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ON THE SIMULATION TECHNIQUE OF STOCHASTIC PROCESSES AND NONLINEAR VIBRATIONS

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SUMMARY. In this paper the procedure and program for simulation of stochastic processes are represented. The program is applied to nonlinear mechanical systems subjected to stochastic stationary excitation. The results obtained are compared with the ones from other methods which are used for estimating the exactitude of simulation technique.

§1. INTRODUCTION

The investigation of random vibration of non-linear dynamical systems is usually carried out by some following methods: the method of Fokker-Planck-Kolmogorov equation(FPK) gives equations for the probability density function of solutions of the systems, which are excited directly or indirectly by white noises. In proper cases it is possible to find stationary solutions of FPK equation. Therefore, it is difficult to apply this method to general dynamical systems [3].

The statistical linearization method is widely used for nonlinear dynamical system but at greater nonlinearity the exactitude of this method is worse [3, 4].

The perturbation method is also used widely but in practice it is able to find solution in the first approximation order [1, 3, 4, 6].

In order to overcome above-mentioned difficulties for more general dynamic systems it is necessary to use numerical method for simulation of stochastic processes and looking for solutions of nonlinear stochastic systems. The main difficulties of the method are to create a reliable computer program for obtaining quite exact results.

In this paper the justification and procedure of simulation of stochastic processes are represented. This is the basis of creating the program for simulation and solving random differential equation.

§2. SIMULATION OF A STOCHASTIC PROCESS

2.1. Simulation formula

Suppose that $\{x(t)\}$ is a stationary Gaussian stochastic process with zero mean value ($< x(t) >= 0$) and $S_X(\omega)$ is its power spectral density function.

It is necessary to create sample functions of the above process in the numerical and graphic forms so that from the sample functions it is able to find again the power spectral density function $S_X(\omega)$ and other probability characteristics of the given process $\{x(t)\}$.

At first, assume that $x(t)$ is a sample of the stochastic process $\{x(t)\}$ given by the numerical series $x_0, x_1, \dots, x_N$, which are corresponded to the regular points of time $0 = t_0, t_1, \dots, t_{N-1} = T$.

The number N is selected in the form $N = 2^\ell$, where ℓ is a positive integer number. Using the finite Fourier transform it is possible to obtain the following results:

$$X(\omega_k) = X_k = \frac{T}{2\pi N} \sum_{j=0}^{N-1} x_j \exp\left(-i2\pi jk/N\right) \tag{2.1}$$

where $i = \sqrt{-1}$ and

$$x_j = \frac{2\pi}{T} \sum_{k=0}^{N-1} X_k \exp\left(i2\pi kj/N\right) = \sum_{k=0}^{N-1} A_k \exp\left(i2\pi kj/N\right) \tag{2.2}$$

$A_k = \frac{2\pi}{T} X_k$ .The coeficients satisfy the following properties:

$$A_{\frac{N}{2}-k} = A^*_{\frac{N}{2}+k}$$

here (*) denotes the complex conjugate, and

$$\Delta\omega = \frac{2\pi}{T} = \frac{2\pi}{N\Delta T}, \quad \omega_k = k\Delta\omega \tag{2.3}$$

The spectral density function is determined by the following formula [5]:

$$S_X(\omega_k) = \frac{2\pi}{T}\langle |X_k|^2\rangle = \frac{T}{2\pi}\langle |A_k|^2\rangle \tag{2.4}$$

Therefore in a formula for calculating spectra it yields:

$$S_X(\omega_{\frac{N}{2}+k}) = S_X(\omega_{\frac{N}{2}-k}) \tag{2.5}$$

From the formula (2.4) it is able to write:

$$\langle |A_k|^2\rangle = \frac{2\pi}{T} S_X(\omega_k) \tag{2.6}$$

Hence x_j can be found from (2.2) where A_k satisfies (2.6).

As the stochastic process $\{x(t)\}$ has the zero mean value, from (2.2) we have:

$$\langle x_j\rangle = \langle \sum_{k=0}^{N-1} A_k \exp\left(\frac{i2\pi kj}{N}\right)\rangle = 0$$

Therefore it is necessary to take A_k so that

$$\langle A_k\rangle = 0 \tag{2.7}$$

Thus, A_k have to satisfy two conditions (2.6) and (2.7), and therefore it can be selected as follows:

$$A_k = \alpha_k \exp\left(i\beta_k\right); \quad k = 0, 1, 2, \ldots, N-1, \tag{2.8}$$

where β_k are independent random variables identically distributed with the uniform density $(1/2\pi)$ between 0 and 2π, and

$$\alpha_k^2 = \frac{2\pi S_X(\omega_k)}{T} \tag{2.9}$$

It is easy to verify that A_k satisfy (2.6) and (2.7).

Thus the sample function in the numerical form of stochastic process $\{x(t)\}$ is as follows:

$$x(t_j) = \sum_{k=0}^{N-1} \sqrt{\frac{2\pi \overline{S}_X(\omega_k)}{T}} \exp(i\beta_k) \exp(i2\pi kj/N)$$

$$j = 0, 1, 2, \ldots, N-1 \tag{2.10}$$

where the frequency domain of the given function $S_X(\omega)$ is divided into $N/2+1$ points. We take

$$\overline{S}_X(\omega_k) = S_X(\omega_k) \text{ with } k = 0, 1, 2, \ldots, N/2,$$

$$\overline{S}_X(\omega_{\frac{N}{2}+k}) = S_X(\omega_{\frac{N}{2}-k}) \text{ with } k = 0, 1, 2, \ldots, N/2-1.$$

In order to use the FFT (Fast Fourier Transform) [5] formula (2.10) has to be rewritten as

$$x(t_j) = \sum_{k=0}^{N-1} XC_k \exp\left(\frac{i2\pi kj}{N}\right); j = 0, 1, 2, \ldots, N-1, \tag{2.11}$$

where

$$XC_k = \sqrt{\frac{2\pi S_X(\omega_k)}{T}} \exp(i\beta_k); \quad k = 0, 1, 2, \ldots, N/2, \tag{2.12}$$

$$XC_{\frac{N}{2}+k} = \sqrt{\frac{2\pi S_X(\omega_{\frac{N}{2}-k})}{T}} \exp(i\beta_k); \quad k = 0, 1, 2, \ldots, N/2-1.$$

2.2. Steps of realization

(1) Discrete domain of frequencies.

Suppose that the function $S_X(\omega)$ is given, because the random processe $\{x(t)\}$ has a finite variance then with a number $\epsilon > 0$, which is given as a sufficently small number, we can find ω_{Max} so that $S_X(\omega_{Max}) < \epsilon$. In practice, when the process is a white noise process, we can take the enough large $\omega_{\max}$.

We take $N = 2^k$ (k is a positive integer number), which is the number of divided points, the step of frequencies is the following:

$$\Delta\omega = \frac{2\omega_{\max}}{N}, \quad \omega_k = k\Delta\omega.$$

$$k = 0, 1, 2, \ldots, N/2$$

(2) Discrete domain of time.

$$\Delta t = \frac{\pi}{\omega_{\max}}, t_j = j\Delta t, j = 0, 1, 2, \ldots, N-1.$$

$$T = N\Delta t, \Delta\omega = \frac{2\pi}{T}, \Delta\omega\Delta t = 2\pi/N.$$

(3) Calculate $\beta_k = 2\pi\overline{\beta}_k$, $k = 0, 1, 2, \ldots, N/2$, where $\overline{\beta}_k$ are the independent random variables identically distributed with the uniform density 1 between 0 and 1.

(4) Calculate: XC_k from (2.2) and use XC_k as inputs for the FFT, it's outputs are $x(t_j)$ according to (2.11).

2.3 Examples

Example 1.

$S_X(\omega) = \text{const} = 1/2\pi$. We take $\omega_{\max} = 200$, $N = 512$. The 10-th sample is shown in Fig. 1. The spectral density function is calculated from 10 sambles and is compared with the initial spectral density function $S_X(\omega)$. Their graphics are shown in Fig. 2. The exact variance $\sigma_x^2 = 64.1503$, the variance is calculated by the simulation method $\sigma_{xsim}^2 = 63.6620$. The error of standard mean square between two spectral density functions ϵ_r = 0.000133985. The graphics of the probability density function is shown in Fig. 3.

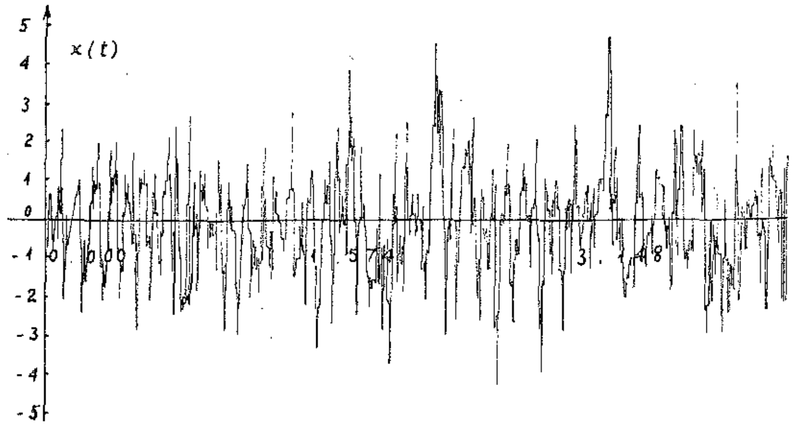

Fig..*1.* 10-th Sample from spectral density $N = 279$, $dy = 5.029E + 0000$

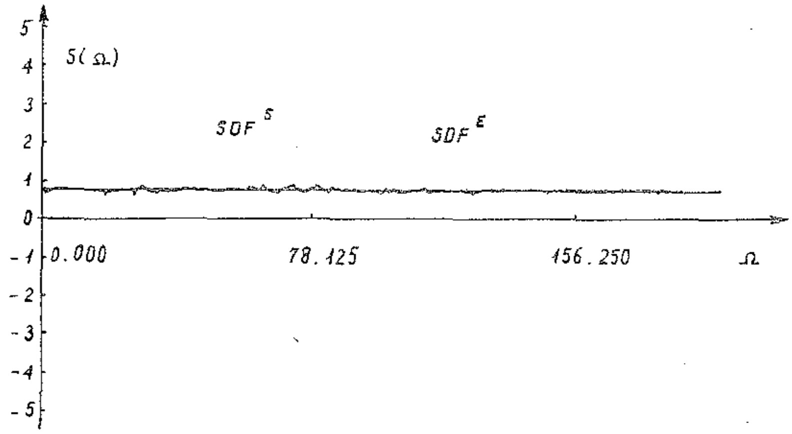

Fig. 2. Exact and simulating SDFs $N = 256$, $dy = 2.132E - 0001$

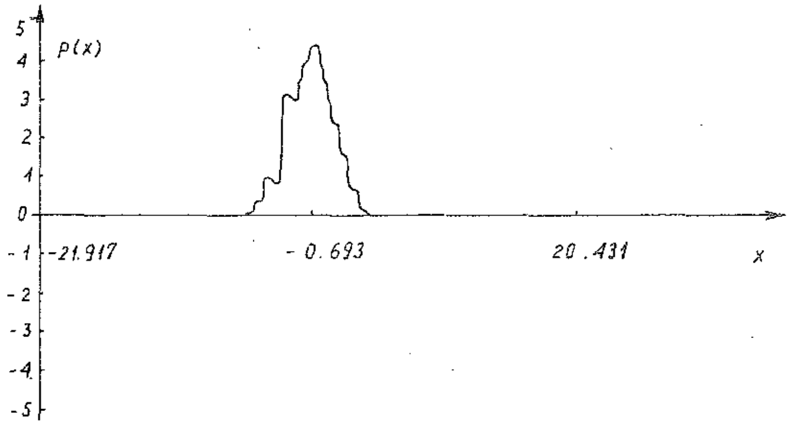

Fig. 3. Test probability density function $N = 65$, $dy = 4.041E + 0002$

Example 2.

The spectral density function $S_X(\omega)$ of a stationary Gaussian random process is given by

$$S_X(\omega) = \frac{a}{2\pi}\left[\frac{1}{a^2+(\omega+\omega_0)^2} + \frac{1}{a^2+(\omega-\omega_0)^2}\right]$$

$a = 3, \omega_0 = 10$. The graphic of $S_X(\omega)$ is shown in Fig. 4, the 10-th sample is shown in Fig. 5. The graphics of the initial spectral density function and of the simulation spectral density function are shown in Fig. 6.

The exact variance $\sigma_x^2 = 0.959449$ and the simulation variance $\sigma_{xsim}^2 = 0.940795$.The error of standard mean square between two spectral density functions $\epsilon_r = 0.000002041$.

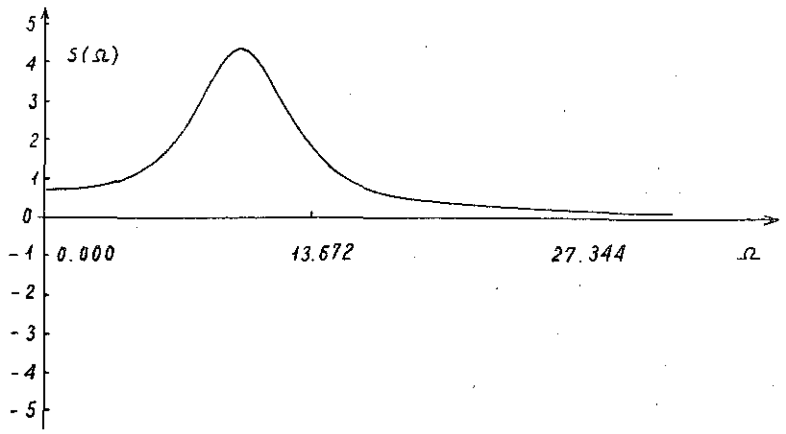

Fig. 4. Initial spectral density function $N = 256,\ dy = 1.246E - 0002$

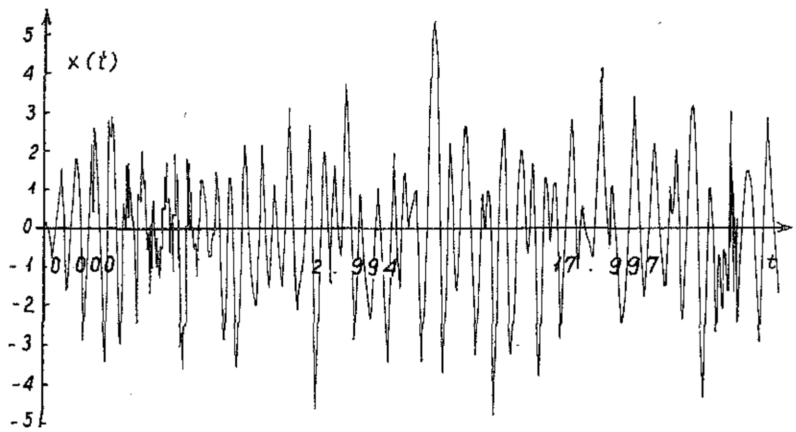

Fig. 5. 10-th Sample from spectral density $N = 279,\ dy = 5.747E - 0001$

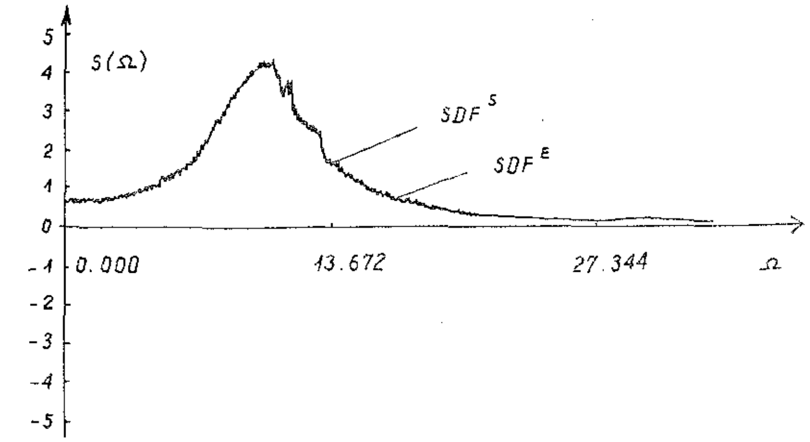

Fig. 6. Exact and simulation SDFs $N = 256$, $dy = 1.334E - 0002$

§3. APPLICATION TO DUFFING EQUATION

Les us apply the simulation technique to Duffing equations with a white noise excitation in order to compare the results of the simulation program and the program for solving non-linear differential equations with the exact solution, which has been found before from the method of FPK equations and the statistical linearization method with the change of the non-linear coefficent. Therefore it is able to estimate the computer program for solving the non-linear random systems.

3.1. Example 1

Consider the dynamical system governed by Duffing equation:

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \mu x^3 = \xi(t) \tag{3.1}$$

where h, μ, ω_0 are constants, $\xi(t)$ is a white noise process with intensity D (i.e. $S_\xi = D/2\pi$).

This problem will be solved by three methods: FPK equation, statistical linearization and simulation.

(1) Use FPK equation

It is easy to find the following results [1].

$$p_X(x) = c\exp\left[-\frac{4h}{D}\left(\frac{\omega_0^2}{2}x^2 + \frac{\mu}{4}x^4\right)\right] \tag{3.2}$$

$$c = 1/\int_{-\infty}^{\infty} p_X(x)dx \tag{3.3}$$

and $< x >= 0$. Therefore

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 p_X(x)dx \tag{3.4}$$

will also be found by using a numerical integral computer program.

(2) Use simulation technique

After the samples of a white noise process has been created the equation (3.1) has become deterministic one with respect to each sample. The fourth-order Runge-Kutta formula can be used for solving this equation. The solution corresponds only one sample. Therefore using the program for spectra it is able to find $S_{xsim}(\omega), \sigma^2_{xsim}$ and probability density function $p_{xsim}(x)$.

(3) Use statistical linearization method

From equation (3.1), according to the statistical linearization method we have

$$\ddot{x} + 2h\dot{x} + \lambda^2 x = \xi(t) \tag{3.5}$$

where

$$\lambda^2 = \omega_0^2 + 3\mu\sigma^2 \tag{3.6}$$

After solving the linear equation (3.5) one can find σ^2 depend on λ^2 (i.e. it had become a equation of σ^2). After solving the obtained equation the result is as follows:

$$\sigma^2 = \frac{\omega_0^2}{6\mu}\left[\left(1 + 12\mu\sigma_0^2/\omega_0^2\right)^{1/2} - 1\right] \tag{3.7}$$

where

$$\sigma_0^2 = \frac{D}{4h\omega_0^2} \tag{3.8}$$

is the variance of the solution of system (3.1) with $\mu = 0$.

After using all three above method for solving equation (3.1) with $S_\xi(\omega) = 2/\pi$ (i.e. $D = 4$), $h = 1$, $\omega_0^2 = 1$ and μ takes the following values 0.001; 0.1; 1.0; 10.0; It is possible to obtain the results described in the table 1, according to the three above methods. From the table one finds the results of the simulation method are rather close to a exact ones. Therefore, the computer program for simulating and solving random differential equations can be acceptable.

Table 1

μ	σ_X^2	σ^2_{xsim}	σ^2_{xlinz}
0.01	0.972143572	0.972561427	0.971675407
0.1	0.817567495	0.817133622	0.805399495
1.0	0.467924062	0.460545075	0.434258545
10.0	0.188904231	0.188416542	0.166666667

It is possible to find that the solution which has been found according to the simulation method quite closes to the exact one. Therefore the computer program for simulation and solution of random differential equation can give the reliable results.

For random non-linear dynamical systems subjected to non-white noise excitation the method of FPK equation cannot be used. In this case the spectral density function of solution process could be found by the methods of simulation and statistical linearization. The reliability of the simulation computer program is described in the previous part, the difference of the rusults obtained by the two methods with various values of non-linear coefficient estimates the effect of the simulation technique.

3.2. Example 2

Let us consider the equation (3.1), $\xi(t)$ is a Gaussian stationary random process with zero mean value, and the spectral density function $S_X(\omega)$ is given by

$$S_\xi(\omega) = \frac{a}{\pi(\omega^2 + \omega_0^2)} \quad (a > 0) \tag{3.9}$$

The graphics of the function $S_\xi(\omega)$ are given in Fig. 7. According to the statistical linearization method the equation which defines the variance has the following form:

$$\sigma_2 = \int_{-\infty}^{\infty} \frac{S_\xi(\omega)}{(\lambda^2-\omega^2)^2+4h^2\omega^2} d\omega \tag{3.10}$$

where

$$\lambda^2 = \omega_0^2 + 3\mu\sigma^2 \tag{3.11}$$

Equation (3.10) had been solved by means of a numerical method, variance σ^2 is found, and we have the following spectral density function $S_X(\omega)$ of the solution processes

$$S_X(\omega) = \frac{S_\xi(\omega)}{(\lambda^2-\omega^2)^2+4h^2\omega^2} \tag{3.12}$$

If the above simulation computer program is used for solving equation (3.1)-(3.9) then we can find the spectral graphics of the solution process and the corresponding variance.

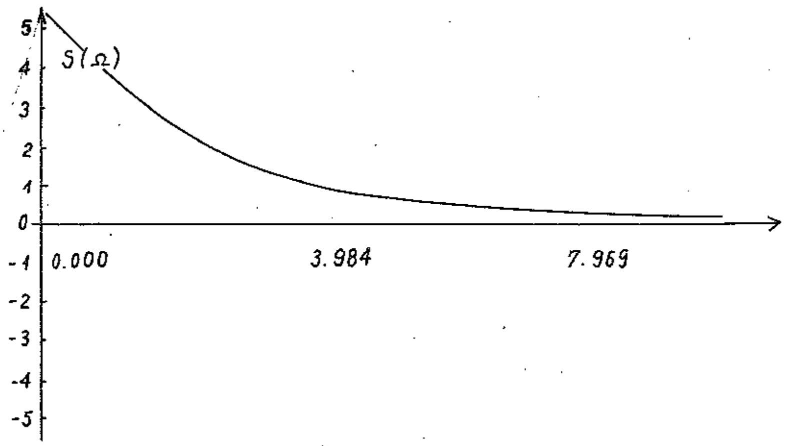

Fig. 7. SDF of random excitation $N = 256$, $dy = 3.252E - 0002$

The graphic of the spectral density function from two above-mentioned methods are given in Fig. 8 and Fig. 9 with $h = 1$, $\omega_0^2 = 1$, $a = 2$ and with $\mu = 0.1$ for the Fig. 8, with $\mu = 10$ for Fig. 9.

The values of variance, correspond to different values of μ, are calculated by mean of the two above methods, are given in table 2.

Table 2

μ	σ^2_{xsim}	σ^2_{xlinz}
0.01	0.221473099	0.220598963
0.1	0.215265430	0.207746310
1.0	0.177855675	0.147007730
10.0	0.097373894	0.063299285

Thus, the results which had been found by the statistical linearization method have quite a great difference with those of the simulation method when the non-linear coefficient μ is not small.

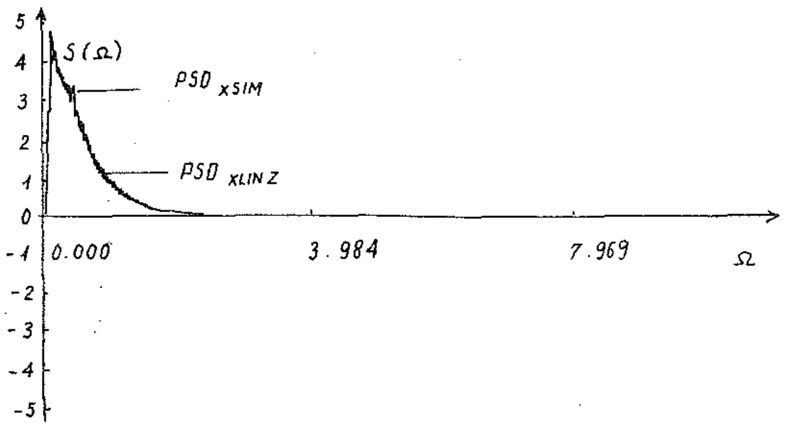

Fig. 8. ($\mu = 0.04$) Lin. and sim. SDFs of responses $N = 256$, $dy = 3.973E - 0002$

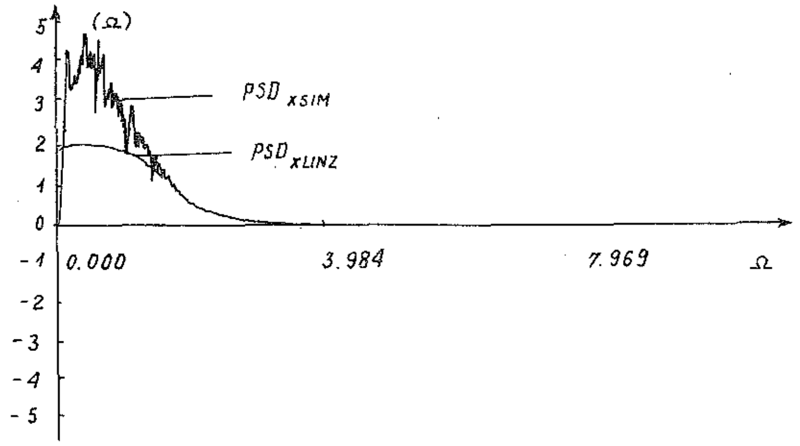

Fig. 9. ($\mu = 10.0$) Lin. and sim. SDFs of responses $N = 256$, $dy = 9.243E - 0003$

CONCLUSIONS

On the basis of above presented results it is able to find that

- The simulation technique can be applied to wider class of the dynamical systems, which are subjected to both white noise excitation and arbitrary stationary stochastic one.

- The computer program presented in previous sections for solving random differential equations give the results with high exactitude.

In these cases one can obtain power spectral density function, probability density and other probability characteristics such as variance, mean value, ... of solution process.

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