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A NUMERICAL METHOD FOR STUDYING HOPF BIFURCATION OF PLANE POISEUILLE FLOW

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SUMMARY. The Hopf bifurcation of plane Poiseuille flows is studied numerically. On the base of the rigorous theory worked out by Joseph and Sattinger [1], a detailed effective and simple algorithm for numerical determination of the type of the above mentioned bifurcation is proposed. The obtained result shows that the bifurcation at the lower branch of the linear stability neutral curve is supercritical meantime at the upper branch the one is subscritical.

1. INTRODUCTION

In the hydrodynamical stability theory the plane Poiseuille flow is often taken as a good example for treatment of different approaches and methods of investigation. As is known, the linear stability problem for this flow has been studied very well for long time by many authors using a vast set of both analytical and numerical methods, for example [2, 3, 4].

The nonlinear stability analysis has been started by the works of Stuart [5, 6] and his method of formal amplitude expension has been developed in [7, 8, 9, 10]. These works relate to Landau conception of bifurcation and the authors's effort concentrates on calculation of the Landau's constant.

From the early seventies the Hopf bifurcation theory has been developed for the Navier-Stokes equations. Many interesting and important results have been obtained [1, 11, 12]. It is excellent that these theoretical achievements have been confirmed experimentally, for example [13]. The matter is formed so good that the nature of the transition to turbulence in fluid motions seems to be understood with the aid of the bifurcation theory [14].

The determination of the type of the Hopf bifurcation for concrete fluid flows is very interesting problem. It is necessary to note that for doing this a very cumbersume computational procedure is needed. The method given in [15] is a way for direct application of rigorous theory [1] to solving the above mentioned problem for the Poiseuille flow. The conclusion made in [8] is that: on the upper branch of the neutral curve the bifurcation is subcritical while on the lower branch it turns out supercritical. In [15] the calculation shows that subcritical bifurcation takes place at some neighbourhood of the minimal critical point and at the both parts adjacent to this arc on the upper and lower branches the Hopf bifurcation is supercritical. In this paper we use a method similar to one of [15] and the obtained result here allows to make the same conclution (see Tab. 1) as in [8]. The disagreement with [15] concerning the type of the bifurcation in the domain of the minimal critical point and at the upper branch of the neutral curve is caused by the fact that the solution of the first order problem in [15] contains only terms $\sin 2\theta$ and $\cos 2\theta$ (see (8.7) of [15]) whiles in the present paper this solution includes also terms $\sin^2\theta$ and $\cos^2\theta$ (see (18)). The method presented here may be used for flows with a free surface or interface too.

2. PROBLEM FORMULATION

Our analysis will be restricted to two-dimensional model. As is known, if we choose a co-ordinate system located at the middle of the channel with x-axis directed downstream, y-axis directed perpendicular to the flow and use the maximal velocity and a half-height of the channel as reference values then velocity profile of the stationary plane Poiseuille flow is written in the form (fig. 1)

$$U(y) = 1 - y^2; \qquad V = 0 \tag{2.1}$$

It is known that at small Reynolds numbers Re this flow is stable and at moderate Re it may lose its stability. Then a secondary flow which bifurcates from (2.1) may appear to be either stationary or time periodic. As is proven in [1], near the critical Reynolds numbers of the linear stability time periodic motions bifurcating from (2.1) exist and they are stable with respect to small disturbances if the bifurcation occurs at $Re > Re_c$ (supercritical) and in the opposite case they will be unstable. Our aim is to determine the type of this bifurcation at a region near the neutral curve.

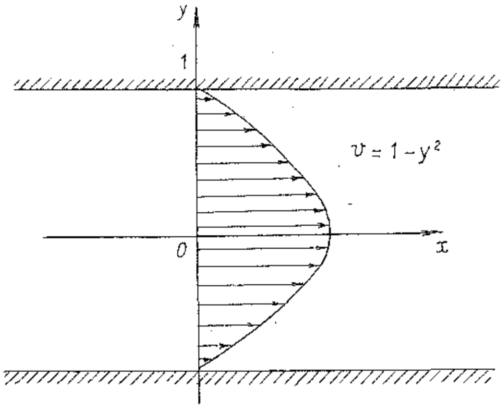

Fig. 1: Stationary plane Poiseuille flow

Suppose that $U(y)+u(x,y,t)$; $v(x,y,t)$; $p(x,y,t)$ is a new motion bifurcating from (2.1). Then the problem for finding it is written as follows:

$$\frac{\partial u}{\partial t} + (U+u)\frac{\partial u}{\partial x} + v\frac{\partial(u+U)}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{Re}\Delta u = 0 \tag{2.2}$$

$$\frac{\partial v}{\partial t} + (U+u)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} - \frac{1}{Re}\Delta v = 0 \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.4}$$

$$u(\pm 1) = v(\pm 1) = 0 \tag{2.5}$$

$$u(x,y,t) = u(x,y,t+T); \quad v(x,y,t) = v(x,y,t+T); \quad p(x,y,t) = p(x,y,t+T) \tag{2.6}$$

3. METHOD OF SOLUTION

As is indicated in [1, 16] the solutions of the above problem exist in a neighbourhood of every critical Reynolds number Re_c at which the spectral problem of (2.1)-(2.6) has a pair of eigenvalues $\pm i\omega_0$ and they can be presented in the form:

$$u = \sum_{n=0}^{\infty} \varepsilon^{n+1} u^n(x, y, s); \quad v = \sum_{n=0}^{\infty} \varepsilon^{n+1} v^n(x, y, s); \quad p = \sum_{n=0}^{\infty} \varepsilon^{n+1} p^n(x, y, s);$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots; \quad Re = Re_c(1 + \varepsilon R_1 + \varepsilon^2 R_2 + \dots) \tag{3.1}$$

where $s = \omega t$, ε is a value similar to the disturbance amplitude and defined by formular (9.3) in [16]. It is proven for this case that $\omega_{2k+1} = 0$ and $R_{2k+1} = 0$ for $k = 0, 1, 2, 3, \dots$ so

$$\omega = \omega_0 + \varepsilon^2\omega_2 + o(\varepsilon^4); \quad Re = Re_c[1 + \varepsilon^2 R_2 + o(\varepsilon^4)] \tag{3.2}$$

hence the type of the Hopf bifurcation should be defined by R_2. If R_2 is positive then the bifurcation is supercritical and in the opposite case one has subcritical bifurcation.

Substituting now (3.1) and (3.2) into (2.2) - (2.6) and collecting all terms of the same order of ε we get a sequence of linear nonhomogeneous problems for u^n, v^n, p^n. They are:

At zero-th order:

$$\omega_0 u_s^0 + U u_x^0 + U' v^0 - \frac{\Delta u^0}{Re_c} + p_x^0 = 0 \tag{3.3a}$$

$$\omega_0 v_s^0 + U v_x^0 - \frac{\Delta v^0}{Re_c} + p_y^0 = 0 \tag{3.3b}$$

$$u_x^0 + v_y^0 = 0 \tag{3.3c}$$

$$u^0(\pm 1) = v^0(\pm 1) = 0 \tag{3.3d}$$

At first order:

$$\omega_0 u_s^1 + U u_x^1 + U' v^1 - \frac{\Delta u^1}{Re_c} + p_x^1 = -(u^0 u_x^0 + v^0 u_y^0) \tag{3.4a}$$

$$\omega_0 v_s^1 + U v_x^1 - \frac{\Delta v^1}{Re_c} + p_y^1 = -(u_0 v_x^0 + v^0 v_y^0) \tag{3.4b}$$

$$u_x^1 + v_y^1 = 0 \tag{3.4c}$$

$$u^1(\pm 1) = v^1(\pm 1) = 0 \tag{3.4d}$$

At second order:

$$\omega_0 u_s^2 + U u_x^2 + U' v^2 - \frac{\Delta u^2}{Re_c} + p_x^2 = -\left(\omega_2 u_s^0 + u^1 u_x^0 + u^0 u_x^1 + v^1 u_y^0 + v^0 u_y^1 + R_2 \frac{\Delta u^0}{Re_c}\right) \tag{3.5a}$$

$$\omega_0 v_s^2 + U v_x^2 - \frac{\Delta v^2}{Re_c} + p_y^2 = -\left(\omega_2 v_s^0 + u^1 v_x^0 + u^0 v_x^1 + v^1 v_y^0 + v^0 v_y^1 + R_2 \frac{\Delta v^0}{Re_c}\right) \tag{3.5b}$$

$$u_x^2 + v_y^2 = 0 \tag{3.5c}$$

$$u^2(\pm 1) = v^2(\pm 1) = 0 \tag{3.5d}$$

Here the lower index denotes derivation with respect to the corresponding variable. As is known the necessary and sufficient condition for solvability of these non-homogeneous problems is that their right-hand side must be orthogonal to solution of the adjoint problem. For determining R_2 in this paper the following procedure is worked out:

3.1. Integration of the linear problem and its adjoint one

The zero-th order problem coincides with the linear stability problem so if the pertubed stream function is presented in the form [2]:

$$\psi(x,y,t) = \varphi(y)\ \exp\{i\alpha(x-ct)\}$$

then we get

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi - i\alpha Re_c[(U-c)(\varphi'' - \alpha^2\varphi) - U''\varphi] = 0 \tag{3.6}$$

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0 \tag{3.7}$$

where (3.6) is the Orr-Sommerfeld equation and it has been studied very well. Here we use the Thomas's finite difference method [3] to calculate the eigenfunction $\varphi(y)$ for every pair $(\alpha, Re_c(\alpha))$ of the neutral curve. It is necessary to note that for plane Poiseuille flow symmetry disturbances are most unstable so insted of (3.7) here we use the condition:

$$\varphi(-1) = \varphi'(-1) = \varphi(0) = \varphi''(0) = 0 \tag{3.7'}$$

Now following Thomas we divide interval $[-1, 0]$ into N equal subintervals using grid points $y_0 = -1, y_1, \ldots, y_N = 0$ and present $\varphi(y)$ through a function $g(y)$ as follows:

$$\varphi(y) = \left(1 + \frac{\delta^2}{6} + \frac{\delta^4}{360}\right)g(y) + O(h^4)$$

Then derivatives of $\varphi(y)$ can be expressed in the form:

$$D\varphi = \frac{1}{h}\mu\delta g + O(h^4); \quad D^2\varphi = \frac{1}{h^2}\left(\delta^2 + \frac{\delta^2}{h^4}\right)g + O(h^6); \quad D^4\varphi = \frac{1}{h^4}\delta^4 g + O(h^4)$$

where h is a step of the finite difference scheme $(h = 1/N)$, and

$$\delta^2 g(y) \equiv g(y-h) - 2g(y) + g(y+h); \quad \mu\delta g(y) \equiv 0.5[g(y+h) - g(y-h)]; \quad \delta^4 = (\delta^2)^2$$

Using these formulae for approximating (3.6) one obtains in every grid point $y_k (k = 0, 1, \ldots, N)$ a finite difference equation of the form:

$$A_k g_{k-2} + B_k g_{k-1} + C_k g_k + D_k g_{k+1} + E_k g_{k+2} = 0$$

where $g_{k\pm\ell}$ means $g(y_k \pm \ell * h)$ $(\ell = 0, 1, 2)$. It is obviously that for approximation (3.6) and (3.7') at boundary points y_0, y_1, y_{N-1} and y_N one must take some fictitious grid points, which are denoted by y_{-2}, y_{-1}, y_{N+1} and y_{N+2}. The first two points of them of course lie lower than y_0 while the last ones are arranged above y_N. Thus we obtain a system of $N+5$ algebraic equations of the above form for detemining $N+5$ values of $g(y)$ at $N+5$ grid points including 4 fictitious ones. The determinant of this system must be equal zero in order to exist a nontrivial solution. This condition is used to find eigenvalue c. Here the line inversion method is applied for calculating the system determinant and the Newton method is used to find c as a root of it. After c is found, by back line iversion one computes eigenfunction $\varphi(y)$ and its derivatives easily.

Now we denote:

$\varphi(y) = a(y) + ib(y)$ with $i = \sqrt{-1}$. Then u^0, v^0 can be written in the form:

$$u^0 = a'\cos\theta - b'\sin\theta; \qquad v^0 = \alpha(a\sin\theta + b\cos\theta) \tag{3.8}$$

where $\theta = \alpha x + s_0$; $s_0 = \omega_0 t$ and the prime means derivation with respect to variable y.

Next we must derive the adjoint problem from the linear one. Its equations have been conducted in [16] (see (11.2)) so taking again the stream function of the adjoint velocity in the form

$$\psi^*(x,y,t) = \varphi^*(y)\exp[i\alpha(x-ct)]$$

we have

$$\varphi^{*IV} - 2\alpha^2\varphi^{*''} + \alpha^4\varphi^* + i\alpha Re_c[(U+c)(\varphi^{*''} - \alpha^2\varphi^*) + 2U'\varphi^{*'}] = 0 \tag{3.9}$$

$$\varphi^*(\pm 1) = \varphi^{*'}(\pm 1) = 0 \tag{3.10}$$

It is necessary to note that if at $Re = Re_c(\alpha)$ we have $c = c_r$ as a eigenvalue for the linear problem then in (3.9) one must take $c = -c_r$. For checking this we have calculated determinant of the both systems of finite difference equations for (3.6), (3.7) and (3.9), (3.10) using Thomas's scheme [3]. The obtained results show that the value $c = -c_r$ satisfies the adjoint problem very well (see table 1, where parameters α, Re_c, c_r are taken from table 1 of [15], Δ and Δ^* are the above mentioned determinant for the linear and adjoint problem respectively). To compute φ^* we use again Thomas's method. Now writing $\varphi^* = a^*(y) + ib^*(y)$, one gets for complex conjugate of the adjoint velocity:

$$\begin{aligned} u^* &= a^{*'}\cos\theta + b^{*'}\sin\theta + i(a^{*'}\sin\theta - b^{*'}\cos\theta) \\ v^* &= \alpha[-a^*\sin\theta + b^*\cos\theta + i(a^*\cos\theta + b^*\sin\theta)] \end{aligned} \tag{3.11}$$

3.2. Integration of the first order problem

Substitution of (14) into the right-hand side of (3.4a) and (3.4b) yields:

$$\frac{\alpha}{2}(a'^2 - b'^2 + b'b'' - a'a'')\sin 2\theta + a'b'\cos 2\theta + ab''\sin^2\theta - a''b\cos^2\theta;$$

and $-\alpha^2(aa' + bb')$ respectively. These expressions suggest to find a partial solution of (3.4) in the form:

$$\begin{aligned} u^1 &= m(y)\sin 2\theta + n(y)\cos 2\theta + h(y)\sin^2\theta + g(y)\cos^2\theta \\ v^1 &= -2\alpha M(y)\cos 2\theta + \alpha W(y)\sin 2\theta \\ p^1 &= p(y)\sin 2\theta + g(y)\cos 2\theta + \xi(y) \\ M(y) &= \int_{-1}^{y} m(\varsigma)d\varsigma; \quad W(y) = \int_{-1}^{y} \big(2n(\varsigma) + g(\varsigma) - h(\varsigma)\big)d\varsigma \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.4a) and (3.4b) we can obtain the equations for determining functions-coefficients as follows:

$$\xi = -\alpha^2(a^2 + b^2)/2$$

$$h'' = -\alpha Re_c ab'' \tag{3.13a}$$

$$g'' = \alpha Re_c a''b \tag{3.13b}$$

$$h(\pm 1) = g(\pm 1) = 0 \tag{3.13c}$$

There are also four equations for m, n, p and q. By eliminating p and q in these equations we can reduce them to the two following ones:

$$M^{IV} - 8\alpha^2 M'' + 16\alpha^4 M + Re_c U_1 W'' + 2\alpha Re_c(1 - 2\alpha U_1)W = \frac{\alpha}{2}Re_c(b'b'' - a'a'' + aa''' - bb''') \tag{3.14a}$$

$$W^{IV} - 8\alpha^2 W'' + 16\alpha^4 W - 4Re_c U_1 M'' - 8\alpha Re_c(1 - 2\alpha U_1)M = \alpha Re_c(ab''' + a'''b - a''b' - a'b'') \tag{3.14b}$$

$$M(\pm 1) = M'(\pm 1) = W(\pm 1) = W'(\pm 1) = 0 \tag{3.14c}$$

where $U_1 = \alpha(U - c_n)$.

As mentioned above for the two-dimensional Poiseuille stability problem symmetrical disturbances are most unstable, so here we take $a(y)$ and $b(y)$ even functions. Hence $h(y)$ and $g(y)$ are even functions too but M and W are always odd functions. The integration of (3.13) is no problem but that of (3.14) should meet with great difficulties because of the large term Re in the left-hand side of the equations. To overcome this here following Orszag [17] we apply the Galerkin method to integrate (3.14). To do this we need to construct a system of basic functions. Here the following functions are chosen:

$$q_{2n+1}(y) = T_{2n+1}(y) - 0.5n(n+1)T_3(y) + [0.5n(n+1) - 1]T_1(y) \tag{3.15}$$

where $n = 2, 3, \ldots$ and $T_k(y)$ are Chebyshev polinominals. It is obviously that these $q_k(y)$ are odd functions and they satisfy (3.14c).

Next substituting the formal expansions:

$$M = \sum_{n=2}^{N} a_n q_{2n+1}; \quad W = \sum_{n=2}^{N} b_n q_{2n+1} \tag{3.16}$$

into the left-hand side of (3.14) and demanding that the difference between the resulting expression and the right-hand side be orthogonal to q_{2n+1} $(n = 2, 3, \ldots, N)$ with respect to the inner product:

$$(f, g) = \int_{-1}^{1} f(y)g(y)(1 - y^2)^{-1/2} dy$$

one obtains $2N - 2$ Galerkin equations for $2N - 2$ coefficients a_n and b_n. After these coefficients are found we use (3.16) to calculate M, W and their derivatives.

3.3. Calculation of R_2

The equations for determination of R_2 and ω_2 are obtained by multiplying the right-hand sides of (3.5a) and (3.5b) by u^* and v^* respectively and integrating the sum of the resulting expressions over domain $\{x \in [0, 2\pi/\alpha]; y \in [-1, 1]\}$. We have:

$$R_2 \int_{-1}^{1} \int_{0}^{2\pi/\alpha} (u^* \Delta u^0 + v^* \Delta v^0) dx dy + \omega_2 \int_{-1}^{1} \int_{0}^{2\pi/\alpha} (u^* u^0_s + v^* v^0_s) dx dy =$$

$$= - \int_{-1}^{1} \int_{0}^{2\pi/\alpha} \left[u^*(u^1 u^0_x + u^0 u^1_x + v^1 u^0_y + v^0 u^1_y) + v^*(u^1 v^0_x + u^0 v^1_x + v^1 v^0_y + v^0 v^1_y) \right] dx dy$$

It should be noted that although general solution of the first order problem is a sum of the partial solution just found above and a solution of the zero-th order problem given by (3.8) but under the above mentioned integration one can take the partial part only because the second part gives nothing in the result of this integration.

4. NUMERICAL RESULT AND CONCLUSION

The calculation has been carried out for parameters of the neutral curve given in (3.9). For integration of equation (3.14) by Galerkin method described in §3.2, here 21 ($N = 23$) functions $q_k(y)$ are taken. The obtained results presented in table 1.

The obtained result shows that R_2 is negative (subcritical bifurcation) on the upper branch while it is positive (supercritical bifurcation) on the lower branch (Fig. 2). It is interesting to note that the bifurcation on the both branches occurs in a domain lying completely in the unstable region of Poiseuille flow. May it be a reason of the fact that no finite-amplitude stable equilibrium motion has been observed for plane Poiseuille flow in practice?.

Table 1

α	RE_c	c_r	Δ	Δ^*	R_2/Re_c	ω_2
0.650	22424	0.1656	-1.E-7,-3.E-7	-4.E-6, 1.E-6	5.64	-35.59
0.700	16355	0.1823	-4.E-7,-3.E-6	-3.E-6, 2.E-6	8.00	-38.78
0.750	12461	0.1983	8.E-7, 1.E-6	-1.E-6,-2.E-6	11.20	-41.87
0.800	9882	0.2136	1.E-7,-1.E-6	-2.E-6, 6.E-7	13.40	-46.15
0.850	8141	0.2278	5.E-7, 1.E-6	-1.E-6, 1.E-6	15.97	-50.65
0.900	6965	0.2408	3.E-7, 5.E-7	-1.E-6,-8.E-7	16.80	-55.72
0.925	6540	0.2467	3.E-7, 9.E-7	-1.E-6,-1.E-6	16.53	-58.43
0.950	6208	0.2522	7.E-8,-2.E-6	-1.E-6, 2.E-6	15.36	-61.36
1.000	5815	0.2612	2.E-7, 1.E-6	-1.E-6, 1.E-6	11.72	-67.93
1.021	5772	0.2642	1.E-7, 3.E-6	-1.E-6,-3.E-6	9.16	-71.38
1.050	5890	0.2664	2.E-7, 1.E-6	-2.E-6,-8.E-7	4.92	-77.55
1.075	6314	0.2658	-3.E-7, 3.E-6	-3.E-6,-3.E-6	0.67	-86.28
1.090	7024	0.2624	1.E-6,-3.E-6	-2.E-6, 5.E-6	-2.36	-96.58
1.095	7613	0.2591	1.E-6,-2.E-6	-3.E-6, 4.E-6	-3.48	-104.8
1.096	7947	0.2572	6.E-7, 6.E-7	-4.E-6, 1.E-6	-3.76	-109.8
1.096	9356	0.2497	1.E-6,-7.E-7	-5.E-6, 4.E-6	-4.69	-128.7
1.095	9895	0.2470	-4.E-7, 4.E-6	-8.E-6,-8.E-8	-4.65	-137.0
1.090	11217	0.2410	2.E-6,-2.E-6	-7.E-6, 7.E-6	-4.97	-153.3
1.075	14307	0.2292	3.E-6,-2.E-6	-1.E-5, 1.E-5	-4.51	-195.8
1.050	19360	0.2147	-5.E-6, 1.E-5	-2.E-5, 4.E-6	-4.36	-236.3
1.020	26360	0.2005	-5.E-6, 1.E-5	-4.E-5, 1.E-5	-5.06	-307.3
1.000	31896	0.1921	9.E-6,-1.E-6	-3.E-5, 4.E-5	-4.43	-335.6
0.980	38329	0.1842	8.E-6, 2.E-6	-5.E-5, 6.E-5	-3.46	-391.9

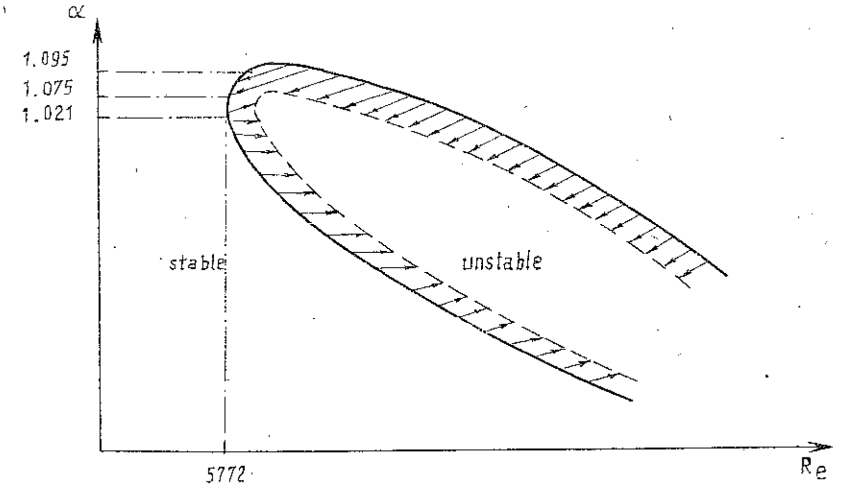

Fig. 2: Bifurcation diagramma

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VỀ MỘT PHƯƠNG PHÁP SỐ NGHIÊN CỨU PHÂN NHÁNH HOPF CỦA DÒNG CHẢY PUADEL TRONG KÊNH PHẲNG

Dựa trên cơ sở lý thuyết chặt chẽ của các tác giả Joseph và Sattinger đưa ra, trong bài này trình bày một thuật toán giải số đơn giản và tiện lợi để xác định dạng của phân nhánh Hopf cho dòng chảy Puadel phẳng. Kết quả nhận được cho thấy phân nhánh ở nhánh trên của đường cong trung gian là trước tới hạn còn ở nhánh dưới là trên giới hạn. Kết quả phù hợp với các tác giả trước nhận được bằng phương pháp khác.