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QUADRATIC AND CUBIC NON-LINEARITIES IN A QUASI-LINEAR PARAMETRICALLY-EXCITED SYSTEM

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SUMMARY. A quasi-linear oscillating system always contains weak factors (negative and positive frictions, exciting forces in resonances, non-linearities of restoring forces) usually characterized by a small parameters ε. Recently, some articles have distinguished the degree of smallness of different factors and interesting phenomena have been found [3, 4]. In the present paper, using the same manner, we shall consider a quasi-linear parametrically-excited system and the difference between the quadratic non-linearity and the cubic one, will be observed.

§1. SYSTEM UNDER CONSIDERATION AND DIFFERENT FORMS OF ITS DIFFERENTIAL EQUATION

Let us consider a quasi-linear parametrically-excited system described by the following differential equation:

$$\ddot{x} + \nu^2 x = \beta x^2 - \gamma x^3 + \Delta x - h\dot{x} + 2qx\cos 2\nu t, \tag{1.1}$$

where x is standard oscillatory variable; $2q > 0$, $2\nu > 0$ are intensity and frequency of the parametric excitation; $\Delta = (\nu^2 - 1)$ is detuning parameter; $h > 0$ is linear damping coefficient; β, γ are coefficients of the quadratic and cubic non-linearities, respectively; overdots denote differentiation with respect to time t.

By standard variable we mean the "original" one, divided by an appropriate positive number so that the maximum of the absolute value $|x|$ is close to unity. Let us denote the small parameter by ε and suppose that h and q are of order of smallness ε^2. Depending on the order of smallness of β, γ, Δ, the differential equation (1.1) can be written in different forms, among them, the most interesting forms are:

- if β, γ, Δ are of the same order ε^2 as h and q, we have:

$$\ddot{x} + \nu^2 x = \varepsilon^2\{\beta x^2 - \gamma x^3 + \Delta x - h\dot{x} + 2qx\cos 2\nu t\}, \tag{1.2}$$

- if β, γ, Δ are of the same order ε, the differential equation (1.1) becomes:

$$\ddot{x} + \nu^2 x = \varepsilon\{\beta x^2 - \gamma x^3 + \Delta x\} + \varepsilon^2\{-h\dot{x} + 2qx\cos 2\nu t\}, \tag{1.3}$$

- at last if β is of order ε while γ, Δ are of order ε^2, we have:

$$\ddot{x} + \nu^2 x = \varepsilon\{\beta x^2\} + \varepsilon^2\{-\gamma x^3 + \Delta x - h\dot{x} + 2qx\cos 2\nu t\}, \tag{1.4}$$

For the sake of simplicity, γ is assumed to be positive while β may by either positive or negative. The case in which γ and Δ are of different orders must be omitted by the reason presented below, at the end of §4.

§2. SYSTEM WITH THE NON-LINEARITIES OF ORDER ε^2

First, we study the case, described by the differential equation (1.2). Using the asymptotic method [1], the solution of (1.2) is found in the form:

$$\begin{aligned} &x = a\cos\psi + \varepsilon u_1(a,\psi,\theta) + \varepsilon^2 u_2(a,\psi,\theta),\\ &\dot{a} = \varepsilon A_1(a,\theta) + \varepsilon^2 A_2(a,\theta),\\ &\dot{\theta} = \varepsilon B_1(a,\theta) + \varepsilon^2 B_2(a,\theta), \quad \psi = \nu t + \theta. \end{aligned} \tag{2.1}$$

where a, θ are amplitude and phase of the oscillatory regime, respectively; u_1, u_2 (A_1, B_1, A_2, B_2) are unknown functions of a, ψ, $\theta(a,\theta)$, which are periodic with period 2π relative to ψ, $\theta(\theta)$.

Substituting (2.1) into (1.2), equating the coefficients of like powers of ε in both sides, then indentifying the terms with the same harmonics, we obtain:

$$\begin{aligned} &A_1 = 0, \quad B_1 = 0, \quad u_1 = 0,\\ &-2\nu A_2 = ha\nu + qa\sin 2\theta,\\ &-2\nu a B_2 = \Delta a - \frac{3}{4}\gamma a^3 + qa\cos 2\theta. \end{aligned} \tag{2.2}$$

Thus, for new variables a and θ, in the second approximation, the differential equation are of the form:

$$\begin{aligned} &\dot{a} = -\varepsilon^2\frac{a}{2\nu}\{h\nu + q\sin 2\theta\},\\ &\dot{\theta} = -\frac{1}{2\nu}\varepsilon^2\{\Delta - \frac{3}{4}\gamma a^2 + q\cos 2\theta\}. \end{aligned} \tag{2.3}$$

The stationary oscillation is determined by equating zero the right-hand sides of the system (2.3):

$$\begin{aligned} &h\nu + q\sin 2\theta = 0,\\ &\Delta - \frac{3}{4}\gamma a^2 + q\cos 2\theta = 0. \end{aligned} \tag{2.4}$$

Eliminating θ from (2.4) leads to a relationship between the amplitude a and the frequency ν:

$$W(a,\nu^2) = \left[\frac{3}{4}\gamma a^2 - \Delta\right]^2 - \Delta_1 = 0, \quad \Delta_1 = q^2 - h^2\nu^2, \tag{2.5a}$$

or

$$\frac{3}{4}\gamma a^2 = (\nu^2 - 1) \pm \sqrt{\Delta_1}, \tag{2.5b}$$

In order to study the stability of the stationary oscillation (a_0, θ_0), the pertubations δa, $\delta\theta$ are introduced, namely:

$$a = a_0 + \delta a, \qquad \theta = \theta_0 + \delta\theta. \tag{2.6}$$

Substituting (2.6) and into (2.3), using (2.4) and linearizing the resulting equations, we obtain the variational equations:

$$\begin{aligned}(\delta a)^{\cdot} &= -\frac{a_0}{\nu} q\cos 2\theta_0 \cdot \delta\theta,\\ (\delta\theta)^{\cdot} &= \frac{1}{\nu a_0}\cdot\frac{3}{4}\gamma a_0^2\cdot\delta a + \frac{1}{\nu} q\sin 2\theta_0\cdot\delta\theta.\end{aligned} \tag{2.7}$$

The characteristic equation of the variational system (2.7) is:

$$\begin{vmatrix} \varrho & -\dfrac{a_0}{\nu}\cos 2\theta \\ \dfrac{1}{\nu a_0}\cdot\dfrac{3}{4}\gamma a_0^2 & \dfrac{q}{\nu}\sin 2\theta_0 - \varrho \end{vmatrix} = \varrho^2 + h\varrho + \frac{a_0}{4\nu^2}\frac{\partial W(a_0,\nu^2)}{\partial a_0} = 0 \tag{2.8}$$

Since $h > 0$, the condition for stability is given by the inequality:

$$\frac{\partial W(a_0,\nu^2)}{\partial a_0} > 0 \quad \text{or} \quad \pm\sqrt{\Delta_1} > 0 \tag{2.9}$$

From (2.9) and (2.5), it follows that the oscillation corresponding to the amplitude a_0 and the phase θ_0 as the stability condition of the stationary oscillation do not depend on the quadratic non-linearity; only the cubic one is the decisive factor so that the present system is nearly identical to the classical parametrically-excited one [1, 2] with only a little difference in their forms: ε is replaced by ε^2.

§3. SYSTEM WITH THE NON-LINEARITIES OF ORDER ε

For the second case, described by the differential equation (1.3), the same asymptotic method is applied. The unknown functions in the asymptotic expansion are:

$$\begin{aligned} &A_1 = 0, \quad B_1 = -\frac{1}{2\nu}\Big\{\Delta - \frac{3}{4}\gamma a^2\Big\},\\ &u_1 = \frac{\beta a^2}{2\nu^2} - \frac{\beta a^2}{6\nu^2}\cos 2\psi + \frac{\gamma a^3}{32\nu^2}\cos 3\psi,\\ &-2\nu A_2 = h\nu a + qa\sin 2\theta,\\ &-2\nu a B_2 = \frac{a}{4\nu^2}\Big[\Delta - \frac{3}{4}\gamma a^2\Big]^2 + \frac{5\beta^2}{6\nu^2}a^3 - \frac{3\gamma^2}{128\nu^2}a^5 + qa\cos 2\theta, \end{aligned} \tag{3.1}$$

and the differential equations, in the second approximation, for a and θ of the form:

$$\begin{aligned} \dot a &= -\frac{\varepsilon^2 a}{2\nu}\{h\nu + q\sin 2\theta\},\\ \dot\theta &= -\frac{\varepsilon}{2\nu}\Big\{\Delta - \frac{3}{4}\gamma a^2\Big\} - \frac{\varepsilon^2}{2\nu}\Big\{\frac{1}{4\nu^2}\Big[\Delta - \frac{3}{4}\gamma a^2\Big]^2 + \frac{5\beta^2}{6\nu^2}a^2 - \frac{3\gamma^2}{128\nu^2}a^4 + q\cos 2\theta\Big\}, \end{aligned} \tag{3.2}$$

where $\sigma = \dfrac{5\beta^2}{6} : \dfrac{3}{4}\gamma$.

The amplitude a and the phase θ of the stationary oscillation satisfy the following equations:

$$\begin{aligned} &h\nu + q\sin 2\theta = 0,\\ &\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2 - \frac{1}{2\nu^2}(3\nu^2 - 1 - 2\sigma)\Big(\frac{3}{4}\gamma a^2\Big) +\\ &+ (\nu^2 - 1) + \frac{(\nu^2-1)^2}{4\nu^2} + q\cos 2\theta = 0, \end{aligned} \tag{3.3}$$

and, after eliminating θ, the relationship between a and ν is obtained:

$$W(a,\nu^2)=\Big\{\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2-\frac{1}{2\nu^2}(3\nu^2-1-2\sigma)\Big(\frac{3}{4}\gamma a^2\Big)+ \\ +(\nu^2-1)+\frac{(\nu^2-1)^2}{4\nu^2}\Big\}^2-\Delta_1=0 \tag{3.4a}$$

or

$$\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2-\frac{1}{2\nu^2}(3\nu^2-1-2\sigma)\Big(\frac{3}{4}\gamma a^2\Big)+ \\ +(\nu^2-1)+\frac{(\nu^2-1)^2}{4\nu^2}\mp\sqrt{\Delta_1}=0. \tag{4.3b}$$

The equation (3.4b) admits the solution:

$$\frac{3}{4}\gamma a^2=\frac{6}{5}(3\nu^2-1-2\sigma)(1-\sqrt{\Delta_2}), \\ \Delta_2=1-\frac{10\nu^2}{3(3\nu^2-1-2\sigma)^2}\Big\{(\nu^2-1)\frac{(\nu^2-1)^2}{4\nu^2}\pm\sqrt{\Delta_1}\Big\}. \tag{3.5}$$

(The other solutions of the equation (3.4b) (sign + before $\sqrt{\Delta_1}$) must be rejected as meaningless, since the corresponding amplitude is too large for standard variable). Expanding $\sqrt{\Delta_2}$ and neglecting the terms of third and higher powers with respect to (ν^2-1), σ, $\sqrt{\Delta_1}$, we obtain a more simple relationship:

$$\frac{3}{4}\gamma a^2=(\nu^2-1)-\frac{(\nu^2-1)^2}{24}+(\nu^2-1)\sigma\pm\sqrt{\Delta_1}. \tag{3.6}$$

As in §2, to analyse the stability of the stationary oscillation (a_0,θ_0), we use:
- the variational equations:

$$(\partial a)^{\cdot}=-\frac{a_0}{\nu}q\cos 2\theta_0\cdot\delta\theta, \\ (\partial\theta)^{\cdot}=-\frac{1}{\nu a_0}\Big\{2\cdot\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a_0^2\Big)^2-\frac{1}{2\nu^2}(3\nu^2-1-\sigma)\Big(\frac{3}{4}\gamma a_0^2\Big)\Big\}\delta a^{\cdot}+\frac{q}{\nu}\sin 2\theta_0\cdot\delta\theta, \tag{3.7}$$

- then, the characteristic equation:

$$\varrho^2+h\varrho+\frac{a_0}{4\nu^2}\frac{\partial W(a_0,\nu^2)}{\partial a_0}=0 \tag{3.8}$$

The stability condition is as previously:

$$\frac{\partial W(a_0,\nu^2)}{\partial a_0}>0\quad\text{or}\quad\pm\sqrt{\Delta_1}>0 \tag{3.9}$$

Thus, the same conclusion as in §2 is obtained: the oscillation corresponding to the larger (smaller) amplitude is stable (unstable).

The relationship (3.6) shows that, in the first approximation, the system described by the differential equation (1.3) belongs to hard kind and the quadratic non-linearity (the term $(\nu^2-1)\sigma$) makes it less hard.

Figure 1 shows the resonant curves $a=a(\nu^2)$ for a typical case: $h^2=0,00010$, $q^2=0,00011$, $\gamma=0,16$, $\sigma=0(a)$ and $\sigma=0,2(b)$ (the formulae (3.5) and (3.6) give practically the same curves).

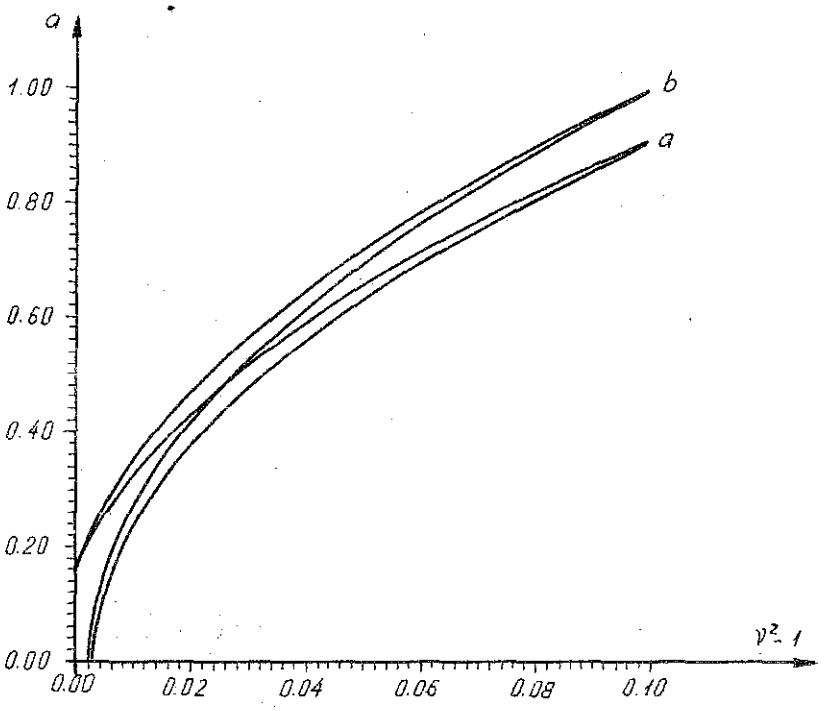

Fig. 1

§4. SYSTEM WITH THE NON-LINEARITIES OF DIFFERENT ORDERS

The third case, described by the differential equation (1.4), is more interesting: the quadratic non-linearity is now of order ϵ while the cubic is of order ϵ^2.

The asymptotic method gives us successively:

$$\begin{aligned} &A_1 = 0, \quad B_1 = 0, \quad u_1 = \frac{\beta a^2}{2\nu^2} - \frac{\beta a^2}{6\nu^2}\cos 2\psi, \\ &-2\nu A_2 = h\nu a + qa\sin 2\theta, \\ &-2\nu a B_2 = \Big(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^3 + \Delta a + qa\cos 2\theta, \end{aligned} \tag{4.1}$$

where

$$\frac{1}{\nu_*^2} = \frac{3}{4}\gamma \ : \ \frac{5}{6}\beta^2,$$

and

$$\begin{aligned} &\dot{a} = -\frac{\epsilon^2 a}{2\nu}\{h\nu + q\sin 2\theta\}, \\ &\dot{\theta} = -\frac{\epsilon^2}{2\nu}\Big\{\Big(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\Big)\frac{5}{6}\beta^2 a^2 + \Delta + q\cos 2\theta\Big\}, \end{aligned} \tag{4.2}$$

The amplitude a and the phase θ of the stationary regime satisfy the equations:

$$\begin{aligned} &h\nu + q\sin 2\theta = 0, \\ &\Big(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^2 + \Delta + q\cos 2\theta = 0, \end{aligned} \tag{4.3}$$

from which, after eliminating θ, we obtain:

$$W(a,\nu^2)=\Big\{\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)\frac{5}{6}\beta^2a^2+\Delta\Big\}^2-\Delta_1=0, \tag{4.4a}$$

or:

$$\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^2=-(\nu^2-1)\pm\sqrt{\Delta_1}. \tag{4.4b}$$

It is easy to form the variational system:

$$\begin{aligned}(\delta a)^{\cdot}&=-\frac{a_0}{\nu}q\cos 2\theta_0\cdot\delta\theta,\\ (\delta\theta)^{\cdot}&=-\frac{1}{\nu a_0}\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^2\delta a+\frac{1}{\nu}q\sin 2\theta_0\cdot\delta\theta,\end{aligned} \tag{4.5}$$

and its characteristic equation:

$$\varrho^2+h\varrho+\frac{a_0}{4\nu^2}\frac{\partial W(a_0,\nu^2)}{\partial a_0}=0 \tag{4.6}$$

The stability condition is:

$$\frac{\partial W(a_0,\nu^2)}{\partial a_0}>0\quad\text{or}\quad\pm\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)\sqrt{\Delta_1}>0 \tag{4.7}$$

From (4.4b) it follows that the oscillation with the larger (smaller) amplitude corresponds to the sign + (−) before $\sqrt{\Delta_1}$ if $\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)>0$ and to the sign − (+) before $\sqrt{\Delta_1}$ if $\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)<0$. Hence, the inequality (4.7) leads to the same conclusion as in §2, §3: the oscillation corresponding to the larger (smaller) amplitude is stable (unstable).

The interesting phenomenon in the third case is that the character (soft-hardness) of the system under consideration depends on ν_*^2, i.e. on both coefficients β and γ simultaneously (see (4.4b)):

- if ν_*^2 is sufficiently large (γ is sufficiently small relative to β^2), then $\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)>0$ for all values $\nu^2\approx 1$, therefore the system belongs to soft kind,
- if ν_*^2 is sufficiently small (γ is sufficiently large relative to β^2), then $\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)<0$ for all values $\nu^2\approx 1$, therefore the system becomes hard one,
- if $\nu_*^2\approx 1$, the $\Big(\frac{1}{\nu^2}-\frac{1}{\nu_*^2}\Big)$ is close to zero, the system is neutralized: in the plane (ν^2,a), the resonant curve $a=a(\nu^2)$ disappears (since a is too large or does not existe) or degenerates into two straigh lines, nearly parallel to the ordinate axis.

Figure 2 shows the resonant curves for fixed values $h^2=0,000050$, $q^2=0,00051$, $\beta^2=0,0240$ and for various values $\nu_*^2=\infty(a)$, $\nu_*^2=2(b)$, $\nu_*^2=\frac{3}{4}(c)$ (the resonant curves lean to the left), $\nu_*^2=\frac{1}{2}(d)$, $\nu_*^2=\frac{1}{4}(e)$ (the resonant curves lean to the right).

Remark. We have omitted the case in which γ and Δ are of different orders. To clear up this question, let us consider the following differential equations:

$$\ddot{x}+\nu^2x=\varepsilon\{\beta x^2-\gamma x^3\}+\varepsilon^2\{\Delta x-h\dot{x}+2qx\cos 2\nu t\} \tag{4.8a}$$

$$\ddot{x}+\nu^2x=\varepsilon\{\beta x^2+\delta x\}+\varepsilon^2\{-\gamma x^3-h\dot{x}+2qx\cos 2\nu t\} \tag{4.8b}$$

For (4.8a), we obtain $B_1=3\gamma a^2/8\nu$ and the equation $\dot{\theta}=\varepsilon B_1+\varepsilon^2B_2=0$ is of the form:

$$\varepsilon\frac{3\gamma}{8\nu}a^2+\varepsilon^2B_2=0 \tag{4.9}$$

From (4.9), it follows that a^2 must be of order ε. Therefore, it is necessary to standardize again the oscillatory variable x. Indeed, dividing the both sides of (4.8a) by a small appropriately chosen positive numbers a_* (of order $\max|x|$), we obtain, for new variable $\chi = x/a_*$, a differential equation in which γ is replaced by $\gamma_* = \gamma a_*^2$. Since a_*^2 is small, the order of smallness of γ_* may be ε^2 and the differential equation (4.8a) may be transformed to another one of type (1.4).

Analogously for (4.8b) we obtain $B_1 = -\Delta/2\nu$ and the equation $\dot{\theta}=0$ becomes:

$$-\varepsilon\frac{\Delta}{2\nu}+\varepsilon^2B_2=0 \tag{4.10}$$

The latter equation may only be statisfied with very larger value a so that the "restandardization" is also necessary. Here, a_* must be large, γ_* may be of order ε and the differential equation (4.8b) is transformed to the type (1.3).

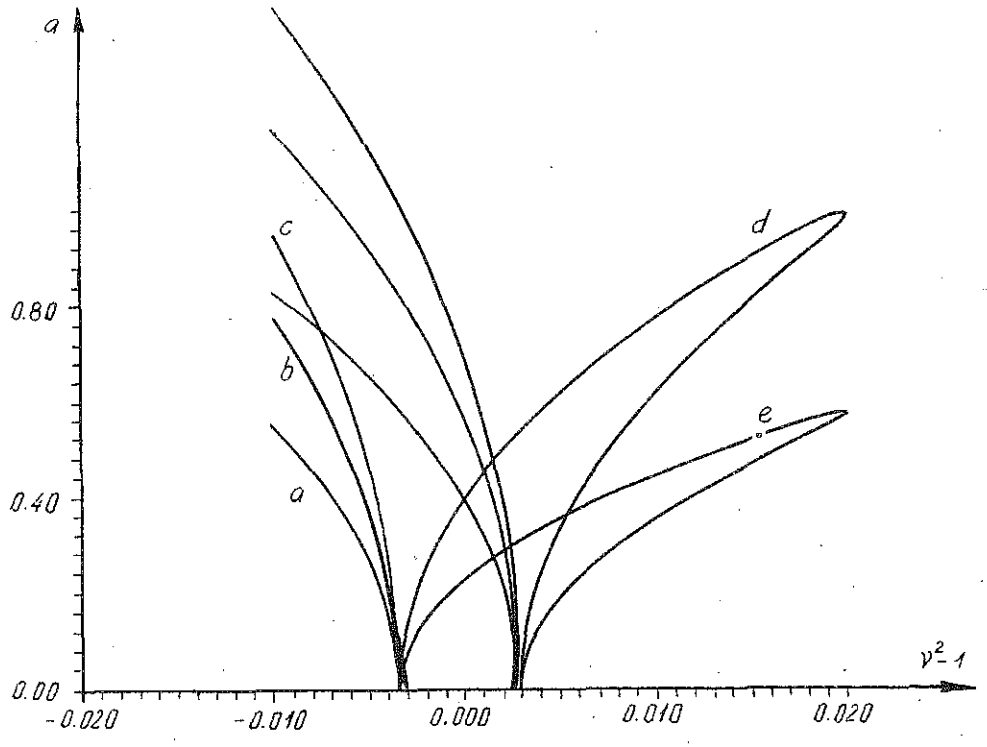

Fig. 2

CONCLUSION

The above analyses reveal the difference between the quadratic non-linearity and the cubic one in a quasi-linear parametrically-excited system. If the non-linearities are of the same order of smallness, the cubic is the dominant factor ($\gamma > 0$, the system belongs to hard kind). On the contrary, if the non-linearities are of different orders of smallness, the character (soft-hardness) of the system depends on both them, in equal degree.

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PHI TUYẾN BẬC HAI VÀ BẬC BA TRONG MỘT HỆ DAO ĐỘNG Á TUYẾN THÔNG SỐ

Khảo sát hệ dao động á tuyến thông số trong đó yếu tố đàn hồi phi tuyến được biểu diễn bởi hai số hạng bậc hai và bậc ba. Nếu hai số hạng trên tương ứng ở cấp ε và ε^2, tác động của chúng là tương đương và hệ có thể có tính mềm hoặc cứng. Ngược lại, nếu chúng ở cùng cấp, số hạng bậc ba quyết định tính mềm hoặc cứng của hệ.

ĐẠI HỘI ĐẠI BIỂU TOÀN QUỐC LẦN THỨ 3 HỘI CƠ HỌC VIỆT NAM

Ngày 6 tháng 12 năm 1992 Hội Cơ học Việt Nam đã tổ chức thành công Đại hội đại biểu toàn quốc lần thứ 3. Sau khi kiểm điểm lại các hoạt động của Hội trong 5 năm qua (1987-1992), thảo luận và quyết định phương hướng công tác Hội trong 5 năm tới (1992-1997) Đại hội đã bầu Bac chấp hành Trung ương hội nhiệm kỳ 3 (1992-1997) : *

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* Trong số 1, 1993 do thiếu xót khi soạn thảo chúng tôi đưa thiếu hai trường hợp phân công là Phó chủ tịch - Xin thành thật xin lỗi độc giả và BCH Trung Ương Hội.