A NEW APPROACH TO THE STABILITY PROBLEM OF PLATES SUBJECTED TO ARBITRARY COMPLEX LOADING

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SUMMARY. The pre-buckling and post-buckling deformation processes are assumed to be less complicated, i.e. processes of average curvature, the influence of complex loading on the stability of plates was analysed in [1]. In this paper eliminating this restriction, post-buckling process may be arbitrary complicated, a generalized expression for determining critical force is formulated by Bubnov-Galiorkin's method and loading parameter method.

1. PRE-BUCKLING PROCESS

Let consider a rectangular plate subjected to biaxial compressions of intensities $p(t)$ and $q(t)$. At any moment t there exists a membrane plane stress state in the plate

$$\sigma_{11} = -p(t), \quad \sigma_{22} = -q(t), \quad \sigma_{12} = \sigma_{33} = \sigma_{32} = \sigma_{31} = 0,$$

so that

$$\sigma = -\frac{1}{3}(p+q), \qquad \sigma_u = (p^2 - pq + q^2)^{1/2}. \tag{1.1}$$

The strain velocity tensor is determined from the following equations

$$\dot{\varepsilon}_{11} = \frac{1}{\sigma_u/s}\left(-\dot{p} + \frac{1}{2}\dot{q}\right) - \left(\frac{1}{\phi'(s)} - \frac{1}{\sigma_u/s}\right)\frac{\left(p\dot{p} + q\dot{q} - \frac{1}{2}p\dot{q} - \frac{1}{2}q\dot{p}\right)\left(p - \frac{1}{2}q\right)}{p^2 - pq + q^2},$$

$$\dot{\varepsilon}_{22} = \frac{1}{\sigma_u/s}\left(-\dot{q} + \frac{1}{2}\dot{p}\right) - \left(\frac{1}{\phi'(s)} - \frac{1}{\sigma_u/s}\right)\frac{\left(p\dot{p} + q\dot{q} - \frac{1}{2}p\dot{q} - \frac{1}{2}q\dot{p}\right)\left(q - \frac{1}{2}p\right)}{p^2 - pq + q^2}. \tag{1.2}$$

and the arc-length of deformation trajectory is evaluated from

$$\frac{ds}{dt} = \frac{2}{\sqrt{3}}\left(\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{22}^2 + \dot{\varepsilon}_{11}\dot{\varepsilon}_{22}\right)^{1/2} \equiv F(s, p, q). \tag{1.3}$$

2. POST-BUCKLING PROCESS

Suppose that external forces depend on a loading parameter t. The parameter t increases and will reach some value t_k. At this moment t_k a bifurcation of equilibrium states is assumed to appear: with an infinitesimal small increment of external force the plate is buckled and receives possible increments of deformation

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}^* - Z\delta\chi_{ij},$$

where

$$\delta\chi_{ij} = \frac{\partial^2 \delta w}{\partial x_i \partial x_j} \equiv \delta w_{,ij}.$$

These post-buckling deformation processes may be arbitrary complicated, so that the stress-strain relations are defined by the elastoplastic deformation process theory [2]

$$\delta\sigma_{ij} = \frac{2}{3}A\big(\delta\varepsilon_{ij} + \delta_{ij}\delta\varepsilon_{kk}\big) + \big(P - A\big)\frac{\sigma_{k\ell}\delta\varepsilon_{k\ell}}{\sigma_u^2}\sigma_{ij}\,, \tag{2.1}$$

where

$$
\begin{aligned}
A &= -\frac{\sigma_u f}{\sin\theta} = \frac{\sigma_u}{s}\left[1 + \left(\frac{3Gs}{\sigma_u} - 1\right)\frac{1 - \cos\theta}{2}\right] = \frac{1}{2}\left(3G + \frac{\sigma_u}{s} - \frac{1}{2}\right)\left(3G - \frac{\sigma_u}{s}\right)\cos\theta, \\
P &= \frac{\psi}{\cos\theta} = \phi'(s) - \frac{3G - \phi\prime}{\cos\theta}\frac{1 - \cos\theta}{2} = \frac{1}{2}(3G + \phi') - \frac{1}{2}\frac{3G - \phi\prime}{\cos\theta}\,,
\end{aligned} \tag{2.2}
$$

with

$$\cos\theta = \frac{\sigma_{ij}\delta\varepsilon_{ij}}{\sigma_u\delta s}\,, \quad \delta s = \frac{2}{\sqrt{3}}\big(\delta\varepsilon_{11}^2 + \delta\varepsilon_{22}^2 + \delta\varepsilon_{11}\delta\varepsilon_{22} + \delta\varepsilon_{12}^2\big)^{1/2}.$$

According to Ilyushin's approximate statement $\delta T_{ij} = 0$ and not accounting the unloading, we obtain

$$\delta\varepsilon_{ij}^* = 0, \quad \cos\theta = -\frac{Z\sigma_{ij}\delta w_{,ij}}{\sigma_u\delta s}\,, \quad \delta s = \frac{2}{\sqrt{3}}Z\big(\delta w_{,11}^2 + \delta w_{,22}^2 + \delta w_{,11}\delta w_{,22} + \delta w_{,12}^2\big)^{1/2}.$$

Hence the quantity $\cos\theta$ does not depend on Z, such that A and P do not depend on Z as well. The bending moments are of the form

$$\delta M_{ij} = \int_{-h/2}^{h/2} \delta\sigma_{ij}Z\,dZ = \frac{h^3}{12}\left[-\frac{2}{3}A\big(\delta w_{,ij} + \delta_{ij}\delta w_{,kk}\big) + \big(A - P\big)\frac{\sigma_{ij}\sigma_{k\ell}}{\sigma_u^2}\delta w_{,k\ell}\right]. \tag{2.3}$$

Substituting δM_{ij} by (2.3), where A, P contain δw, into the stability equation

$$\frac{\partial^2 \delta M_{ij}}{\partial x_i \partial x_j} + T_{ij}\delta\chi_{ij} = 0,$$

we obtain

$$
\begin{aligned}
&\left[A - \frac{3}{4}(A - P)\frac{p^2}{p^2 - pq + q^2}\right]\frac{\partial^4 \delta w}{\partial x^4} + 2\left[A - \frac{3}{4}(A - P)\frac{pq}{p^2 - pq + q^2}\right]\frac{\partial^4 \delta w}{\partial x^2 \partial y^2} + \\
&+ \left[A - \frac{3}{4}(A - P)\frac{q^2}{p^2 - pq + q^2}\right]\frac{\partial^4 \delta w}{\partial y^4} + \frac{9p}{h^2}\frac{\partial^2 \delta w}{\partial x^2} + \frac{9p}{h^2}\frac{\partial^2 \delta w}{\partial y^2} + \\
&+ \frac{1}{2}\frac{\partial^2 A}{\partial x^2}\left(2\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 \delta w}{\partial y^2}\right) + \frac{1}{2}\frac{\partial^2 A}{\partial y^2}\left(2\frac{\partial^2 \delta w}{\partial y^2} + \frac{\partial^2 \delta w}{\partial x^2}\right) + \frac{\partial^2 A}{\partial x\partial y}\frac{\partial^2 \delta w}{\partial x\partial y} - \\
&- \frac{3}{4}\frac{p\frac{\partial^2 \delta w}{\partial x^2} + q\frac{\partial^2 \delta w}{\partial y^2}}{p^2 - pq + q^2}\left[p\left(\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 P}{\partial x^2}\right) + q\left(\frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 P}{\partial y^2}\right)\right] = 0.
\end{aligned} \tag{2.4}
$$

Satisfying kinematic boundary conditions with edges simply supported we can find a solution of the form

$$\delta w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

Notice that $\cos \theta$ does not contain C, so that A, P also do not contain C. Substituting δw into the stability equation and applying Bubnov-Galiorkin's method [3]:

$$\int_0^a \int_0^b X \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dx dy = 0, \tag{2.5}$$

where X is the expression in the left hand side of the stability equation (2.4), give from (2.5) an equation for finding a critical force.

Consider *the case of a square plate*. In this case

$$\delta w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}, \tag{2.6}$$

$$\cos \theta = -\frac{p + q}{(p^2 - pq + q^2)^{1/2} \frac{2}{\sqrt{3}} \left(3 + \operatorname{ctg}^2 \frac{\pi x}{a} \operatorname{ctg}^2 \frac{\pi y}{a}\right)^{1/2}}. \tag{2.7}$$

According to (2.6), the equation (2.4) reduces to

$$X \equiv \left[\frac{\pi^4}{a^4}\left(4 - \frac{3}{4}\frac{(p+q)^2}{p^2 - pq + q^2}\right)A + \frac{3\pi^4}{4a^4}\frac{(p+q)^2}{p^2 - pq + q^2}P - \frac{9\pi^2}{a^2 h^2}(p+q)+\right.$$
$$+ \frac{3\pi^2}{4a^2}\left(-2 + \frac{p(p+q)}{p^2 - pq + q^2}\right)\frac{\partial^2 A}{\partial x^2} + \frac{3\pi^2}{4a^2}\left(-2 + \frac{q(p+q)}{p^2 - pq + q^2}\right)\frac{\partial^2 A}{\partial y^2} - \frac{3\pi^2}{4a^2}\frac{p(p+q)}{p^2 - pq + q^2}\frac{\partial^2 P}{\partial x^2} -$$
$$\left. - \frac{3\pi^2}{4a^2}\frac{q(p+q)}{p^2 - pq + q^2}\frac{\partial^2 P}{\partial y^2}\right]\sin \frac{\pi x}{a}\sin \frac{\pi y}{a} + \frac{\pi^2}{a^2}\frac{\partial^2 A}{\partial x \partial y}\cos \frac{\pi x}{a}\cos \frac{\pi y}{a} = 0. \tag{2.8}$$

The expressions of A and P from (2.2) and (2.7) are written in the form

$$A = A_1(s, p, q) + A_2(s, q, p)f(x, y). \tag{2.9}$$

where

$$A_1 = \frac{1}{2}\left(3G + \frac{\sigma_u}{s}\right), \quad A_2 = \frac{1}{2}\left(3G - \frac{\sigma_u}{s}\right)\frac{p+q}{\sigma_u},$$

$$\sigma_u = (p^2 - pq + q^2)^{1/2}, \quad f(x, y) = \frac{\sqrt{3}}{2}\left(3 + \operatorname{ctg}^2 \frac{\pi x}{a}\operatorname{ctg}^2 \frac{\pi y}{a}\right)^{-1/2},$$

and

$$P = P_1(s, p, q) + P_2(s, p, q)g(x, y), \tag{2.10}$$

where

$$P_1 = \frac{1}{2}(3G + \phi'), \quad P_2 = \frac{1}{2}(3G - \phi')\frac{\sigma_u}{p+q}, \quad g(x, y) = \frac{2}{\sqrt{3}}\left(3 + \operatorname{ctg}^2 \frac{\pi x}{a}\operatorname{ctg}^2 \frac{\pi y}{a}\right)^{1/2} = \frac{1}{f(x, y)}.$$

Hence

$$\frac{\partial^2 A}{\partial x^2} = A_2 f''_{xx}, \quad \frac{\partial^2 A}{\partial y^2} = A_2 f''_{yy}, \quad \frac{\partial^2 A}{\partial x \partial y} = A_2 f''_{xy}, \quad \frac{\partial^2 P}{\partial x^2} = P_2 g''_{xx}, \quad \frac{\partial^2 P}{\partial y^2} = P_2 g''_{yy}.$$

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Applying Bubnov-Galiorkin's method

$$\int\limits_0^a \int\limits_0^a X \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}\, dxdy = 0,$$

we obtain

$$\frac{\pi^2}{4}\left(4 - \frac{3}{4}\frac{(p+q)^2}{p^2-pq+q^2}\right)A_1 + \frac{3\pi^2}{16}\frac{(p+q)^2}{p^2-pq+q^2}P_1 + \frac{\pi^2}{a^2}\left(4 - \frac{3}{4}\frac{(p+q)^2}{p^2-pq+q^2}\right)C_1 A_2 +$$

$$+ \frac{3\pi^2}{4a^2}\frac{(p+q)^2}{p^2-pq+q^2}C_5 P_2 - \frac{9(p+q)}{4}\frac{a^2}{h^2} + C_4 A_2 + \frac{3}{4}\left(-2 + \frac{p(p+q)}{p^2-pq+q^2}\right)C_2 A_2 +$$

$$+ \frac{3}{4}\left(-2 + \frac{q(p+q)}{p^2-pq+q^2}\right)C_3 A_2 - \frac{3}{4}\frac{p(p+q)}{p^2-pq+q^2}C_6 P_2 - \frac{3}{4}\frac{q(p+q)}{p^2-pq+q^2}C_7 P_2 = 0. \tag{2.11}$$

where $C_i (i = 1 \div 7)$ are constants, evaluated by the following integrals

$$C_1 = \int\limits_0^a \int\limits_0^a f(x,y)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy; \quad C_2 = \int\limits_0^a \int\limits_0^a f''_{xx}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy;$$

$$C_3 = \int\limits_0^a \int\limits_0^a f''_{yy}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy; \quad C_4 = \int\limits_0^a \int\limits_0^a f''_{xy}\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\cos\frac{\pi y}{a}\, dxdy;$$

$$C_5 = \int\limits_0^a \int\limits_0^a g(x,y)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy;$$

$$C_6 = \int\limits_0^a \int\limits_0^a g''_{xx}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy; \quad C_7 = \int\limits_0^a \int\limits_0^a g''_{yy}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\, dxdy.$$

Substituting the expressions of A_1, A_2, P_1, P_2 into the equation (2.11), from which we get the formula for determining a critical force

$$i^2 \equiv \frac{9a^2}{h^2} = \frac{4}{p+q}\left\{\frac{\pi^2}{2}\left(3G + \frac{\sigma_u}{s}\right) + \frac{3\pi^2}{32}\left(\phi' - \frac{\sigma_u}{s}\right)\frac{(p+q)^2}{p^2-pq+q^2} + \right.$$

$$+ \frac{1}{2}\left(3G - \frac{\sigma_u}{s}\right)\frac{p+q}{\sigma_u}\left[\frac{\pi^2}{a^2}\left(4 - \frac{3}{4}\frac{(p+q)^2}{p^2-pq+q^2}\right)C_1 + \frac{3}{4}\left(-2 + \frac{p(p+q)}{p^2-pq+q^2}\right)C_2 + \right.$$

$$+ \frac{3}{4}\left(-2 + \frac{q(p+q)}{p^2-pq+q^2}\right)C_3 + C_4\right] + \frac{1}{2}(3G - \phi')\frac{\sigma_u}{p+q}\left[\frac{3\pi^2}{4a^2}\frac{(p+q)^2}{p^2-pq+q^2}C_5 - \right.$$

$$\left. - \frac{3}{4}\frac{p(p+q)}{p^2-pq+q^2}C_6 - \frac{3}{4}\frac{q(p+q)}{p^2-pq+q^2}C_7\right]\right\} \equiv H(s,p,q), \tag{2.12}$$

where $\sigma_u = \left(p^2 - pq + q^2\right)^{1/2}$.

Now, suppose that $p \equiv p(t)$, $q \equiv q(t)$ are known as functions of loading parameter t. The equations (1.3) and (2.12) are satisfied simultaneously, from that we can determine a critical value t_k of the loading parameter. Then the critical forces are as follows

$$p_{th} = p(t_k); \qquad q_{th} = q(t_k).$$

In particular case when post-buckling process is a process of average curvature, we have

$$A = \frac{\sigma_u}{s}, \qquad P = \phi'(s)$$

from (2.2) to get

$$\cos\theta = -\frac{p+q}{\sigma_u} f(x,y) = 1,$$

the functions $f(x,y)$ and $g(x,y)$ must be constant. Hence we obtain

$$-\frac{p+q}{\sigma_u}C_1 = \frac{a^2}{4}, \qquad -\frac{\sigma_u}{p+q}C_5 = \frac{a^2}{4}, \qquad C_2 = C_3 = C_4 = C_6 = C_7 = 0.$$

Finally, from (2.12) we get the known expression of the critical force in [1]

$$i^2 \equiv \frac{9a^2}{h^2} = \frac{4}{p+q}\left[\pi^2\frac{\sigma_u}{s} + \frac{3\pi^2}{16}\left(\phi' - \frac{\sigma_u}{s}\right)\frac{(p+q)^2}{p^2 - pq + q^2}\right],$$

or

$$i^2 \equiv \frac{9a^2}{h^2} = \frac{3G\pi^2\varphi_N}{p+q}\left[4 - \frac{3}{4}\left(1 - \frac{\varphi_t}{\varphi_N}\right)\frac{(p+q)^2}{p^2 - pq + q^2}\right],$$

where

$$\varphi_N = \frac{\sigma_u/s}{3G} = \frac{(p^2 - pq + q^2)^{1/2}}{3Gs}, \qquad \varphi_t = \frac{\phi'(s)}{3G}.$$

CONCLUSIONS

1. A typical example on the application of the general elastoplastic deformation process theory in the stability problem of plates is given.

2. Establishing a method for formulating an expression of critical force in the general case without any restriction on the arbitrary complex loading, this approach has a practical meaning in engineering calculation.

3. Received result in [1] is a particular case of the generalized expression (2.12) shown in this paper.

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MỘT CÁCH TIẾP CẬN MỚI BÀI TOÁN ỔN ĐỊNH CỦA BẢN CHỊU TẢI PHỨC TẠP BẤT KỲ

Gỡ bỏ mọi giả thiết hạn chế lên quá trình biến dạng sau khi mất ổn định, ở đây có thể xem đó là quá trình phức tạp bất kỳ. Bằng phương pháp Bubnov - Galiorkin và phương pháp tham số tải đã thiết lập được công thức hiển xác định lực tới hạn tổng quát hơn kết quả nhận được trước đây.