Tạp chí Cơ Học   Journal of Mechanics, NCSR of Vietnam T. XIV, 1992, No 2 (23 — 27)

AN EXTENSION OF THE E. MELAN'S SHAKEDOWN THEOREM OF THE ELASTO-PLASTIC STRUCTURE

NGUYEN VAN PHO

Hanoi University of Civil Engineering

SUMMARY. In this paper, the extended quasi-static shakedown theorem is proved. The condition of the E. Melan's theorem

$$\exists\rho_{ij}(\vec{x}) \quad \forall\vec{p}\in G$$

is replaced by $\exists\Delta P$ and $\exists\rho_{ij}^{(\vec{P})}(\vec{x})$ for at each point of G.

A numerical example is given.

1. INTRODUCTION

In 1938 E. Melan has been proved the quasi-static shakedown theorem for 3-dimensional elasto-perfectly plastic solid is subjected to loadss [1]. In 1957 W. Prager has been proved the shakdown theorem in the case of body subjected combined to loads and temperatures [2].

In 1975 we have extended E. Melan's theorem by replacing the conditions

$$\exists\rho_{ij}(\vec{x}) \quad \forall\vec{P}\in G \quad \forall\vec{x}\in V \tag{1.1}$$

by the lighter conditions. Here $\rho_{ij}(\vec{x})$ stands for the time independent of residual stress field (satisfying the equilibrium equations for vanishing loads), G is the domain of loads; V - the domain occupied by the body; $\vec{x}=\{x_i\}$ - the space variables [3].

In this paper the author shall extand the obtained results by looking at the extensions of E. Melan's theorem under an unifield point of view, and then we prove a more extended shakedown critere, and its consequences.

A numerical example is given, in virtue of illustration and comparison.

2. EXTENDED SHAKEDOWN THEOREM

The E. Melan's and Prager shakedown theorems have two noticeable points.

Firstly, the proof is of local nature, i.e. the proof for the stop plastic deformation at point of the shakedown domain G.

Secondly, there exists a time independent residual stress field with respect to all point of the domain G.

The load, varying in G, is a function $\vec{P}=\vec{P}(t)$ of the time t, while for a elasto-perfectly plastic body, G is a determined domain (not depending on t).

From these two notices we have the following extension theorem.

1. Theorem: The elasto-perfectly plastic structure is quasi-static shakedown in the open domain G of loads wherever at each point $\vec{P} \in G$ there exists a time independent residual stress field $\rho_{ij}^{(\vec{P})}(\vec{x})$ and there exists a vicinity ΔP of $\vec{P}$ such as the total stress field satisfies

$$f\left[\sigma_{ij}^{(e)}(\vec{x}, \vec{P}) + \rho_{ij}(\vec{P})(\vec{x})\right] < C \quad \forall \vec{x} \in V,$$

Here $f = C$ is the yield condition, $\sigma_{ij}^{(e)}(\vec{x}, \vec{P})$ is the perfectly elastic stress corresponding to the load $\vec{P}$.

2. Proof: Let P_0 be a point of the open domain G. By virtue of the hypothesis of the extension theorem the shakedown sufficient condition is

$$(I) \quad \left\{ \exists \Delta P_0 \quad \text{and} \quad \exists \rho_{ij}^{(\vec{P})}(\vec{x}) \Rightarrow \left\{ \begin{array}{ll} L\rho_{ij}(\vec{P}_0)(\vec{x}) = 0 \quad \forall x \in V, \notin S_P & (2.1) \\ N\rho_{ij}(\vec{P}_0)(\vec{x}) = 0 \quad \forall x \in S_P & (2.2) \\ f\left[\sigma_{ij}^{(e)}(\vec{x}, \vec{P}) + \rho_{ij}^{(\vec{P})}(\vec{x})\right] < C \quad \forall P \in \Delta P_0 & (2.3) \end{array} \right. \right.$$

Here L is the equilibrium operator of all points in the body, N is the equilibrium operator of all points of the loaded boundaries S_P.

Condition (I) expresses the shakedown condition (after E. Melan's theorem) of body in vicinity ΔP. Every ΔP satisfying (I) will be called an E. Melan vicinity.

Now we shall prove that if E. Melan's shakedown condition is satisfying at each point of G, then the structure is shakedown on G.

In fact, let us cover the open domain G by a set of E. Melan vicinities $\{\Delta P_i\}$, i.e. our system is E. Melan shakedown in each vicinity ΔP_i.

Consequently for each ΔP_i there exists a bounded moment t_{0i} from which the system is shakedown on ΔP_i. The moment from which the whole system is shakedown on G is choosen as

$$T_0 = \max_{\{i\}} t_{0i}$$

Evidently T_0 is finitary. To achieve the proof we still have to show that under the hypothesis of the extension theorem we shall be able to find a set of E. Melan vicinities $\{\Delta P_i\}$ covering G.

From condition (2.3) we see that once

$$f\left[\sigma_{ij}^{(e)}(\vec{x}, \vec{P}) + \rho_{ij}^{(P)}(\vec{x})\right] < C$$

is satisfied, then condition

$$f\left[\sigma_{ij}^{(e)}(\vec{x}, \vec{P}) + \varepsilon_{ij}^{(e)} + \rho_{ij}^{(P)}(\vec{x})\right] < C$$

with $|\varepsilon_{ij}| > 0$ small enough, is satisfied too, because $f < C$ means an open domain (not including its boundary).

Here $\sigma_{ij}^{(e)}(\vec{x}, \vec{P})$ is the elastic stress corresponding to the load $\vec{P}$, while $\sigma_{ij}^{(e)}(\vec{x}, \vec{P}) + \varepsilon_{ij}^{(e)}$ is the elastic stress corresponding to the load $\vec{P} + \delta\vec{P}$ belong to the vicinity ΔP of $\vec{P}$. This means that at whatever $\vec{P}$ of G, by the hypothesis of the extension theorem we always can pick out a vicinity ΔP of $\vec{P}$. In other words we have proved the existence of a set of vicinities $\{\Delta P_i\}$

3. CONSEQUENCES

Consequence 1. After the E. Melan's extension theorem the shakedown domain G is a convex one in the space of loads.

Proof. Let $P_1 \in G$, $\vec{P}_2 \in G$, we have to prove that

$$\mu \vec{P}_1 + (1-\mu)\vec{P}_2 \in G \qquad (0 \le \mu \le 1)$$

In fact, since $\vec{P}_1 \in G \Rightarrow \exists \rho_{ij}^{(1)}(\vec{x})$ and $\exists \Delta P_1 \Rightarrow$

$$\begin{cases} L\rho_{ij}^{(1)} = 0 & \forall \vec{x} \in V, \notin S_P \\ N\rho_{ij}^{(1)} = 0 & \forall \vec{x} \in S_P \\ f\left[\sigma_{ij1}^{(e)}(\vec{x}, \vec{P}_1) + \rho_{ij}^{(1)}(\vec{x})\right] < C & \forall \vec{x} \in V,\ \forall \vec{P} \in \Delta P_1 \end{cases}$$

and since $\vec{P}_2 \in G \Rightarrow \exists \rho_{ij}^{(2)}(\vec{x})$ and $\exists \Delta P_2 \Rightarrow$

$$\begin{cases} L\rho_{ij}^{(2)} = 0 & \forall \vec{x} \in V, \notin S_P \\ N\rho_{ij}^{(2)} = 0 & \forall \vec{x} \in S_P \\ f\left[\sigma_{ij1}^{(e)}(\vec{x}, \vec{P}_2) + \rho_{ij}^{(2)}(\vec{x})\right] < C & \forall \vec{x} \in V,\ \forall \vec{P} \in \Delta P_2 \end{cases}$$

L and N being linear homogeneous operators, we have

$$L\left[\mu\rho_{ij}^{(1)} + (1-\mu)\rho_{ij}^{(2)}\right] = 0 \qquad \forall \vec{x} \in V,\ \notin S_P$$
$$N\left[\mu\rho_{ij}^{(1)} + (1-\mu)\rho_{ij}^{(2)}\right] = 0 \qquad \forall \vec{x} \in S_P$$

As $R \equiv \left\{\sigma_{ij}\ :\ f(\sigma_{ij}) < C\right\}$ is a convex domain, we have

$$f\left\{\left[\mu\sigma_{ij1}^{(e)} + (1-\mu)\sigma_{ij2}^{(e)}\right] + \left[\mu\rho_{ij}^{(1)} + (1-\mu)\rho_{ij}^{(2)}\right]\right\} < C \quad (0 < \mu < 1)$$

The loads in ΔP_1 and ΔP_2 can be represented as

$$\vec{P} = \vec{P}_1 + \delta\vec{P}_1$$
$$\vec{P} = \vec{P}_2 + \delta\vec{P}_2$$

Here $\delta\vec{P}_1$, $\delta\vec{P}_2$ are small enough vectors
The elastic stresses in ΔP_1 and ΔP_2 can be represented as

$$\sigma_{ij1}^{(e)}(\vec{x}, \vec{P}_1, \delta\vec{P}_1) \doteq \sigma_{ij}^{(e)}(\vec{x}, \vec{P}_1) + \sigma_{ij1}^{(e)}(\vec{x}, \delta\vec{P}_1)$$
$$\sigma_{ij2}^{(e)}(\vec{x}, \vec{P}_2, \delta\vec{P}_2) = \sigma_{ij}^{(e)}(\vec{x}, \vec{P}_2) + \sigma_{ij2}^{(e)}(\vec{x}, \delta\vec{P}_2)$$

We note

$$\rho_{ij}^{(3)} \equiv \mu\rho_{ij}^{(1)}(\vec{x}) + (1-\mu)\rho_{ij}^{(2)}(\vec{x})$$
$$\sigma_{ij3}^{(e)} \equiv \mu\sigma_{ij1}^{(e)}(\vec{x}, \vec{P}_1) + (1-\mu)\sigma_{ij2}^{(e)}(\vec{x})$$

Since $f < C$ is convex we have

$$f(\sigma_{ij3}^{(e)} + \rho_{ij}^{(3)}) < C$$

By virtue of the solution of a linear elasticity problem we can pose

$$\sigma_{ij}^{(e)}(\vec{x}, \delta\vec{P}_1) = \varepsilon_{ij}^{1)}, \qquad \sigma_{ij}^{(e)}(\vec{x}, \delta P_2) = \varepsilon_{ij}^{(2)} \qquad (0 < \mu < 1)$$

where $\varepsilon_{ij}^{(1)}$, $\varepsilon_{ij}^{(2)}$ are infinitesimal elastic stress.

From these we have

$$f\left[\sigma_{ij3}(e) + \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} + \rho_{ij}^{(3)}\right] < C$$

Now let the value of μ be fixed, $\mu = \mu_0$, $\mu_0 \in (0,1)$; then $\rho_{ij}^{(3)} = \rho_{ij}^{(3)}(\vec{x})$ is a function not depending on time, while $\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} = \varepsilon_{ij}$ is also infinitesimal elastic stress.

Thus, whenever $\delta\vec{P}_1$ and $\delta\vec{P}_2$ vary in the vicinities ΔP_1 and ΔP_2 of the points $\vec{P}_1$ and $\vec{P}_2$ respectively, then $\varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$ varies correspondingly in the vicinity ΔP_3 of $\vec{P}_3$. This means that to any convex combination $\vec{P}_3$ of $\vec{P}_1$ and $\vec{P}_2$

$$\vec{P}_3 = \mu_0\vec{P}_1 + (1-\mu_0)\vec{P}_2$$

there exists a residual stress not depending on time

$$\rho_{ij}^{(3)}(\vec{x}) = \mu_0\rho_{ij}(1)(\vec{x}) + (1-\mu_0)\rho_{ij}(2)(\vec{x})$$

and there exists a vicinity of ΔP_3 of $\vec{P}_3$, satisfying the conditions of the extended theorem, i.e. the shakedown domain based on the extension theorem is a convex domain.

Consequence 2: If G is covered by a set of open subdomain G_i, $(G = \bigcup_i G_i)$, and if the system is shakedown in each G_i, then it will be shakedown in G.

In fact, wherever G is covered by a set of open subdomains G_i it will be easy to extract from it a set of E. Melan vicinities $\{\Delta P_i\}$ to let the system shakedown on G.

Consequence 1 helps us to set up and find G approximately, Consequence 2 allows us to prolounge the shakedown domain from a subdomain G_i to a subdomain G_j whenever $G_i \cap G_j \neq \phi$

4. EXAMPLE

Consider the shaft subjected to tensional and torsional forces [4]

The elastic solution are

$$\tau_{\varphi z}^* = \frac{2\pi}{r}, \qquad \sigma_z^* = \frac{P}{\pi a^2}$$

where a is the radius of the shaft, M is the torsional moment, P is the tensional force.

Consider the dimensionless variables.

$$\rho = \frac{r}{a}, \quad \tau^* = \frac{\tau_{\varphi z}^*}{\tau_s}, \quad \sigma^* = \frac{\sigma_z^*}{\sigma_s}, \quad m = \frac{2M}{\pi a^3 \tau_s}, \quad p = \frac{P}{\pi a^2 \sigma_s}$$

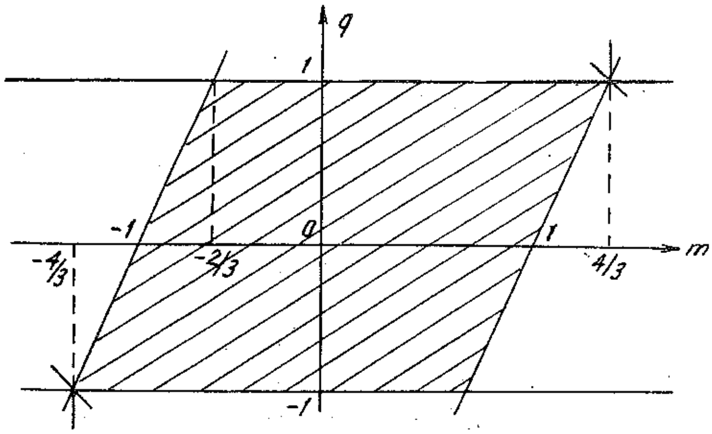
The residual stress is selected $\sigma^0 = 0$, $\tau^0 = q(1 - \frac{4}{3}\rho)$ q is not determined yet.

The case of only acting moment M $(P = 0)$

Choose the yield condition $|\tau| \leq 1 \Rightarrow$

$$|M\rho + q(1 - \frac{4}{3}\rho)| \leq 1 \qquad \forall\rho \in [0,1]$$

On the plane (m, q), we graphically determine the admissible domain of M with corresponding q is $\quad -\frac{4}{3} \leq m \leq \frac{4}{3}$



If we use the E. Melan's theorem, then with a determined value of $q \in [-1, 1]$, we have the corresponding variable domain of M, while always keeping $\Delta m = 2$.

Choosing $q = 1$, we have the shakedown domain

$$G_1 = \left\{ -\frac{2}{3} \leq m \leq \frac{4}{3} \right\}$$

Choosing $q = -1$, we have the shakedown domain

$$G_2 = \left\{ -\frac{4}{3} \leq m \leq \frac{2}{3} \right\}$$

here G_1, G_2 are subdomains.

From Consequence two, we have shakedown domain is

$$-\frac{4}{3} \leq m \leq \frac{4}{3} \qquad (\Delta m = \frac{8}{3})$$

Case when $p \neq 0 \Rightarrow$

$$|m\rho + q(1 - \frac{4}{3}\rho)| \leq \sqrt{1 - p^2} \quad \forall \rho \in [,1] \quad \Rightarrow \quad |m| \leq \frac{4}{3}\sqrt{1 - p^2}$$

REFERENCES

1. Koiter W. T. General theorem of elastic-plastic Solids, in Progress in Solid Mechanics. (North-Holland, Amsterdam 1960)
2. Prager W. Shakedown in elastic-plastic media subjected to cycles of load and temperature. Symposium su la plasticita nella scienza delle constutioni in onore de Arturo Danusso Varenna, Stettember 1956, Bologna, 1957.
3. Pho Nguyen van. An extension of the E. Melan's theorem and the method of solving the control problem and optimal design of the elasto-plastic system subjected combined to variable loads and temperatures. Journal of Science and Technology No 3, 1976 (in Vietnamese)
4. Rozenblum V. I. Shakedown theory of elasto-plastic Solid. News of the Academy of Science USSR STD No 6, 1958.

Received January 11, 1992
(xem tiếp trang 32)