal{C}_0 \times \mathcal{Z}(z_t), \exists r^* > 0, \forall h \in [0, r^*), (u_t + hv, z_t + h\beta) \in \mathcal{C}_t \times \mathcal{Z}_0, \\ \mathcal{P}_t(u_t, z_t) \leq \mathcal{E}_t(u_t + hv, z_t + h\beta) + \mathsf{Diss}(z_t, [0,t]) + \int_0^L |\hat{\mathsf{R}}(z_t + h\beta) - \hat{\mathsf{R}}(z_t)|dx, \end{cases} \tag{33}$$

$$\textbf{(E)} : \quad \mathcal{P}_t(u_t, z_t) = \mathcal{P}_0(u_0, z_0) + \int_0^t \sigma_{t^*} \dot{U}_{t^*} dt^*. \tag{34}$$

The first item **(S)** of the evolution problem is a directional stability condition which consists in selecting the *local* minimizers of the total energy as the stable states. Indeed, the right-hand side of (33) can be seen as the total energy of a neighboring state $(u_t + hv, z_t + h\beta)$ under specific assumptions. Let us assume a *virtual* evolution of the system at *fixed* loading, starting from the state (u_t, z_t) and ending at the state $(u_t + hv, z_t + h\beta)$. Let us call t and $t + \Delta t^*$, the starting and finishing time of this virtual evolution. We assume this evolution is short enough ($\Delta t^* \ll 1$) so that the phase transformation has a monotonous evolution. As a result, the dissipated energy due to the phase transformation between t and $t + \Delta t^*$ reads $\int_0^L |\hat{\mathsf{R}}(z_t + h\beta) - \hat{\mathsf{R}}(z_t)|dx$. Therefore, the stability criterion consists in selecting the states that minimize locally the total energy of the system among the neighboring states that can be reached monotonously with respect to the phase transformation variable.

The second item **(E)** is an energy balance condition that enforces the total energy of the system (the sum of the total free energy and the total dissipated energy) to be an absolutely continuous function of time equals to the sum of the initial total energy and the total work of external loads from 0 to t. This energetic formulation consisting in **(S)** and **(E)** will be shown to be particularly suited for handling softening behavior as well as non-smooth evolution.

Given the stability criterion **(S)**, we now characterize further the stable states by doing a Taylor expansion of the stability criterion around the state (u_t, z_t) for a given test direction $(v, \beta) \in \mathcal{C}_0 \times \mathcal{Z}(z_t)$. Firstly, the stability criterion is equivalent to

$$\mathcal{E}_t(u_t, z_t) - \mathcal{E}_t(u_t + hv, z_t + h\beta) \leq \int_0^L |\hat{\mathsf{R}}(z_t + h\beta) - \hat{\mathsf{R}}(z_t)| dx. \tag{35}$$

Performing a Taylor expansion of the left hand side around the state (u_t, z_t), we have

$$\mathcal{E}_t(u_t, z_t) - \mathcal{E}_t(u_t + hv, z_t + h\beta) = -hD\mathcal{E}_t(u_t, z_t)(v, \beta) - \frac{h^2}{2} D^2\mathcal{E}_t(u_t, z_t)(v, \beta) + o(h^2), \tag{36}$$

where $D\mathcal{E}_t(u_t, z_t)(v, \beta)$ and $D^2\mathcal{E}_t(u_t, z_t)(v, \beta)$ denote the first and second order derivatives of $\mathcal{E}_t$ at (u_t, z_t) in the direction (v, β), respectively. Applying the Taylor expansion to the right hand side of (33) gives

$$\int_0^L |\hat{\mathsf{R}}(z_t + h\beta) - \hat{\mathsf{R}}(z_t)| dx = \int_0^L h|\beta| \left| \hat{\mathsf{R}}'(z_t) + \frac{h}{2} \hat{\mathsf{R}}''(z_t)\beta \right| dx + o(h^2). \tag{37}$$

Since $\hat{\mathsf{R}}'(z_t) > 0$, we have for sufficiently small h

$$\int_0^L h|\beta| \left| \hat{\mathsf{R}}'(z_t) + \frac{h}{2} \hat{\mathsf{R}}''(z_t)\beta \right| dx = \int_0^L h|\beta| \left(\hat{\mathsf{R}}'(z_t) + \frac{h}{2} \hat{\mathsf{R}}''(z_t)\beta \right) dx. \tag{38}$$

Therefore we deduce the state (u_t, z_t) is stable if and only if for any $(v, \beta) \in \mathcal{C}_0 \times \mathcal{Z}(z_t)$ and h sufficiently small

$$\begin{aligned} 0 \leq \quad & \left(D\mathcal{E}_t(u_t, z_t)(v, \beta) + \int_\Omega \hat{\mathsf{R}}'(z_t)|\beta| dx \right) \\ & + \frac{h}{2} \left(D^2\mathcal{E}_t(u_t, z_t)(v, \beta) + \int_\Omega \operatorname{sgn}(\beta) \hat{\mathsf{R}}''(z_t)\beta^2 dx \right) + o(h). \end{aligned} \tag{39}$$

Let us introduce the two following quantities

$$D\mathcal{P}_t^*(u_t, z_t)(v, \beta) = D\mathcal{E}_t(u_t, z_t)(v, \beta) + \int_\Omega \hat{\mathsf{R}}'(z_t)|\beta| dx, \tag{40}$$

$$D^2\mathcal{P}_t^*(u_t, z_t)(v, \beta) = D^2\mathcal{E}_t(u_t, z_t)(v, \beta) + \int_\Omega \operatorname{sgn}(\beta) \hat{\mathsf{R}}''(z_t)\beta^2 dx, \tag{41}$$

that we call the "pseudo" first and second order derivative of the total energy[3] in the test direction (v, β), respectively. We have then shown that

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[3] These are not strictly speaking true directional derivatives of the total energy. Indeed, an admissible direction of perturbation for the total energy should depend not only on the spatial coordinate but also on time as the total energy involves the whole history of the process through the dissipated energy. As a result $D\mathcal{P}_t^*(u_t, z_t)(v, \beta)$ and $D^2\mathcal{P}_t^*(u_t, z_t)(v, \beta)$ will be regarded only as "pseudo" derivatives.

Proposition 3.1. *The state (u_t, z_t) is stable in the test direction (v, β) if* (resp. *only if) it satisfies the first order condition of stability*

$$D\mathcal{P}_t^*(u_t, z_t)(v, \beta) > 0 \quad (\text{resps.} \geq 0).$$

If $D\mathcal{P}_t^*(u_t, z_t)(v, \beta) = 0$, then the state (u_t, z_t) is stable in the test direction (v, β) if (resp. *only if) it satisfies the second order condition of stability*

$$D\mathcal{P}_t^*(u_t, z_t)(v, \beta) > 0 \quad (\text{resps.} \geq 0).$$

3.2. First order condition of stability

Let us first characterize the properties on (u_t, z_t) implied by the first necessary order optimality condition

$$\forall (v, \beta) \in \mathcal{C}_0 \times \mathcal{Z}(z_t), \quad D\mathcal{P}_t^*(u_t, z_t)(v, \beta) = D\mathcal{E}_t(u_t, z_t)(v, \beta) + \int_0^L \hat{\mathsf{R}}'(z_t)|\beta| dx \geq 0, \tag{42}$$

where the first order derivative of the total free energy $D\mathcal{E}_t(u_t, z_t)(v, \beta)$ reads

$$D\mathcal{E}_t(u_t, z_t)(v, \beta) = \int_0^L \Big(\frac{\partial \hat{\phi}}{\partial \varepsilon}(u_t', z_t) v' + \frac{\partial \hat{\phi}}{\partial z}(u_t', z_t) \beta \Big) dx. \tag{43}$$

By putting $\beta = 0$ in (42), we obtain the weak form of the mechanical equilibrium

$$\forall v \in \mathcal{C}_0, \quad \int_0^L \sigma_t v' dx = 0, \tag{44}$$

where $\sigma_t = \dfrac{\partial \hat{\phi}}{\partial \varepsilon}(u_t', z_t) = \hat{\mathsf{E}}(z_t)(u_t' - \mathsf{p}_1 z_t)$ corresponds to the stress field in the bar. Integrating by part the weak form of the equilibrium, we find

$$-\int_0^L \sigma_t' v dx + [\sigma_t v]_0^L = 0. \tag{45}$$

Since $v \in \mathcal{C}_0$, we have $v(0) = v(L) = 0$. Therefore, we deduce by a classical argument of calculus of variation that $\sigma_t' = 0$ for any $x \in (0, L)$. Thus, the stress σ_t is spatially constant and only depends on time t. Let us now derive the phase transformation laws from the first order optimality condition (42) and see if they are consistent with the local form of the elastic domain (15) deduced from the work property. By putting $v = 0$ in (42), we obtain the weak form of the phase transformation criterion

$$\forall \beta \in \mathcal{Z}(z_t), \quad \int_0^L \Big(\frac{\partial \hat{\phi}}{\partial z}(u_t', z_t) \beta + \hat{\mathsf{R}}'(z_t)|\beta| \Big) dx \geq 0. \tag{46}$$

To derive the strong form of the criterion, let us first consider test direction $\beta \in \mathcal{Z}(z_t)$ such that $\beta \geq 0$ on $(0, L)$. On the subsets of $(0, L)$ where $z_t = 1$, we have necessarily $\beta = 0$ in virtue of the definition (27) of $\mathcal{Z}(z_t)$. Injecting β in (46), we find

$$\int_{0 \leq z_t < 1} \Big(\frac{\partial \hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t) \Big) \beta dx \geq 0. \tag{47}$$

As this variational inequality holds for any $\beta \geq 0$ in $\mathcal{Z}(z_t)$, we deduce that

$$-\frac{\partial \hat{\phi}}{\partial z}(u_t'(x), z_t(x)) \leq \hat{\mathsf{R}}'(z_t(x)) \quad \text{if} \quad 0 \leq z_t(x) < 1. \tag{48}$$

Reciprocally, let us consider test direction $\beta \in \mathcal{Z}(z_t)$ such that $\beta \leq 0$ on $(0, L)$. On the subsets of $(0, L)$ where $z_t = 0$, we have necessarily $\beta = 0$ in virtue of the definition (27) of $\mathcal{Z}(z_t)$. Injecting β in (46), we find

$$\int_{0<z_t\leq 1} \Big(-\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)(-\beta)dx \geq 0. \tag{49}$$

Since this variational inequality holds for any $(-\beta) \geq 0$ in $\mathcal{Z}(z_t)$, we deduce that

$$-\hat{\mathsf{R}}'(z_t(x)) \leq -\frac{\partial\hat{\phi}}{\partial z}(u_t'(x), z_t(x)) \quad \text{if} \quad 0 < z_t(x) \leq 1. \tag{50}$$

We have then shown that the first order condition of stability is totally consistent with the local phase transformation criteria deduced from the work property.
Now let us show, *for sufficiently smooth evolutions in time*, how the energy balance condition **(E)** leads to the complementary Kuhn-Tucker consistency condition (16) at the local level. Deriving (34) in time gives

$$\int_0^L \Big(\frac{\partial\hat{\phi}}{\partial\varepsilon}(u_t', z_t)\dot{u}_t' + \frac{\partial\hat{\phi}}{\partial z}(u_t', z_t)\dot{z}_t\Big)dx + \int_0^L \hat{\mathsf{R}}'(z_t)|\dot{z}_t|dx = \sigma_t\dot{U}_t. \tag{51}$$

Since the following relation holds in virtue of the mechanical equilibrium

$$\int_0^L \frac{\partial\hat{\phi}}{\partial\varepsilon}(u_t', z_t)\dot{u}_t'dx = -\int_0^L \sigma_t'\dot{u}_t'dx + [\sigma_t\dot{u}_t]_0^L = \sigma_t\dot{U}_t, \tag{52}$$

we deduce that the energy balance condition conversely reads

$$\int_0^L \Big(\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t)\dot{z}_t + \hat{\mathsf{R}}'(z_t)|\dot{z}_t|\Big)dx = 0. \tag{53}$$

On the subset of $(0, L)$ where $\dot{z}_t = 0$, the contribution of the integrand of (53) is zero. Moreover, since z_t is bound to stay in the interval $[0, 1]$, we have $0 \leq z_t < 1$ (*resp.* $0 < z_t \leq 1$) when $\dot{z}_t > 0$ (*resp.* $\dot{z}_t < 0$). But in virtue of (48) and (50), we have

$$\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t)\dot{z}_t + \hat{\mathsf{R}}'(z_t)|\dot{z}_t| = \Big(\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)|\dot{z}_t| \geq 0 \quad \text{if} \quad 0 \leq z_t < 1, \tag{54}$$

$$\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t)\dot{z}_t + \hat{\mathsf{R}}'(z_t)|\dot{z}_t| = \Big(-\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)|\dot{z}_t| \geq 0 \quad \text{if} \quad 0 < z_t \leq 1. \tag{55}$$

Thus, the energy balance condition reads

$$\int_{\dot{z}_t>0} \underbrace{\Big(\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)|\dot{z}_t|}_{\geq 0}\,dx + \int_{\dot{z}_t<0} \underbrace{\Big(-\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)|\dot{z}_t|}_{\geq 0}\,dx = 0. \tag{56}$$

We deduce that the integrand of each integral is necessarily zero and we have for any $x \in (0, L)$

$$\Big(\mathrm{sgn}(\dot{z}_t)\frac{\partial\hat{\phi}}{\partial z}(u_t', z_t) + \hat{\mathsf{R}}'(z_t)\Big)|\dot{z}_t| = 0. \tag{57}$$

Therefore, we have shown that the solutions of the variational evolution problem **(S)** and **(E)** necessarily satisfy the Kuhn-Tucker conditions (15)-(16) when the evolution is smooth enough with respect to time. It is no coincidence since *standard* models first introduced by [25] are indeed models whose evolution laws admit a variational formulation. However, their original formulation which is local does not include stability concepts which are necessary when it comes to softening materials. On the other hand, the energetic formulation **(S)** and **(E)** does introduce such stability arguments which are fundamental as we will see in the next section.

3.3. The issue of stability for the homogeneous phase transformation

We investigate in this section the stability of *spatially homogeneous* states under displacement controlled loading. The loading is *fixed*[4], we will omit in this section the subscript t related to a time evolution. Let us call $(x \mapsto (\bar{u}(x), \bar{z}(x))$ the displacement and the phase transformation fields defined over $[0, L]$ of such a state. Note that since the strain field $x \mapsto \partial\bar{u}/\partial x$ is supposed to be homogeneous, it is equal to the nominal strain $\bar{\varepsilon} = U/L$ where U is the applied displacement at $x = L$. To investigate the stability issue, we will discriminate different cases, depending if the forward or backward phase transformation criteria (48) or (50) are activated or not. Note that since the state $(\bar{u}, \bar{z})$ is homogeneous, the phase transformation criteria do not depend on the spatial coordinate x. The homogeneous states for which the phase transformation criteria remain *strict* inequalities will be labelled as *elastic* states while the states for which the forward (or backward) phase transformation criterion is an equality will be labelled as *phase transforming* states.

3.3.1. *Elastic states*

We first consider elastic states for which the phase transformation criteria are strict inequalities. The pseudo first order derivative of the total energy calculated in the admissible direction $(v, \beta) \in \mathcal{C}_0 \times \mathcal{Z}(\bar{z})$ is given by

$$D\mathcal{P}^*(\bar{u}, \bar{z})(v, \beta) = \int_0^L \frac{\partial\hat{\phi}}{\partial\varepsilon}(\bar{\varepsilon}, \bar{z})v' dx + \int_0^L \left(\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon}, \bar{z})\beta + \hat{\mathsf{R}}'(\bar{z})|\beta| \right) dx. \tag{58}$$

The first integral is zero since $\frac{\partial\hat{\phi}}{\partial\varepsilon}(\bar{\varepsilon}, \bar{z})$ is homogeneous in $[0, L]$ and $v(0) = v(L) = 0$. Now let us discriminate three different cases.

Case 1 If $\bar{z} = 0$, then $\beta \geq 0$ over $[0, L]$ in virtue (27) and only the forward phase transformation criterion needs to be satisfied. The upper criterion reads $\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon}, 0) + \hat{\mathsf{R}}'(0) > 0$ and we deduce that $D\mathcal{P}^*(\bar{u}, 0)(v, \beta) = \left(\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon}, 0) + \hat{\mathsf{R}}'(0) \right) \int_0^L \beta dx > 0$ for non trivial β in $\mathcal{Z}(0)$ and any $v \in \mathcal{C}_0$.

Case 2 If $\bar{z} = 1$, then $\beta \leq 0$ over $[0, L]$ in virtue (27) and only the backward phase transformation criterion needs to be satisfied. The lower criterion reads $-\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon}, 1) + \hat{\mathsf{R}}'(1) > 0$

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[4] studying energetic stability of a state consists indeed in comparing the total energy of this state to neighboring states with the same applied loading

and we deduce that $D\mathcal{P}^*(\bar{u},1)(v,\beta) = \Big(-\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon},1)+\hat{\mathsf{R}}'(1)\Big)\int_0^L(-\beta)dx > 0$ for non trivial β in $\mathcal{Z}(1)$ and any $v \in \mathcal{C}_0$.

Case 3 If $0 < \bar{z} < 1$, both the forward and backward phase transformation criterion needs to be satisfied and $\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon},\bar{z})\beta(x)+\hat{\mathsf{R}}'(\bar{z})|\beta(x)| > 0$ whenever $\beta(x) \neq 0$. Thus we deduce again that $D\mathcal{P}^*(\bar{u}_t,\bar{z})(v,\beta) > 0$ for any non-trivial β in $\mathcal{Z}(\bar{z})$ and any $v \in \mathcal{C}_0$.

To complete the study of the elastic states, it remains to study the case of a test direction $\beta = 0$ in $[0,L]$ for which $D\hat{\mathcal{P}}^*(\bar{u}_t,\bar{z})(v,0) = 0$. In that case, the stability of the state is given by the sign of the second order derivative $D^2\hat{\mathcal{P}}^*(\bar{u},\bar{z})(v,0)$ according to Proposition 3.1. A straightforward calculation shows that $D^2\hat{\mathcal{P}}^*(\bar{u},\bar{z})(v,0) = \int_0^L \hat{\mathsf{E}}(\bar{z})v'^2dx$ and the pseudo second order derivative is non negative for non trivial $v \in \mathcal{C}_0$. We have then shown that homogeneous elastic states are always stable.

3.3.2. *Phase transforming states*

We are now investigating the case of homogeneous states for which the phase transformation criteria are reached. Let us assume first that the forward phase transformation criterion is an equality *i.e.* $\hat{\mathsf{R}}'(\bar{z}) = -\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon},\bar{z})$ over $(0,L)$. Since the state is homogeneous, the global response of the bar corresponds to the intrinsic response of the material. Based on Proposition 2.5, the nominal strain is given by $\bar{\varepsilon} = \bar{\varepsilon}_{\mathsf{AM}}(\bar{z})$ while the stress reads $\bar{\sigma} = \bar{\sigma}_{\mathsf{AM}}(\bar{z})$. Under this assumption, we have

$$
\begin{aligned}
D\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) &= \int_0^L \frac{\partial\hat{\phi}}{\partial\varepsilon}(\bar{\varepsilon},\bar{z})v'dx + \int_0^L\Big(\frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon},\bar{z})\beta+\hat{\mathsf{R}}'(\bar{z})|\beta|\Big)dx \\
&= \int_0^L \underbrace{\hat{\mathsf{R}}'(\bar{z})}_{>0}\underbrace{\Big(|\beta|-\beta\Big)}_{\geq 0}dx \geq 0. \qquad (59)
\end{aligned}
$$

For forward phase transforming states such as $\bar{z} = 1$, then $\beta \leq 0$ over $(0,L)$ and $D\mathcal{P}^*(\bar{u},1)(v,\beta) = 0$ if and only if $\beta = 0$ over $(0,L)$. Since $D^2\hat{\mathcal{P}}^*(\bar{u},1)(v,0) > 0$ for non trivial $v \in \mathcal{C}_0$, then we deduce that the homogeneous state is stable when $\bar{z} = 1$ *i.e.* at the end of the forward homogeneous phase transformation.

Now let us focus on forward phase transforming states such as $0 \leq \bar{z} < 1$, we have that $D\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = 0$ if and only if $\beta \geq 0$ over $(0,L)$. Therefore, the sign of $D^2\hat{\mathcal{P}}^*(\bar{u},\bar{z})(v,\beta)$ must be investigated only in the following test direction space

$$
\mathcal{Z}^+ = \{\beta \in L^2(0,L) : \beta \geq 0 \text{ on } (0,L)\}, \qquad (60)
$$

where $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ is given after a direct calculation by

$$
\begin{aligned}
D^2\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = \int_0^L \Big(&\frac{1}{2}\hat{\mathsf{E}}''(\bar{z})(\bar{\varepsilon}_{\mathsf{AM}}(\bar{z})-\mathsf{p}_1 z)^2\beta^2 + 2\hat{\mathsf{E}}'(\bar{z})(\bar{\varepsilon}_{\mathsf{AM}}(\bar{z})-z\mathsf{p}_1)(v'-\mathsf{p}_1\beta)\beta \\
&+\hat{\mathsf{E}}(\bar{z})(v'-\mathsf{p}_1\beta)^2+\hat{\mathsf{G}}''(\bar{z})\beta^2+\hat{\mathsf{R}}''(\bar{z})\beta^2\Big)dx. \qquad (61)
\end{aligned}
$$

Let us rewrite $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ by introducing the compliance $\hat{\mathsf{S}}$ and the stress value $\bar{\sigma}_{\mathsf{AM}}(\bar{z})$. After some calculations, we obtain the condensed form

$$D^2\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = \int_0^L \hat{\mathsf{E}}(\bar{z})\Big(v' - (\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1)\beta\Big)^2 dx \\ - \int_0^L \left(\frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) - \hat{\mathsf{R}}''(\bar{z})\right)\beta^2 dx. \tag{62}$$

To investigate the sign of $D^2\mathcal{P}^*(\bar{u},\bar{z})(v,\beta)$, let us rewrite the forward phase transformation criterion in terms of stress, we have

$$\frac{1}{2}\hat{\mathsf{S}}'(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}'(\bar{z}) - \hat{\mathsf{R}}'(\bar{z}) = 0. \tag{63}$$

By taking the right derivative with respect to $\bar{z} \in [0,1)$ of this expression, we find

$$\Big(\bar{\sigma}'_{\mathsf{AM}}(\bar{z})(\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1) + \frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) - \hat{\mathsf{R}}''(\bar{z})\Big) = 0. \tag{64}$$

We then deduce that

$$\bar{\sigma}'_{\mathsf{AM}}(\bar{z}) = -\frac{\frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) - \hat{\mathsf{R}}''(\bar{z})}{\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1}, \tag{65}$$

where the denominator is non negative ($\hat{\mathsf{S}}' = -\hat{\mathsf{E}}'/\mathsf{E}^2 > 0$). Now let us discriminate two cases. For SMA with a *forward stress hardening* behavior the stress $z \mapsto \bar{\sigma}_{\mathsf{AM}}(z)$ is a non decreasing function and hence $\bar{\sigma}'_{\mathsf{AM}} > 0$. In this case, in virtue of (65), we deduce that $\frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) - \hat{\mathsf{R}}''(\bar{z}) < 0$ and we find that $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ is non negative for non trivial (v,β) according to (62). We deduce that the forward phase transforming homogeneous state is stable for SMA with a forward stress hardening character.
Let us consider the converse case of SMA with a *forward stress softening* behavior. In this case, we have $\bar{\sigma}'_{\mathsf{AM}} < 0$ and in virtue of (65), $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ is now the difference of two positive integrals. To know its sign, let us first bound $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ from below by minimizing it with respect to v in $\mathcal{C}_0$ for a given β in $\mathcal{Z}^+(\bar{z})$. Let us call $\bar{v}_\beta$ the unique minimizer over $\mathcal{C}_0$ of $v \mapsto D^2\mathcal{P}_t^*(\bar{u},\bar{z})(v,\beta)$ (it is a linear elastic problem for a fixed β). It satisfies the first order optimality condition

$$\int_0^L \hat{\mathsf{E}}(\bar{z})\big(\bar{v}'_\beta - (\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1)\beta\big)w' dx = 0, \quad \forall w \in \mathcal{C}_0. \tag{66}$$

Integrating by parts and making use of the boundary conditions $\bar{v}_\beta(0) = \bar{v}_\beta(1) = 0$, we get

$$x \in [0,L], \quad \bar{v}_\beta(x) = -x\int_0^L \big(\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1\big)\beta dy + \int_0^x \big(\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z}) + \mathsf{p}_1\big)\beta dy. \tag{67}$$

Inserting this optimal displacement test direction in (62) gives

$$D^2\mathcal{P}_t^*(\bar{u},\bar{z})(\bar{v}_\beta,\beta) = (L\hat{\mathsf{S}}(\bar{z}))^{-1}\big(\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{AM}}(\bar{z})+\mathsf{p}_1\big)^2\bigg(\int_0^L \beta dx\bigg)^2 - \bigg(\frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{AM}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) - \hat{\mathsf{R}}''(\bar{z})\bigg)\int_0^L \beta^2 dx. \tag{68}$$

Now let us consider a sequence (β_n) of piecewise functions such that $\beta_n(x) = 1$ for $x \in [0, L/n)$ and $\beta_n(x) = 0$ otherwise. For such sequence, we have $(\int_0^L \beta_n dx)^2 = (L/n)^2$ and $\int_0^L \beta_n^2 dx = L/n$. Therefore, for sufficiently large n, we will have $D^2\mathcal{P}_t^*(\bar{u},\bar{z})(\bar{v}_{\beta_n},\beta_n) < 0$. The homogeneous forward phase transforming state such that $\bar{z} \in [0,1)$ is then unstable for forward stress softening SMA.

The same analysis can be carried for the backward phase transforming states by assuming that the backward phase transformation criterion is an equality *i.e.* $\hat{\mathsf{R}}'(\bar{z}) = \frac{\partial\hat{\phi}}{\partial z}(\bar{\varepsilon},\bar{z})$ over $(0,L)$. This case is analog to the forward phase transformation case and we will simply show the main steps of the study. Firstly, based on Proposition 2.5, we have $\bar{\varepsilon} = \bar{\varepsilon}_{\mathsf{MA}}(\bar{z})$ and the stress is $\bar{\sigma}_{\mathsf{MA}}(\bar{z})$. The pseudo first variation of the total energy reads

$$D\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = \int_0^L \underbrace{\hat{\mathsf{R}}'(\bar{z})}_{>0}\underbrace{\Big(|\beta|+\beta\Big)}_{\geq 0} dx \geq 0. \tag{69}$$

In this case, the stability of the homogeneous state is ensured for $\bar{z} = 0$ since $D\mathcal{P}^*(\bar{u},0)(v,\beta) = 0$ if and only if $\beta = 0$ over $(0,L)$ and $D^2\hat{\mathcal{P}}^*(\bar{u},0)(v,0) > 0$ for non trivial $v \in \mathcal{C}_0$. As a result the homogeneous state is stable at the end of the backward homogeneous phase transformation.

For $\bar{z} \in (0,1]$, $D\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = 0$ if and only if $\beta \leq 0$ over $(0,L)$. In this case, the second variation of the total energy given by

$$D^2\mathcal{P}^*(\bar{u},\bar{z})(v,\beta) = \int_0^L \hat{\mathsf{E}}(\bar{z})\Big(v' - (\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{MA}}(\bar{z})+\mathsf{p}_1)\beta\Big)^2 dx - \int_0^L \bigg(\frac{1}{2}\hat{\mathsf{S}}''(\bar{z})(\bar{\sigma}_{\mathsf{MA}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) + \hat{\mathsf{R}}''(\bar{z})\bigg)\beta^2 dx, \tag{70}$$

must be studied over $\mathcal{C}_0 \times \mathcal{Z}^-$ where $\mathcal{Z}^-$ reads

$$\mathcal{Z}^- = \{\beta \in L^2(0,L) : \beta \leq 0 \text{ on } (0,L)\}. \tag{71}$$

Since by deriving the backward phase transformation criterion we have

$$\bar{\sigma}'_{\mathsf{MA}}(\bar{z}) = -\frac{\frac{1}{2}\mathsf{S}''(\bar{z})(\bar{\sigma}_{\mathsf{MA}}(\bar{z}))^2 - \hat{\mathsf{G}}''(\bar{z}) + \hat{\mathsf{R}}''(\bar{z})}{\hat{\mathsf{S}}'(\bar{z})\bar{\sigma}_{\mathsf{MA}}(\bar{z}) + \mathsf{p}_1}, \tag{72}$$

we deduce that the homogeneous state is stable for SMA with a backward stress hardening character *i.e.* $\bar{\sigma}'_{\mathsf{MA}}(\bar{z}) > 0$. For SMA with a backward stress softening character *i.e.* $\bar{\sigma}'_{\mathsf{MA}}(\bar{z}) < 0$, by considering the same sequence (β_n) than for the forward stress softening case, one can show that $D^2\mathcal{P}^*(\bar{u},\bar{z})(\bar{v}_{\beta_n},\beta_n) < 0$ for sufficiently large n. Therefore the homogeneous state is unstable in this case. Finally, we can summarize the stability results as follows

Proposition 3.2. *The homogeneous state $(\bar{u},\bar{z})$ for this SMA model enjoys the following stability properties:*

- *The elastic homogeneous states are* stable*;*
- *The forward* (resp. *backward) phase transforming homogeneous states are* stable *for materials with a forward* (resp. *backward) stress hardening character;*
- *The forward* (resp. *backward) phase transforming homogeneous states are* unstable *if $\bar{z} \in [0,1)$* (resp. $\bar{z} \in (0,1]$) *and* stable *if $\bar{z} = 1$* (resp. $\bar{z} = 0$) *for materials with a forward* (resp. *backward) stress softening character.*

3.4. A stable evolution path for SMA with a softening behavior

In the previous section, we showed the unstability of homogeneous states during the phase transformation for materials with a softening behavior. In this case, we have to investigate the existence non-homogeneous paths that ensures the stability criterion **(S)** and the energy balance condition **(E)**. We will assume in the rest of this section that the material has a forward and backward stress softening character. As a result, the stress related to the homogeneous response is monotonically increasing from 0 to $\bar{\sigma}_{\mathsf{AM}}(0)$ when the nominal strain $\bar{\varepsilon}$ goes from 0 to $\bar{\varepsilon}_{\mathsf{AM}}(0)$, monotonically decreasing from $\bar{\sigma}_{\mathsf{AM}}(0)$ to $\bar{\sigma}_{\mathsf{AM}}(1)$ when the nominal strain $\bar{\varepsilon}$ goes from $\bar{\varepsilon}_{\mathsf{AM}}(0)$ to $\bar{\varepsilon}_{\mathsf{AM}}(1)$ and monotonically increasing from $\bar{\sigma}_{\mathsf{AM}}(1)$ to $+\infty$ when the nominal strain $\bar{\varepsilon}$ goes from $\bar{\varepsilon}_{\mathsf{AM}}(1)$ to $+\infty$. Thus, for a given $\tilde{\sigma}_{\mathsf{AM}}$ such that $\bar{\sigma}_{\mathsf{AM}}(0) < \tilde{\sigma}_{\mathsf{AM}} < \bar{\sigma}_{\mathsf{AM}}(1))$, the homogeneous stress-strain response crosses the horizontal line $\sigma = \tilde{\sigma}_{\mathsf{AM}}$ at the following strain values

$$\tilde{\varepsilon}_{\mathsf{A}} = \frac{\tilde{\sigma}_{\mathsf{AM}}}{\mathsf{E}_{\mathsf{A}}}, \quad \tilde{\varepsilon}_{\mathsf{AM}} = \bar{\varepsilon}_{\mathsf{AM}}((\bar{\sigma}_{\mathsf{AM}})^{-1}(\tilde{\sigma}_{\mathsf{AM}})), \quad \tilde{\varepsilon}_{\mathsf{M}} = \frac{\tilde{\sigma}_{\mathsf{AM}}}{\mathsf{E}_{\mathsf{M}}} + \mathsf{p}_1. \tag{73}$$

We are now in measure to construct a *non-homogeneous* evolution of the bar $(t,x) \mapsto (\tilde{u}_t(x),\tilde{z}_t(x))$ under an applied increasing nominal strain $\bar{\varepsilon}_t$ with $\dot{\bar{\varepsilon}}_t > 0$. It follows the three following steps:

Step 1: For $0 \le \bar{\varepsilon}_t < \tilde{\varepsilon}_{\mathsf{A}}$, the response is elastic and the whole bar is in the austenite state *i.e.* $\tilde{z}_t(x) = 0$ for $x \in (0,L)$. The strain field $\tilde{\varepsilon}_t = \partial\tilde{u}_t/\partial x$ is homogeneous and its value is given by the nominal strain $\bar{\varepsilon}_t$, while the stress $\tilde{\sigma}_t$ is equal to $\mathsf{E}_{\mathsf{A}}\bar{\varepsilon}_t$;

Step 2: For $\tilde{\varepsilon}_{\mathsf{A}} \le \bar{\varepsilon}_t < \tilde{\varepsilon}_{\mathsf{M}}$, the bar undergoes a *non-homogeneous* phase transformation. The stress $\tilde{\sigma}_t$ is constant in time and is equal to $\tilde{\sigma}_{\mathsf{AM}}$ during the whole phase transformation. The transformation phase field $\tilde{z}_t$ is piecewise constant in the bar while the displacement field $\tilde{u}_t$ is piecewise linear. They are given by

- $\tilde{u}_t(x) = \tilde{\varepsilon}_{\mathsf{A}}x$ and $\tilde{z}_t(x) = 1$ for $x \in (0,\theta_t L)$;
- $\tilde{u}_t(x) = \tilde{\varepsilon}_{\mathsf{M}}(x - \theta_t L) + \tilde{\varepsilon}_{\mathsf{M}}\theta_t L$ and $\tilde{z}_t(x) = 0$ for $x \in [\theta_t L, L)$,

where θ_t corresponds to the volume fraction of the bar that has turned into martensite

$$\theta_t = \frac{\bar{\varepsilon}_t - \tilde{\varepsilon}_{\mathsf{A}}}{\tilde{\varepsilon}_{\mathsf{M}} - \tilde{\varepsilon}_{\mathsf{A}}}, \tag{74}$$

Step 3: For $\tilde{\varepsilon}_{\mathsf{M}} \leq \bar{\varepsilon}_t$, the whole bar has been transformed into martensite *i.e.* $\tilde{z}_t(x) = 1$ for $x \in (0, L)$. The strain field is again homogeneous and is equal to $\bar{\varepsilon}_t$, while the stress is equal to $\mathsf{E}_{\mathsf{M}}(\bar{\varepsilon}_t - \mathsf{p}_1)$.

Stability criterion Let us first examine if the constructed evolution path satisfies the stability criterion **(S)**. During Step 1, the state $(\tilde{u}_t, \tilde{z}_t)$ is in a *homogeneous* austenite state and hence the bar is stable according to Proposition 3.1. The same argument applies also for the Step 3 for which the whole bar is in a homogeneous martensite state. It remains then to check the stability of Step 2. Let us first remark that the forward transformation phase criterion is, by construction, a strict inequality in the bar during this step. Indeed, the two strain states $\tilde{\varepsilon}_{\mathsf{A}}$ and $\tilde{\varepsilon}_{\mathsf{M}}$ introduced in (73) are in the *interior* of the elastic domain $\hat{\mathbb{E}}(0)$ and $\hat{\mathbb{E}}(1)$, respectively. Hence satisfies

$$-\frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_{\mathsf{A}}, 0) < \hat{\mathsf{R}}'(0), \qquad -\frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_{\mathsf{M}}, 1) > -\hat{\mathsf{R}}'(1). \tag{75}$$

Since, $(\tilde{\varepsilon}_t, \tilde{z}_t)$ is equal to $(\tilde{\varepsilon}_{\mathsf{A}}, 0)$ on $[\theta_t L, L)$ and to $(\tilde{\varepsilon}_{\mathsf{M}}, 1)$ on $(0, \theta_t L)$, we deduce that

$$\begin{cases} 0 < -\frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_t(x), \tilde{z}_t(x)) + \hat{\mathsf{R}}'(\tilde{z}_t(x)) & \text{for} \quad x \in (0, \theta_t L), \\ 0 < \frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_t(x), \tilde{z}_t(x)) + \hat{\mathsf{R}}'(\tilde{z}_t(x)) & \text{for} \quad x \in [\theta_t L, L). \end{cases} \tag{76}$$

The first variation of the total energy (42) reads for such state

$$\begin{aligned} D\mathcal{P}^*_t(u_t, z_t)(v, \beta) = &\int_0^{\theta_t L} \Big(-\frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_t(x), \tilde{z}_t(x)) + \hat{\mathsf{R}}'(z_t) \Big) |\beta| dx \\ &+ \int_{\theta_t L}^{L} \Big(\frac{\partial \hat{\phi}}{\partial z}(\tilde{\varepsilon}_t(x), \tilde{z}_t(x)) + \hat{\mathsf{R}}'(z_t) \Big) |\beta| dx, \end{aligned} \tag{77}$$

with β such that $\beta(x) \leq 0$ on $(0, \theta_t L)$ and $\beta \geq 0$ on $(\theta_t L, L)$. Therefore, the pseudo first variation of the total energy (42) is non negative for non trivial test function β. When $\beta = 0$ in $(0, L)$, the second variation is also non negative for non trivial displacement test function $v \in \mathcal{C}_0$. We conclude that the constructed non-homogenous path is stable during the whole evolution.

Energy balance condition It remains to examine if the non-homogeneous path satisfies the energy balance condition **(E)**. During Step 1, since the response is linear elastic and homogeneous, the total energy reads

$$\mathcal{P}_t(\tilde{u}_t, \tilde{z}_t) = \frac{1}{2} \int_0^L \mathsf{E}(0) \bar{\varepsilon}_t^2 dx = \frac{1}{2} \mathsf{E}_{\mathsf{A}} \bar{\varepsilon}_t^2 L = \mathcal{P}_0(\tilde{u}_0, \tilde{z}_0) + \int_0^t \sigma_{t^*}(L) \dot{U}_{t^*} dt^*, \tag{78}$$

with $\mathcal{P}_0(\tilde{u}_0, \tilde{z}_0) = 0$, $\sigma_{t^*}(L) = \mathsf{E}_{\mathsf{A}} \bar{\varepsilon}_{t^*}$ and $\dot{U}_{t^*} = \dot{\bar{\varepsilon}}_{t^*} L$. Thus, the energy balance condition **(E)** is satisfied during this step.

During Step 2, the evolution of a material point $x^* \in [0, L]$ is discontinuous in time as the phase transformation variable $t \mapsto \tilde{z}_t(x^*)$ goes directly from 0 to 1 when the nominal strain reaches the value $\big(\tilde{\varepsilon}_{\mathsf{M}}(\tilde{\sigma}) - \tilde{\varepsilon}_{\mathsf{A}}(\tilde{\sigma})\big)x^*/L + \tilde{\varepsilon}_{\mathsf{A}}(\tilde{\sigma})$. Due to this time discontinuity, we cannot calculate the dissipated energy by means of (29) as it involves the rate of the phase transformation variable. Instead, we have to rely on the more general definition (31) of the total dissipated energy. For a given time discretization $(\tau_i)_{1\leq i\leq M}$ with $\tau_1 = 0$ and $\tau_M = t$, as the phase transformation is time increasing, we have by construction for any i and any $x \in (0, L)$, $\tilde{z}_{\tau_{i-1}}(x) \leq \tilde{z}_{\tau_i}(x)$. Since $z \mapsto \hat{\mathsf{R}}(z)$ is an increasing function, we obtain $\hat{\mathsf{R}}(\tilde{z}_{\tau_{i-1}}(x)) \leq \hat{\mathsf{R}}(\tilde{z}_{\tau_i}(x))$. Then, the total dissipated energy for the time discretization (τ_i) reads

$$\begin{aligned}\sum_{i=1}^{M}\int_0^L |\hat{\mathsf{R}}(z_{\tau_i}) - \hat{\mathsf{R}}(z_{\tau_{i-1}})|dx &= \sum_{i=1}^{M}\int_0^L \hat{\mathsf{R}}(z_{\tau_i}) - \hat{\mathsf{R}}(z_{\tau_{i-1}})dx \\ &= \int_0^L \hat{\mathsf{R}}(z_{\tau_M}) - \hat{\mathsf{R}}(z_{\tau_1})dx \\ &= \int_0^L \hat{\mathsf{R}}(z_t)dx = \hat{\mathsf{R}}(1)\theta_t L.\end{aligned} \tag{79}$$

Therefore the total dissipated energy is independent of the time discretization and we deduce that

$$\mathsf{Diss}(z_t, [0, t]) = \hat{\mathsf{R}}(1)\theta_t L. \tag{80}$$

Let us now compute the total free energy of the state. As $\tilde{\varepsilon}_t$ and $\tilde{z}_t$ are piecewise constant over $(0, L)$, we obtain

$$\mathcal{E}_t(\tilde{u}_t, \tilde{z}_t) = \int_0^L \hat{\phi}(\tilde{\varepsilon}_t, \tilde{z}_t)dx = \theta_t L\hat{\phi}(\tilde{\varepsilon}_{\mathsf{M}}, 1) + (1 - \theta_t)L\hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}}, 0). \tag{81}$$

Combining (80) and (81), we get the value of the total energy of the system

$$\mathcal{P}_t(\tilde{u}_t, \tilde{z}_t) = \theta_t L\hat{\phi}(\tilde{\varepsilon}_{\mathsf{M}}, 1) + (1 - \theta_t)L\hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}}, 0) + \hat{\mathsf{R}}(1)\theta_t L. \tag{82}$$

It remains to establish the total work done by external forces during the non-homogeneous evolution. Since the global stress-nominal strain response of the non-homogeneous path is piecewise linear, it reads

$$\int_0^t \sigma_{t^*}\dot{U}_{t^*}dt^* = \underbrace{\frac{1}{2}\tilde{\sigma}_{\mathsf{AM}}\tilde{\varepsilon}_{\mathsf{A}}L}_{=\hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}},0)} + \theta_t\tilde{\sigma}_{\mathsf{AM}}(\tilde{\varepsilon}_{\mathsf{M}} - \tilde{\varepsilon}_{\mathsf{A}})L. \tag{83}$$

The quantities $\hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}}, 0)$ and $\theta_t\tilde{\sigma}_{\mathsf{AM}}(\tilde{\varepsilon}_{\mathsf{M}} - \tilde{\varepsilon}_{\mathsf{A}})$ are associated to the linear austenite part and phase transformation part occurring at constant stress, respectively. The energy balance condition **(E)** requires equality between (82) and (83). Rearranging the expression, it leads to the following equality

$$\hat{\phi}(\tilde{\varepsilon}_{\mathsf{M}}, 1) + \hat{\mathsf{R}}(1) = \hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}}, 0) + \tilde{\sigma}_{\mathsf{AM}}(\tilde{\varepsilon}_{\mathsf{M}} - \tilde{\varepsilon}_{\mathsf{A}}), \tag{84}$$

which does not involve any time related quantities anymore. The left hand side of (84) represents the work of external forces by unit of length of the homogeneous response from 0 to a nominal strain $\tilde{\varepsilon}_{\mathsf{M}}$. On Fig. 3 (Left), this quantity is represented by the gray area under the stress-nominal strain curve associated to the *homogeneous* response. On the other hand, the right hand side of (84) represents the work of external forces by unit of length of the constructed non-homogeneous response from 0 to a nominal strain $\tilde{\varepsilon}_{\mathsf{M}}$. On Fig. 3 (Right), this quantity is represented by the gray area under the stress-nominal strain curve associated to the *non-homogeneous* response. Requiring equality between these two energy areas leads to the Maxwell line associated to the homogeneous softening response of the SMA. To solve (84) and determine the vertical position of the Maxwell line, let us first remark that

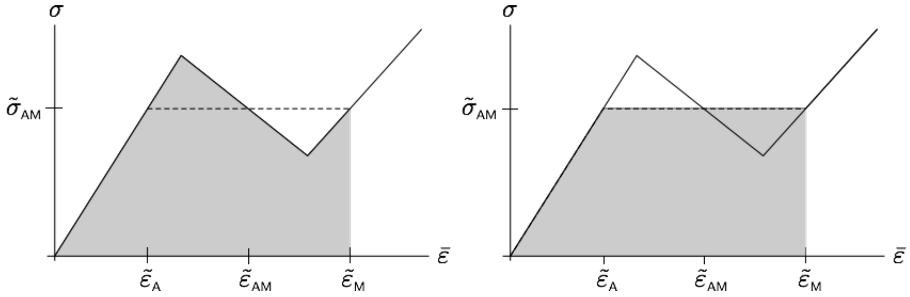

Fig. 3. Stress-nominal strain curves. The plain line and the dashed line represent the homogeneous (Left) and non-homogeneous response (Right), respectively. Grey areas corresponds to the work of external forces done in a homogeneous response (Left) and in a non-homogeneous response (Right) from $\bar{\varepsilon} = 0$ to $\bar{\varepsilon} = \tilde{\varepsilon}_{\mathsf{M}}$

$$\tilde{\varepsilon}_{\mathsf{M}} - \tilde{\varepsilon}_{\mathsf{A}} = (\mathsf{S}_{\mathsf{M}} - \mathsf{S}_{\mathsf{A}})\tilde{\sigma}_{\mathsf{AM}} + \mathsf{p}_1. \tag{85}$$

Since the free energy of the beginning and ending states of the non-homogeneous phase transformation read

$$\hat{\phi}(\tilde{\varepsilon}_{\mathsf{M}}, 1) = \frac{1}{2}\mathsf{S}_{\mathsf{M}}\tilde{\sigma}_{\mathsf{AM}}^2 + \hat{\mathsf{G}}(1), \quad \hat{\phi}(\tilde{\varepsilon}_{\mathsf{A}}, 1) = \frac{1}{2}\mathsf{S}_{\mathsf{A}}\tilde{\sigma}_{\mathsf{AM}}^2 + \hat{\mathsf{G}}(0), \tag{86}$$

we then obtain in virtue of (84)

$$(\mathsf{S}_{\mathsf{M}} - \mathsf{S}_{\mathsf{A}})\tilde{\sigma}_{\mathsf{AM}}^2 + 2\tilde{\sigma}_{\mathsf{AM}}\varepsilon_0 - 2(\hat{\mathsf{G}}(1) + \hat{\mathsf{R}}(1) - \hat{\mathsf{G}}(0) - \hat{\mathsf{R}}(0)) = 0. \tag{87}$$

Given the Hypothesis **2.4**, we have $(\mathsf{G} + \mathsf{R})' > 0$. Then, we deduce that $\hat{\mathsf{G}}(1) + \hat{\mathsf{R}}(1) - \hat{\mathsf{G}}(0) - \hat{\mathsf{R}}(0) > 0$ and there exists a unique $\tilde{\sigma}_{\mathsf{AM}}$ for which (84) holds. Its value is given by

$$\tilde{\sigma}_{\mathsf{AM}} = \frac{\sqrt{\varepsilon_0^2 + 2(\mathsf{S}_{\mathsf{M}} - \mathsf{S}_{\mathsf{A}})(\hat{\mathsf{G}}(1) + \hat{\mathsf{R}}(1))} - \varepsilon_0}{\mathsf{S}_{\mathsf{M}} - \mathsf{S}_{\mathsf{A}}}. \tag{88}$$

Therefore, the non homogeneous phase transformation satisfies the energy balance condition during Step 2 if and only if the phase transformation occurs according to the Maxwell line.

Finally during Step 3, the response of the martensite bar is again linear elastic and homogeneous. Thus, the energy balance condition will hold if and only if it holds at the end of Step 2, which is ensured if the phase transformation occurs according to the Maxwell line.

3.4.1. *Non-homogeneous backward phase transformation*

The same procedure can be applied for the backward phase transformation to construct a non-homogeneous evolution path in case of a stress softening backward phase transformation. In particular, the stress $\tilde{\sigma}_{\mathsf{MA}}$ of the Maxwell line associated to the backward phase transformation reads

$$\tilde{\sigma}_{\mathsf{MA}} = \frac{\sqrt{\varepsilon_0^2 + 2(\mathsf{S_M} - \mathsf{S_A})(\hat{\mathsf{G}}(1) - \hat{\mathsf{R}}(1))} - \varepsilon_0}{\mathsf{S_M} - \mathsf{S_A}}. \tag{89}$$

Remark 2. *Note that for the non-homogeneous evolution path, if one unloads elastically the bar during the forward phase transformation (Step 2), the slope of the unloading in the stress-nominal strain diagram is* $(\theta_t \mathsf{S_A} + (1-\theta_t)\mathsf{S_M})^{-1}$*. The slope* $(\theta_t \mathsf{S_A} + (1-\theta_t)\mathsf{S_M})^{-1}$ *is simply a rule of mixture between the Young modulus of the austenite and martensite phases weighted by the volume fraction of each constituent in the bar. Therefore, on the contrary of the homogeneous response for which successive elastic unloadings give access to the Young's modulus* $\hat{\mathsf{E}}(z)$ *for* $z \in [0,1]$*, no such specific information can be retrieved from the non-homogeneous response. As a result, despite being a stable solution of the evolution problem, the study of the non-homogeneous evolution gives few information for the identification of the SMA model parameters. To get rid of this non-homogeneous behavior, the experimental technique used by [12] consists in stabilizing the homogeneous branch by reinforcing the SMA tensile specimen by stainless steel bars. In that case, the global response exhibits hardening and the intrinsic behavior of the SMA can be indeed extracted. Note that their experimental results agree with the energetic stability criterion: without the reinforcements, the SMA specimen has a non-homogeneous evolution and the phase transformation occurs at a constant stress closed to the Maxwell stress of the softening intrinsic curve.*

4. CONCLUSIONS

This paper dealts with the modeling of the superelastic response of SMA in a one dimensional setting. Departing from the class of rate-independent models with a scalar internal variable, the derivation of the model of SMA is based on a thermodynamic work principle. This leads to a standard evolution for which the phase transformation criterion is of an energy release inequality type. Upon an appropriate change of internal variable, the model depends on three material scalar functions of the internal variable and one material scalar parameter.

The quasi-static evolution problem is formulated in terms of a stability criterion **(S)** and an energy balance condition **(E)**. The stability criterion consists in selecting, *at a given loading*, the states that ensure a local minimality condition. The energy balance conditions enforces the total energy of the system to be an absolutely continuous function of time. Based on the stability criterion, it was shown that the homogeneous phase transformation is stable for stress-hardening materials. However, this stability of the homogeneous states is lost for stress-softening materials. In this case, a stable evolution path that satisfies both **(S)** and **(E)** consists in a non-homogeneous evolution that follows the Maxwell line of the softening intrinsic behavior. Specifically, such a phase transformation field consists in a bar which is partially transformed in martensite with a sharp martensite-austenite front. In particular, the phase front is moving at the rate of the loading. All these results are consistent with those of [16] in the case of non-linear elasticity and hence, this energetic framework might be a good candidate to study the stability of SMA structures in more complex cases.

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