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CONVECTIVE MOTION IN BINARY MIXTURE

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In the paper the problem of convective motion in a binary mixture is studied. In the binary mixture convective motion is caused not only by gradient of temperature but also by gradient of concentration.

1. BASIC EQUATIONS

For the mathematical description of convection in a binary mixture, the Boussinesq approximation of the equation of motion, the heat equation and the diffusion equation will be assumed [1]

$$\frac{\partial v}{\partial t} + (v\nabla)v = -\frac{1}{\rho_0}\nabla p + \nu\Delta v + g(\beta_1 T + \beta_2 C)\gamma \tag{1.1}$$

$$\frac{\partial T}{\partial t} + v\nabla T = (\chi + \alpha^2 DN)\Delta T + \alpha DN\Delta C \tag{1.2}$$

$$\frac{\partial C}{\partial t} + v\nabla C = D\Delta C + \alpha D\Delta T \tag{1.3}$$

$$\operatorname{div} v = 0 \tag{1.4}$$

Where the following notations are used: v denotes the velocity, p - the pressure, T, C - the temperature and the concentration in the mixture, ρ - the equilibrium state density of the mixture, g - the acceleration of gravity, β_1, β_2 - the heat and concentration coefficients of volume expansion of the mixture, D - the diffusion coefficient, ν - the viscous coefficient, χ - the coefficient of heat conductivity, α, N - thermodiffusion and thermodynamics parameters, γ - the unit vector of vertical upward axis x_3 in the cartesian coordinate system $Ox_1x_2x_3$.

We consider the system of equations (1.1) - (1.4) with the following boundary and initial conditions:

$$v = 0, \quad T = T_1, \quad C = C_1 \quad \text{on } S \tag{1.5}$$

$$v\big|_{t=0} = v(0), \quad T\big|_{t=0} = T(0), \quad C\big|_{t=0} = C(0) \tag{1.6}$$

In [1] it has been proved that if

$$\nabla T_0 = -A\gamma, \quad \nabla C_0 = -B\gamma \tag{1.7}$$

where A and B are constants then there exists an equilibrium state.

The problem (1.1) - (1.6) are written in dimensionless form. For this purpose some dimensionless quantities are introduced: L denotes a reference length, L^2/ν - a reference time, χ/L - a reference velocity, AL - a reference temperature, $BL\chi/D$ - a reference concentration, $\rho_0\chi\nu/L^2$ - a reference pressure.

Instead of the system (1.1) - (1.6) we get

$$\frac{\partial v}{\partial t} + P_1^{-1} v \nabla v = \Delta v - \nabla p + R_1 T \gamma + R_2 C \gamma \tag{1.8}$$

$$P_1 \frac{\partial T}{\partial t} + v \nabla T = (1+a)\Delta T + \frac{a}{b}\Delta C \tag{1.9}$$

$$P_2 \frac{\partial C}{\partial t} + P_1^{-1} P_2 v \nabla C = \Delta C + b \Delta T \tag{1.10}$$

$$\operatorname{div} v = 0 \tag{1.11}$$

$$v = 0, \quad T = T_1, \quad C = C_1 \quad \text{on } S \tag{1.12}$$

$$v\big|_{t=0} = v(0), \quad T\big|_{t=0} = T(0), \quad C\big|_{t=0} = C(0) \tag{1.13}$$

Where R_1, R_2 are the heat and diffusion Rayleigh numbers:

$$R_1 = \frac{g\beta_1 A L^4}{\nu\chi}, \qquad R_2 = \frac{g\beta_2 B L^4}{\nu D} \tag{1.14}$$

P_1 is a Prandlt number and P_2 is a Smidth number:

$$P_1 = \frac{\nu}{\chi}, \qquad P_2 = \frac{\nu}{D} \tag{1.15}$$

a, b are defined by:

$$a = \frac{\alpha^2 N D}{\chi}, \qquad b = \frac{\alpha D A}{\chi B} \tag{1.16}$$

Using (1.7) from the system of equations and boundary conditions (1.8) - (1.12) we receive the linear equations describing the small convective motion in the mixture:

$$\frac{\partial v}{\partial t} = \Delta v - \nabla p + R_1 \gamma T + R_2 \gamma C \tag{1.17}$$

$$P_1 \frac{\partial T}{\partial t} = (1+a)\Delta T + \frac{a}{b}\Delta C + v\gamma \tag{1.18}$$

$$P_2 \frac{\partial C}{\partial t} = \Delta C + b\Delta T + (v\gamma) \tag{1.19}$$

$$\operatorname{div} v = 0 \tag{1.20}$$

$$v = 0, \quad T = 0, \quad C = 0 \quad \text{on } S \tag{1.21}$$

$$v\big|_{t=0} = v(0), \quad T\big|_{t=0} = T(0), \quad C\big|_{t=0} = C(0) \tag{1.22}$$

2. EXISTENCE THEOREM

The following Hilbert spaces are used throughout:

$$L_2(\Omega) = H_2(\Omega) \times H_2(\Omega) \times H_2(\Omega),$$

with scalar product and norm:

$$(u, v)_{L_2(\Omega)} = \sum_{i=1}^{3} \int_\Omega u_i v_i \, d\Omega, \quad \|v\|_{L_2(\Omega)} = \left\{ (v, v)_{L_2(\Omega)} \right\}^{1/2},$$

$$\widetilde{L}_2 = \left\{ v \in L_2(\Omega), \quad \operatorname{div} v = 0, \quad v_n = 0 \quad \text{on } S \right\},$$

$$
\begin{aligned}
W_2^1(\Omega) &= H_2^1(\Omega) \times H_2^1(\Omega) \times H_2^1(\Omega),\\
H_2^1(\Omega) &= \{p \in H_2(\Omega),\ \nabla p \in H_2(\Omega)\},\\
H_{2,0}^1(\Omega) &= \{p \in H_2^1(\Omega),\ p = 0 \quad \text{on } S\},\\
W_{2,0}^1(\Omega) &= H_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega),\\
\widetilde{W}_{2,0}^1(\Omega) &= \{v \in W_2^1(\Omega),\ \operatorname{div} v = 0,\ v = 0 \text{ on } S\}.
\end{aligned}
$$

Likely as it has been done in [2] we can prove that the equations (1.17) - (1.20) with the conditions (1.21) are equivelent to the following equations in space $\widetilde{L}_2(\Omega)$ and $H_2(\Omega)$

$$\frac{dv}{dt} = -A_1 v + R_1 B_{12} T + R_2 B_{13} C \tag{2.1}$$

$$P_1 \frac{dT}{dt} = -(1+a) A_2 T - \frac{a}{b} A_2 C + B_{21} v \tag{2.2}$$

$$P_2 \frac{dC}{dt} = -A_2 C - b A_2 T + B_{31} v \tag{2.3}$$

Where the operator A_1 is self-adjoint positive definite in $\widetilde{L}_2(\Omega)$, its inverse operator is positive and compact, the operator A_2 is self-adjoint, positive definite in $H_2(\Omega)$ its inverse operator is positive and compact. The operators B_{12}, B_{13}, B_{21}, B_{31} are defined by

$$
\begin{aligned}
B_{12}T &\equiv \gamma T, \quad & B_{13}C &= \gamma C,\\
B_{21}v &= (v\gamma), \quad & B_{31}v &= (v\gamma).
\end{aligned}
$$

Definition. The generalized solution of problem (1.17) - (1.22) is called the solution of the following problem in space $\widetilde{L}_2(\Omega) \times H_2(\Omega) \times H_2(\Omega)$:

$$\frac{dX}{dt} = -\mathcal{A}X + \mathcal{B}X \tag{2.4}$$

$$X\big|_{t=0} = X(0) \tag{2.5}$$

Where $\quad X = (v, T, C)^{\perp}$

$$
\mathcal{A} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \dfrac{1+a}{P_1} A_2 & \dfrac{a}{P_1 b} A_2 \\ 0 & \dfrac{1}{P_2} b A_2 & \dfrac{1}{P_2} A_2 \end{bmatrix}, \quad
\mathcal{B} = \begin{bmatrix} 0 & R_1 B_{12} & R_2 B_{13} \\ \dfrac{1}{P_1} B_{21} & 0 & 0 \\ \dfrac{1}{P_2} B_{31} & 0 & 0 \end{bmatrix}
$$

It is clear that the equation (2.4) and the system of equations (2.1) - (2.3) are equivalent.

From (1.16) it follows that the constant a is positive, the constant b may be positive may be negative.

We consider the case when $b < 0$. We rewrite the operator $\mathcal{A}$ in the form

$$
\mathcal{A} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \dfrac{1+a}{P_1} A_2 & -\dfrac{a}{P_1 |b|} A_2 \\ 0 & -\dfrac{1}{P_2} |b| A_2 & \dfrac{1}{P_2} A_2 \end{bmatrix}
$$

In the equation (2.4) setting $X = E^{1/2}C$ we get

$$\frac{dY}{dt} = -E^{1/2}\mathcal{A}E^{-1/2}Y + E^{1/2}\mathcal{B}E^{-1/2}Y \tag{2.6}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{P_1|b|}{a} & 0 \\ 0 & 0 & \dfrac{P_2}{|b|} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}.$$

It is easy to see that

$$\mathcal{A}_1 = E^{1/2}\mathcal{A}E^{-1/2} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \dfrac{1+a}{P_1}A_2 & -\left(\dfrac{q}{P_1P_2}\right)^{1/2}A_2 \\ 0 & -\left(\dfrac{a}{P_1P_2}\right)^{1/2}A_2 & \dfrac{1}{P_2}A_2 \end{bmatrix}$$

and

$$(\mathcal{A}_1Y, Y) = (A_1Y_1, Y_1) + \frac{1}{P_1}(A_2Y_2, Y_2) + \left(\left(\frac{a}{P_1}\right)^{1/2}A_2^{1/2}Y_2 - \left(\frac{1}{P_2}\right)^{1/2}A_2^{1/2}Y_3\right)^2 \tag{2.6}$$

$$\mathcal{A}_1^{-1} = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & P_1A_2^{-1} & -(aP_1P_2)^{1/2}A_2^{-1} \\ 0 & -(aP_1P_2)^{1/2}A_2^{-1} & P_2(1+a)A_2^{-1} \end{bmatrix}$$

It is clear that the operator $\mathcal{A}_1^{-1}$ is positive and compact and the operator $\mathcal{A}_1$ is self-adjoint positive definite.

In the case when $b > 0$ the operator $\mathcal{A}_1$ get a form

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \dfrac{1+a}{P_1}A_2 & \left(\dfrac{q}{P_1P_2}\right)^{1/2}A_2 \\ 0 & \left(\dfrac{a}{P_1P_2}\right)^{1/2}A_1 & \dfrac{1}{P_2}A_2 \end{bmatrix}$$

In the same way we can prove that the operator $\mathcal{A}_1$ is self-adjoint positive definite, its inverse operator is positive and compact.

So we obtain [3]

Theorem 2.1. The problem (1.17) - (1.22) get an unique generalized solution.

3. SPECTRUM THEOREM AND CONVECTIVE STABILITY

In (2.1) - (2.3), (2.6) setting

$$v = v_1e^{-\lambda t}, \quad T = T_1e^{-\lambda t}, \quad C = C_1e^{-\lambda t}, \quad Y = Y_1e^{-\lambda t}$$

we get

$$\lambda v_1 = A_1 v_1 - R_1 B_{12} T_1 - R_2 B_{13} C_1 \tag{3.1}$$

$$P_1 \lambda T_1 = (1+a) A_2 T_1 + \frac{a}{b} A_2 C - B_{21} v_1 \tag{3.2}$$

$$P_2 \lambda C_1 = A_2 C_1 + b A_2 T_1 - B_{31} v \tag{3.3}$$

$$\lambda Y = A_1 Y_1 - B_1 Y_1 \tag{3.4}$$

Using the theorem 10.1 in [4] we obtain

Theorem 3.1. The spectrum of the problem (3.1) - (3.3) is discret. Excepting a finite number of the spectrum points others ones are contained in the sector $-\varepsilon < \arg\lambda < \varepsilon$, $\pi - \varepsilon < \arg\lambda < \pi + \varepsilon$. The system of eigenfunctions of the problem is complete in the space $\widehat{W}^1_{2,0}(\Omega) \times H^1_{2,0}(\Omega) \times H^1_{2,0}(\Omega)$.

From the system of equations (3.1) - (3.3) we get

$$\lambda(v_1, v_1) = (A_1 v_1, v_1) - R_1(B_{12} T_1, v_1) - R_2(B_{13} C_1, v_1) \tag{3.5}$$

$$\lambda P_1(T_1, T_1) = (1+a)(A_2 T_1, T_1) + \frac{a}{b}(A_1 C_1, T_1) - (B_{21} v_1, T_1) \tag{3.6}$$

$$\lambda P_2(C_1, C_1) = (A_2 C_1, C_1) + b(A_2 T_1, C_1) - (B_{31} v_1, C_1) \tag{3.7}$$

From (3.5) - (3.7) it follows

$$(\lambda - \lambda^*)\left\{ \|v_1\|^2 + R_1 P_1 \|T_1\|^2 + R_2 P_2 \|C_1\|^2 \right\} = 0 \tag{3.8}$$

$$\begin{aligned} &(\lambda + \lambda^*)\left\{ \|v_1\|^2 - R_1 P_1 \|T_1\|^2 - R_2 P_2 \|C_1\|^2 \right\} = \\ &= 2\|A_1^{1/2} v_1\|^2 - 2R_1(1+a)\|A_2^{1/2} T_1\|^2 - 2R_2\|A_2^{1/2} C_1\|^2 - \\ &- 2\frac{R_1 a + b^2 R_2}{b}\left(A_2^{1/2} C_1, A_2^{1/2} T_1\right) \end{aligned} \tag{3.9}$$

$$\begin{aligned} &(\lambda + \lambda^*)\left\{ \|v_1\|^2 + R_1 P_1 \|T_1\|^2 - R_2 P_2 \|C_1\|^2 \right\} = \\ &= 2\|A_1^{1/2} v_1\|^2 + 2R_1(1+a)\|A_2^{1/2} T_1\|^2 - 2R_2\|A_2^{1/2} C_1\|^2 - \\ &+ 2\frac{aR_1 - R_2 b^2}{b}(A_2^{1/2} C_1, A_2^{1/2} T_1) - 4R_1 Re(B_{12} T_1, v_1) \end{aligned} \tag{3.10}$$

$$\begin{aligned} &(\lambda + \lambda^*)\left\{ \|v_1\|^2 - R_1 P_1 \|T_1\|^2 + R_2 P_2 \|C_1\|^2 \right\} = \\ &= 2\|A_1^{1/2} v_1\|^2 - 2R_1(1+a)\|A_2^{1/2} T_1\|^2 + 2R_2\|A_2^{1/2} C_1\|^2 - \\ &+ 2\frac{-aR_1 + R_2 b^2}{b}(A_2^{1/2} C_1, A_2^{1/2} T_1) - 4R_2 Re(B_{13} C_1, v_1) \end{aligned} \tag{3.11}$$

$$\begin{aligned} &(\lambda + \lambda^*)\left\{ \|v_1\|^2 + R_1 P_1 \|T_1\|^2 + R_2 P_2 \|C_1\|^2 \right\} = \\ &= 2\|A_1^{1/2} v_1\|^2 + 2R_1(1+a)\|A_2^{1/2} T_1\|^2 + 2R_2\|A_2^{1/2} C_1\|^2 - \\ &+ 2\frac{aR_1 + R_2 b^2}{b}(A_2^{1/2} C_1, A_2^{1/2} T_1) - 4R_1 Re(B_{12} T_1, v_1) - \\ &- 4R_2 Re(B_{13} C_1, v_1) \end{aligned} \tag{3.12}$$

The equality (3.8) implies that if $R_1 > 0$, $R_2 > 0$ then λ are real, i.e all the perturbations in the mixture in crease or decrease monotonely.

In the case when $R_1 < 0$, $R_2 < 0$ from (3.9) we get

$$2Re\lambda\Big\{\|v_1\|^2 + |R_1|P_1\|T_1\|^2 + |R_2|P_2\|C_1\|^2\Big\} =$$
$$= 2\|A_1^{1/2}v_1\|^2 + 2|R_1|(1+a)\|A_2^{1/2}T_1\|^2 + 2|R_2|\|A_2^{1/2}C_1\|^2 +$$
$$+ 2\frac{a|R_1| + b^2|R_2|}{b}\big(A_2^{1/2}C_1, A_2^{1/2}T_1\big)$$

This inplies that $Re > 0$ if

$$\frac{a|R_1| + b^2|R_2|}{2b\big(|R_1|\cdot|R_2|(1+a)\big)^{1/2}} < 1 \tag{3.13}$$

In the case when $R_1 > 0$, $R_2 < 0$ from (3.10) it follows that $Re\lambda > 0$ if

$$R_1 < \gamma_1\gamma_2 \quad \text{and} \quad \frac{aR_1 + b^2|R_2|}{|b|\big(R_1|R_2|a\big)^{1/2}} < 1 \tag{3.14}$$

where γ_1, γ_2 are the minimum eigenvalues of the operators A_1 and A_2.

In the case when $R_1 < 0$, $R_2 > 0$ from (3.11) it follows that $Re\lambda > 0$ if

$$R_2 < \frac{\sqrt{2}}{2}\gamma_1\gamma_2 \quad \text{and} \quad \frac{a|R_1| + b^2R_2}{\sqrt{2}|b|\big(R_2|R_1|(1+a)\big)^{1/2}} < 1 \tag{3.15}$$

In the case when $R_1 > 0$, $R_2 > 0$ from (3.12) it follows that $Re\lambda > 0$ if

$$R_1 < \frac{1}{2}\gamma_1\gamma_2\,,\; R_2 < \frac{1}{4}\gamma_1\gamma_2\,,\quad \frac{aR_1 + b^2R_2}{b\sqrt{2aR_1R_2}} < 1 \tag{3.16}$$

(3.13) - (3.16) are sufficient conditions for convective stability in the mixture but not necessary.

Conclusion. In the paper the existence and spectrum theorems have been proved some sufficient conditions for convective stability in a binary mixture are obtained.

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CHUYỂN ĐỘNG ĐỐI LƯU TRONG HỖN HỢP CHẤT LỎNG HAI THÀNH PHẦN

Trong bài báo đã chứng minh định lý tồn tại duy nhất nghiệm và định lý về cấu trúc phổ cho bài toán tuyến tính về chuyển động đối lưu nhiệt trong hỗn hợp chất lỏng hai thành phần. Đã thu được một số điều kiện đủ để chuyển động đối lưu trong hỗn hợp ổn định.