Tạp chí Cơ học Journal of Mechanics, NCNST of Vietnam T. XVI, 1994, No 2 (20 – 25)

QUADRATIC AND CUBIC NON-LINEARITIES IN A QUASI-LINEAR FORCED SYSTEM

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SUMMARY. In [3], the difference between the quadratic non-linearity and the cubic one in a quasi-linear parametrically - excited system has been analyzed. In the present paper, the same question will be examined for a quasi-linear forced system and analogous results as in [3] will be obtained.

§1. SYSTEM UNDER CONSIDERATION AND DIFFERENT FORMS OF ITS DIFFERENTIAL EQUATION

Let us consider a quasi-linear forced system, described by the differential equation:

$$\ddot{x} + \nu^2 x = \beta x^2 - \gamma x^3 + \Delta x - h\dot{x} + q\cos\nu t \tag{1.1}$$

where $q > 0$, $\nu > 0$ are intensity and frequency of forced excitation, respectively; the signification of other symbols has been explained in [3].

Assuming that the order of smallness of h and q is ε^2, the differential equation (1.1) can be written in the following forms, depending on the orders of smallness of β, γ and Δ:

- if β, γ, Δ are of order ε^2, we have:

$$\ddot{x} + \nu^2 x = \varepsilon^2\{\beta x^2 - \gamma x^3 + \Delta x - h\dot{x} + q\cos\nu t\} \tag{1.2}$$

- if β, γ, Δ are of order ε, we have:

$$\ddot{x} + \nu^2 x = \varepsilon\{\beta x^2 - \gamma x^3 + \Delta x\} + \varepsilon^2\{-h\dot{x} + q\cos\nu t\} \tag{1.3}$$

- at last, if β is of order ε while γ and Δ are of order ε^2, we have:

$$\ddot{x} + \nu^2 x = \varepsilon\{\beta x^2\} + \varepsilon^2\{-\gamma x^3 + \Delta x - h\dot{x} + q\cos\nu t\} \tag{1.4}$$

As in [3], the case in which γ and Δ are of different orders is rejected and, for the sake of simplicity, γ is assumed to be positive.

§2. SYSTEM WITH THE NON-LINEARITIES OF ORDER ε^2

First, we shall examine the case described by the differential equation (1.2). As in [3], the asymptotic method is used and we obtain successively

$$
\begin{aligned}
&A_1 = 0, \quad B_1 = 0, \quad u_1 = 0,\\
&-2\nu A_2 = h\nu a + q\sin\theta,\\
&-2\nu a B_2 = \Delta a - \frac{3}{4}\gamma a^3 + q\cos\theta,
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
\dot{a} &= -\frac{a}{2\nu}\Big\{h\nu + \frac{q}{a}\sin\theta\Big\},\\
\dot{\theta} &= -\frac{1}{2\nu}\Big\{\Delta - \frac{3}{4}\gamma a^2 + \frac{q}{a}\cos\theta\Big\}.
\end{aligned}
\tag{2.2}
$$

Setting the right-hand sides of (2.2) equal to zero yields:

$$
\begin{aligned}
&h\nu + \frac{q}{a}\sin\theta = 0,\\
&\Delta - \frac{3}{4}\gamma a^2 + \frac{q}{a}\cos\theta = 0,
\end{aligned}
\tag{2.3}
$$

and, after eliminating θ, the relationship between the amplitude a and the frequency ν of the stationary oscillation is obtained:

$$
W(a,\nu^2) = a^2\Big\{\Big[\frac{3}{4}\gamma a^2 - (\nu^2-1)\Big]^2 + h^2\nu^2\Big\} - q^2 = 0.
\tag{2.4}
$$

To study the stability of the stationary oscillation (a_0, θ_0) the pertubations δa, $\delta\theta$ are introduced, namely:

$$
\delta a = a - a_0, \quad \delta\theta = \theta - \theta_0.
\tag{2.5}
$$

It is easy to establish the variational system:

$$
\begin{aligned}
(\delta a)^{\cdot} &= \frac{q}{2\nu a_0}\sin\theta_0\cdot\delta a - \frac{q}{2\nu}\cos\theta_0\cdot\delta\theta,\\
(\delta\theta)^{\cdot} &= -\frac{1}{2\nu a_0}\Big\{-2\cdot\frac{3}{4}\gamma a_0^2 - \frac{q}{a_0}\cos\theta_0\Big\}\delta a + \frac{q}{2\nu a_0}\sin\theta_0\cdot\delta\theta,
\end{aligned}
\tag{2.6}
$$

and its characteristic equation:

$$
\rho^2 + h\rho + \frac{1}{8\nu^2 a_0}\cdot\frac{\partial W(a_0,\nu^2)}{\partial a_0} = 0
\tag{2.7}
$$

Since $h > 0$, the sufficient condition for stability is:

$$
\frac{\partial W(a_0,\nu^2)}{\partial a_0} > 0
\tag{2.8}
$$

Obviously, in the case considered, the system is nearly identical to the classical forced one [1, 2]: the hardness of the system is determined by the cubic non-linearity and, in the first approximation, the quadratic non-linearity does not affect the amplitude, the phase as the stability of the stationary oscillation.

§3. SYSTEM WITH THE NON-LINEARITIES OF ORDER ε

For the second case, described by the differential equation (1.3), the unknown functions in the asymptotic expansion are:

$$
\begin{aligned}
&A_1 = 0, \quad B_1 = -\frac{1}{2\nu}\{\Delta - \frac{3}{4}\gamma a^2\},\\
&u_1 = \frac{\beta a^2}{2\nu^2} - \frac{\beta a^2}{6\nu^2}\cos 2\psi + \frac{\gamma a^3}{32\nu^2}\cos 3\psi,\\
&-2\nu A_2 = h\nu a + q\sin\theta,\\
&-2\nu a B_2 = \frac{5\beta^2}{6\nu^2}a^3 - \frac{3\gamma^2}{128\nu^2}a^5 + \frac{a}{4\nu^2}\Big[\Delta - \frac{3}{4}\gamma a^2\Big]^2 + q\cos\theta,
\end{aligned}
\tag{3.1}
$$

and the differential equations for a and θ, in the second approximation, are of the form:

$$
\begin{aligned}
&\dot a = -\frac{a}{2\nu}\Big\{h\nu + \frac{q}{a}\sin\theta\Big\},\\
&\dot\theta = -\frac{1}{2\nu}\Big\{\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2}\Big(\frac{3}{4}\gamma a^2\Big) + (\nu^2 - 1) + \frac{(\nu^2-1)^2}{4\nu^2} + \frac{q}{a}\cos\theta\Big\},
\end{aligned}
\tag{3.2}
$$

where $\sigma = \dfrac{5\beta^2}{6} : \dfrac{3}{4}\gamma$.

The amplitude a and the phase θ of the stationary oscillation satisfy the equations:

$$
\begin{aligned}
&h\nu + \frac{q}{a}\sin\theta = 0,\\
&\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2}\Big(\frac{3}{4}\gamma a^2\Big) + (\nu^2 - 1) + \frac{(\nu^2-1)^2}{4\nu^2} + \frac{q}{a}\cos\theta = 0,
\end{aligned}
\tag{3.3}
$$

and the relationship between a and ν is:

$$
W(a,\nu^2) = a^2\Big\{\Big[\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a^2\Big)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2}\Big(\frac{3}{4}\gamma a^2\Big) + (\nu^2 - 1) + \frac{(\nu^2-1)^2}{4\nu^2}\Big]^2 + h^2\nu^2\Big\} - q^2 = 0
\tag{3.4}
$$

It is noted that, for each given value ν^2 in the neighbourhood of unity, the algebraic equation (3.4) of unknown a^2 has no more three acceptable solutions. Indeed, (3.4) can be rewritten in the form:

$$
\Big[\frac{5}{24\nu^2}X^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2}X + (\nu^2 - 1) + \frac{(\nu^2-1)^2}{4\nu^2}\Big]^2 = \frac{Q^2}{X} - h^2\nu^2,
\tag{3.5}
$$

where:

$$
X = \frac{3}{4}\gamma a^2, \qquad Q^2 = \frac{3}{4}\gamma q^2
$$

In the plane (XY) let us draw the graphs Y_1 and Y_2 of the left and right-hand sides of (3.5), respectively (Fig. 1). The graph Y_1 has two minima on the abscissa axis: M_1 near $X = 0$ and M_2 near $X = 24/5$. The graph Y_2 is a hyperbola quite near its asymptotes: the ordinate axis $X = 0$ and the abscisse line $Y = -h^2\nu^2$. For $h = 0$ the mentioned graphs intersect themselves at two points P and Q located in the neighbourhood and on both sides of M_2. When h increases, the graph Y_2 is shifted downward, P and Q approach M_2 then coincide in it and finally diseappear. Obviously, the abscisses of P and Q are two solutions of the equation (3.5). However, these solutions must be rejected since the corresponding values $a^2 = 4X/3\gamma$ are too large (for standard variable). Thus, the equation (3.5) (i.e. the equation (3.4)) admits no more three acceptable solutions near $X = 0$.

To study the stability of the stationary oscillation (a_0, θ_0), we use:

- the variational system:

$$
\begin{aligned}
&(\delta a)^{\cdot} = \frac{q}{2\nu a_0}\sin\theta_0\cdot\delta a - \frac{q}{2\nu}\cos\theta_0\cdot\delta\theta,\\
&(\delta\theta)^{\cdot} = -\frac{1}{2\nu a_0}\Big\{4\cdot\frac{5}{24\nu^2}\Big(\frac{3}{4}\gamma a_0^2\Big)^2 - 2\cdot\frac{3\nu^2 - 1 - 2\delta}{2\nu^2}\Big(\frac{3}{4}\gamma a_0^2\Big) - \frac{q}{a_0}\cos\theta_0\Big\}\delta a + \frac{q}{2\nu a_0}\sin\theta_0\cdot\delta\theta,
\end{aligned}
\tag{3.6}
$$

and its characteristic equation:

$$\rho^2 + h\rho + \frac{1}{8\nu^2 a_0}\frac{\partial W(a_0,\nu^2)}{\partial a_0} = 0. \tag{3.7}$$

From (3.7) the following sufficient stability condition is deduced:

$$\frac{\partial W(a_0,\nu^2)}{\partial a_0} > 0. \tag{3.8}$$

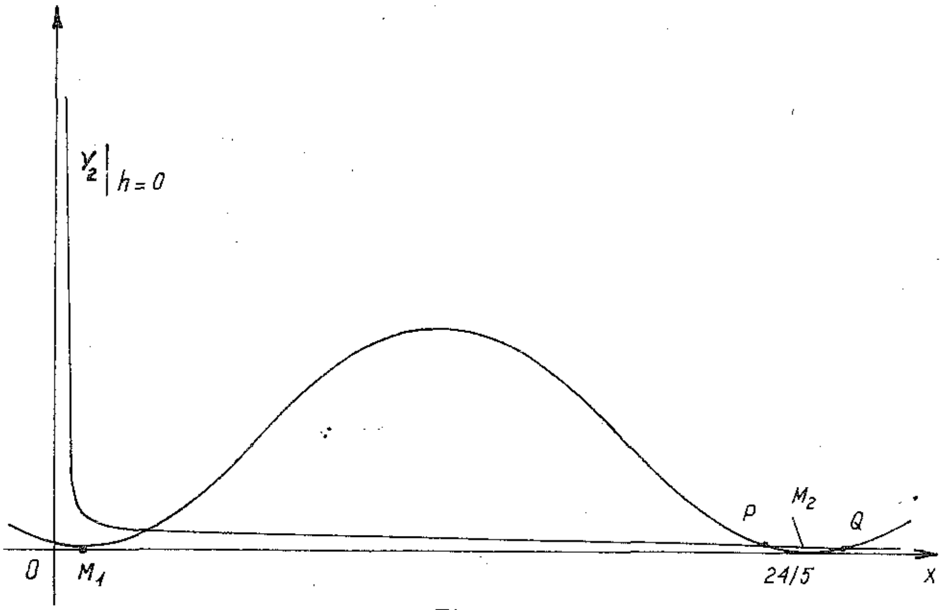

Fig. 1

The structure of (3.4) shows that the character (soft-hardness) of the system is determined by the cubic non-linearity γ ($\gamma > 0$, the system belongs to hard kind) and the quadratic one β (σ - of order ε) plays only a supplementary role (it makes the system less hard). Figure 2 shows the resonance curves for the typical case:

$$h^2 = 0.0002; \quad q^2 = 0.00025; \quad \frac{3}{4}\gamma = 0.24; \quad \sigma = 0 \text{ (a) and } \sigma = 0.05 \text{ (b)}$$

§4. SYSTEM WITH THE NON-LINEARITIES OF DIFFERENT ORDERS

In the third case, described by the differential equation (1.4), the quadratic non-linearity is of order ε while the cubic one is of order ε^2. The asymptotic method gives us successively:

$$\begin{aligned} &A_1 = 0, \quad B_1 = 0, \quad u_1 = \frac{\beta a^2}{2\nu^2} - \frac{\beta a^2}{6\nu^2}\cos 2\psi, \\ &-2\nu A_2 = h\nu a + q\sin\theta, \\ &-2\nu a B_2 = \left(\frac{5\beta^2}{6\nu^2} - \frac{3}{4}\gamma\right)a^3 + \Delta a + q\cos\theta, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \dot a &= -\frac{a}{2\nu}\left\{h\nu + \frac{q}{a}\sin\theta\right\}, \\ \dot\theta &= -\frac{1}{2\nu}\left\{\left(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\right)\frac{5\beta^2}{6}a^2 + (\nu^2 - 1) + \frac{q}{a}\cos\theta\right\}, \end{aligned} \tag{4.2}$$

where

$$\frac{1}{\nu_*^2} = \frac{3}{4}\gamma \ ; \ \frac{5\beta^2}{6}.$$

The equations for a and θ in the stationary oscillation are:

$$\begin{aligned} &h\nu + \frac{q}{a}\sin\theta = 0, \\ &\Big(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^2 + (\nu^2 - 1) + \frac{q}{a}\cos\theta = 0, \end{aligned} \tag{4.3}$$

and the relationship between a and ν is:

$$W(a, \nu^2) = a^2\Big\{\Big[\Big(\frac{1}{\nu^2} - \frac{1}{\nu_*^2}\Big)\frac{5\beta^2}{6}a^2 + (\nu^2 - 1)\Big]^2 + h^2\nu^2\Big\} - q^2 = 0. \tag{4.4}$$

To study the stability of the stationary oscillation (a_0, θ_0) we use:
- the variational system:

$$\begin{aligned} &(\delta a)^{\cdot} = \frac{q}{2\nu a_0}\sin\theta_0 \cdot \delta a - \frac{q}{2\nu}\cos\theta_0 \cdot \delta\theta, \\ &(\delta\theta)^{\cdot} = -\frac{1}{2\nu a_0}\Big\{2\Big(\frac{1}{\nu^2} - \frac{1}{\nu_8^2}\Big)\frac{5\beta^2}{6}a_0^2 - \frac{q}{a_0}\cos\theta_0\Big\}\delta a + \frac{q}{2\nu a_0}\sin\theta_0 \cdot \delta\theta, \end{aligned} \tag{4.5}$$

- and its characteristic equation:

$$\rho^2 + h\rho + \frac{1}{8\nu^2 a_0}\ \frac{\partial W(a_0, \nu^2)}{\partial a_0} = 0 \tag{4.6}$$

The stability condition is as previously:

$$\frac{\partial W(a_0, \nu^2)}{\partial a_0} > 0 \tag{4.7}$$

Analogous to the coresponding case in [3], the interesting phenomenon in the last case is that the character (soft-hardness) of the system under consideration depends on ν_*^2 i.e. on the ratio of β^2 and γ:
- if ν_*^2 is enough greater than 1, the system belongs to soft kind,
- if ν_*^2 is enough less than 1, the system becomes hard one,
- if ν_*^2 is close to 1, the system is neutralized, it becomes a linear one.

Figure 3 shows the resonance curves for the typical case: $h^2 = 0.0003$; $q^2 = 0.0003$; $\frac{5\beta^2}{6} = 0.04$ and $\frac{3\gamma}{4} = 0$ (a) (the resonance curve leans to the left), $\frac{3\gamma}{4} = 0.04$ (b), $\frac{3\gamma}{4} = 0.08$ (c) (the resonance curve leans to the right).

CONCLUSION

The results obtained show the difference between the quadratic non linearity and the cubic one. If the two non-linearities are of the same order of smallness, the cubic non-linearity is the dominant factor, on the contrary, if they are of different orders, the character (soft-hardness) of the system depends on both them, in equal degree.

This publication is completed with financial support from the National Basic Research Program in Natural Sciences.

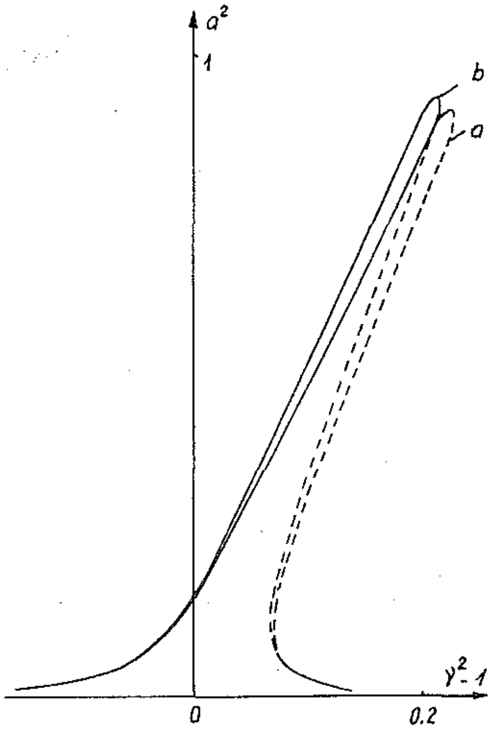

Fig. 2.

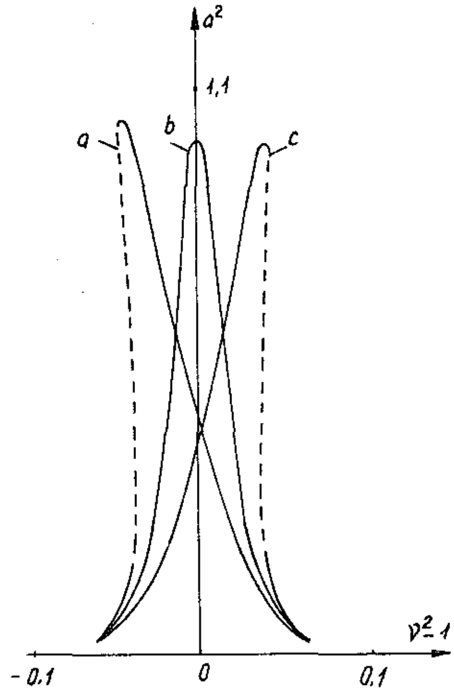

Fig. 3

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Received March 23, 1993

PHI TUYẾN BẬC HAI VÀ BẬC BA TRONG MỘT HỆ Á TUYẾN CƯỠNG BỨC

Tiếp tục vấn đề đặt ra trong [3], bài báo này xét vai trò các số hạng đàn hồi phi tuyến bậc hai và bậc ba trong một hệ dao động á tuyến cưỡng bức. Kết quả thu được cho thấy:

- Nếu hai số hạng phi tuyến nói trên ở cùng cấp (ε hoặc ε^2), số hạng bậc ba quyết định tính cứng mềm của hệ, số hạng bậc hai chỉ có ảnh hưởng bổ sung;

- Nếu số hạng bậc hai ở cấp ε và số hạng bậc ba ở cấp ε^2, tính cứng mềm của hệ phụ thuộc cả vào hai số hạng đó và vì vậy, tùy "tỷ số" giữa chúng, hệ có thể thuộc loại cứng hoặc mềm hoặc bị "trung hòa".