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THE PROBLEM OF LONGITUDINAL SHOCK OF TWO SPHERICAL END ELASTIC BARS WITH VISCO-ELASTIC RESISTANCE FORCE

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Based on the theory of one-dimensional wave together with Dalembert solution and Hertz's law of deformation holds, in [1] and [2] we studied the problem of shock of two elastic bars with free spherical end. In this paper, we continue to study the above problem when the second end of the second bar meets visco-elastic resistance force.

§1. FORMULATION OF THE PROBLEM

The motion equation of the bars is:

$$\frac{\partial^2 U_j}{\partial t^2} = a_j^2 \frac{\partial^2 U_j}{\partial x_j^2} \tag{1.1}$$

where $j = 1, 2;$ $a_j = \sqrt{\dfrac{E_j}{\rho_j}}$ - wave velocity.

Initial conditions: At $t = 0$,

$$\frac{\partial U_1}{\partial t} = V_1; \quad U_1 = 0; \quad \frac{\partial U_1}{\partial x_1} = 0 \tag{1.2a}$$

$$\frac{\partial U_2}{\partial t} = 0; \quad U_2 = 0; \quad \frac{\partial U_2}{\partial x_2} = 0 \tag{1.2b}$$

Boundary conditions:
At the shock end $x_1 = \ell_1;\ x_2 = \ell_2,$

$$E_1 F_1 \frac{\partial U_1}{\partial x_1} = E_2 F_2 \frac{\partial U_2}{\partial x_2} = -K(U_1 + U_2)^{3/2} \tag{1.3}$$

At the free end, $x_1 = 0$,

$$\frac{\partial U_1}{\partial x_1} = 0 \tag{1.4}$$

When the end of the second bar bear on the visco-elastic sole, we obtain:

$$x_2 = 0; \quad \frac{\partial U_2}{\partial x_2} = -K_1 U_2 - \lambda \frac{\partial U_2}{\partial t} \tag{1.5}$$

In this equation k_1, λ are elastic and viscid coefficients respectively. They are considered as constants. A general solution of eq. (1.1) is of the D'Alembert form:

$$U_j = \varphi_j(a_j t - x_j) + \psi_j(a_j t + x_j)$$

§2. DETERMINATION OF WAVE FUNCTIONS OF BARS

Assume that the second bar is in the rest, the first bar centro-longitudinally moves and impacts to the second one with velocity V_1, based on [1] we get:

$$
\begin{aligned}
\varphi_1'(z_1) &= \frac{V_1}{2a_1} \\
\varphi_2'(z_2) &= 0
\end{aligned}
\tag{2.1}
$$

where $-\ell_j < z_j < \ell_j$, with $j = 1, 2$.

According to the boundary condition (1.3) we have:

$$A(-\varphi_2' + \psi_2')^{-1/3} \cdot (-\varphi_2'' + \psi_2'') = 2\varphi_1' + B\varphi_2' + C\psi_2' \tag{2.2}$$

$$\psi_1' = \varphi_1' + \frac{1}{\alpha}(-\varphi_2' + \psi_2') \tag{2.3}$$

where

$$A = \frac{2}{3}\Big(-\frac{1}{\beta}\Big)^{2/3} \cdot \Big(\frac{1}{\alpha}\Big)^{2/3} \cdot \frac{a_2}{a_1}; \quad B = \frac{\alpha a_2 - a_1}{\alpha a_1}; \quad C = \frac{\alpha a_2 + a_1}{\alpha a_1}; \quad \alpha = \frac{E_1 \cdot F_1}{E_2 \cdot F_2}; \quad \beta = \frac{K}{E_1 \cdot F_1}.$$

Consider that $T_2 = iT_1 + qT_1$ with $i = 1, 2, 3, \dots; 0 \le q < 1$.

The wave functions $\varphi_1'(a_1 t - x_1)$; $\psi_1'(a_1 t + x_1)$; $\varphi_2'(a_2 t - x_2)$ and $\psi_2'(a_2 t + x_2)$ with $0 < t < T_2$ are determined as follows:

At first period $T_1 (0 < t < T_1)$, we have $\varphi_2' = 0$ and $\varphi_2'' = 0$, and from eq. (2.1) $\varphi_1' = \dfrac{V_1}{2a_1}$.

Notice that $\dfrac{d\psi_2}{dz} = \psi_2' = y$ with $z = a_2 t + \ell_2$ then eq. (2.2) can be written as follows:

$$y' = \frac{1}{A}\Big(\frac{V_1}{a_1} + cy\Big) \cdot y^{1/3} \tag{2.4}$$

Integrated eq. (2.4) we have:

$$z - z_0 = \frac{3A}{C_1 \cdot C}\Big[\frac{1}{2\sqrt{3}}\ln\frac{y^{2/3} - C_1 y^{1/3} + C_1^2}{(C_1 + y^{1/3})^2} + \mathrm{arctg}\frac{2y^{1/3} - C_1}{\sqrt{3}C_1} - \mathrm{arctg}\Big(-\frac{1}{\sqrt{3}}\Big)\Big]$$

where $C_1 = \dfrac{V_1}{Ca_1}$. From eq. (2.3) we obtain:

$$\psi_1' = \frac{V_1}{2a_1} + \frac{1}{\alpha}y \tag{2.5}$$

Similarly [1] for the i^{th} period of first bar we get:

$$\big(\varphi_1'\big)_{1i} = \frac{V_1}{2a_1} + \frac{1}{\alpha}\sum_{n=1}^{i-1} y_n \tag{2.6}$$

$$y_i' = \frac{1}{A}y_i^{1/3} \cdot \Big[\frac{V_1}{a_1} + \frac{2}{\alpha}\sum_{n=1}^{i-1} y_n + Cy_i\Big] \tag{2.7}$$

Equation (2.7) can be solved by the finite difference method. Value y_i of first bar at the start of period i is equal to that one at the end of period $(i-1)$. From eq. (2.3) we have:

$$\big(\psi_1'\big)_{1i} = \frac{V_1}{2a_1} + \frac{1}{\alpha}\sum_{n=1}^{i} y_n \tag{2.8}$$

Finally in the interval $iT_1 < t < T_2 = iT_1 + qT_1$ we obtain:

$$\varphi_2'(a_2t - \ell_2) = 0 \quad \text{and} \quad \varphi_2''(a_2t - \ell_2) = 0.$$

From eq. (2.2) we have

$$\left(\psi_2''\right)_1 = \frac{1}{A}\left(\psi_2'\right)_1^{1/3} \cdot \left[2\left(\varphi_1'\right)_1 + C\left(\psi_2'\right)_1\right] \tag{2.9}$$

where $\left(\varphi_1'\right)_1 = \left(\psi_1'\right)_{1i}$.

Solving eq. (2.9) the wave functions $\left(\psi_2'\right)_1$ is determined. From eq. (2.3) we obtain:

$$\left(\psi_1'\right)_1 = \left(\varphi_1'\right)_1 + \frac{1}{\alpha}\left(\psi_2'\right)_1 \tag{2.10}$$

when $t < \frac{T_2}{2}$ reflected wave $\varphi_2'(a_2t - x_2)$ does not appear in the second bar.

If $\frac{T_2}{2} < t < T_2$ then the wave function $\varphi_2'(a_2t - \ell_2) = 0$, but the wave function $\varphi_2'(a_2t - x_2)$ appears in the second bar. Determination of the wave function $\varphi_2'(a_2t - x_2)$ with $\frac{T_2}{2} < t < T_2$ is done in the same way described by [1]. According to boundary condition mentioned in eq. (1.5), the following cases are occured.

If $1 - \lambda a_2 \neq 0$ then:

$$\varphi_2'(a_2t) - \frac{K_1}{1 - \lambda a_2}\varphi_2(a_2t) = \frac{1 + \lambda a_2}{1 - \lambda a_2}\psi_2'(a_2t) + \frac{K_1}{1 - \lambda a_2}\psi_2(a_2t)$$

or

$$\varphi_2'(a_2t - x_2) - \frac{K_1}{1 - \lambda a_2}\varphi_2(a_2t - x_2) = \frac{1 + \lambda a_2}{1 - \lambda a_2}\psi_2'(a_2t - x_2) + \frac{K_1}{1 - \lambda a_2}\psi_2(a_2t - x_2) \tag{2.11}$$

with $0 < a_2t - x_2 < 2\ell_2$.

Based on eq. (2.1) when $0 < a_2t - x_2 < \ell_2$ then $\varphi_2'(a_2t - x_2) = 0$. When $\ell_2 < a_2t - x_2 < 2\ell_2$ then $\psi_2'(a_2t - x_2)$ is known and $\psi_2(a_2t - x_2)$ is determined. Integrating eq. (2.11) with the condition of $\varphi_2(\ell_2 - 0) = 0$ we obtain:

$$\varphi_2(a_2t - x_2) = e^{-\frac{K_1}{1-\lambda a_2}(a_2t - x_2)} \cdot \int_{\ell_2}^{(a_2t - x_2)} e^{\frac{K_1}{1-\lambda a_2}\tau} \cdot \left[\frac{1 + \lambda a_2}{1 - \lambda a_2}\psi_2'(\tau) + \frac{K_1}{1 - \lambda a_2}\psi_2(\tau)\right] d\tau \tag{2.12}$$

If $1 - \lambda a_2 = 0$, we have:

$$\varphi_2(a_2t - x_2) = -\frac{2}{K_1}\psi_2'(a_2t - x_2) - \psi_2(a_2t - x_2) \tag{2.13}$$

Based on those mentioned above, we can determine the wave function $\varphi_2'(a_2t - x_2)$ in the second bar. So that we can determine wave function $\varphi_1'(a_1t - x_1)$, $\psi_1'(a_1t + x_1)$, $\varphi_2'(a_2t - x_2)$ and $\psi_2'(a_2t + x_2)$ at each of the sections of the bars in interval $0 < t < \frac{2\ell_2}{a_2}$.

In interval $T_2 < t < 2T_2$ studying each of period T_1 with $T_2 + (n-1)T_1 < t < T_2 + nT_1$. Let $(\ )_{2n}$ be a wave function, that is determined in n^{th} period of first bar and the wave function itself is also determined in the second period of second bar, where $n = 1, 2, \ldots, i$. At the first period of first bar with $T_2 < t < T_2 + T_1$, according boundary condition (1.3) we have:

$$\left(\psi_2''\right)_{21} = \left(\varphi_2''\right)_{21} + \frac{1}{A}\left[2\left(\varphi_1'\right)_{21} + B\left(\varphi_2'\right)_{21} + C\left(\psi_2'\right)_{21}\right] \cdot \left[\left(-\varphi_2'\right)_{21} + \left(\psi_2'\right)_{21}\right]^{1/3} \tag{2.14}$$

$$\left(\psi_1'\right)_{21} = \left(\varphi_1'\right)_{21} + \frac{1}{\alpha}\left[\left(-\varphi_2'\right)_{21} + \left(\psi_2'\right)_{21}\right] \tag{2.15}$$

From condition (1.4)

$$\varphi_1'(a_1t-\ell_1)=\psi_1'(a_1t-\ell_1)=\psi_1'[a_1(t-T_1)+\ell_1],$$

or

$$(\varphi_1')_{21}=(\psi_1')_{20}, \tag{2.16}$$

where $(\psi_1')_{20}$ is the wave function $\psi_1'(a_1t+\ell_1)$ with $(T_2-T_1)<t<T_2$, which was determined, so that the wave function $(\varphi_1')_{21}$ is also determined. According to (1.5) and (2.11) we have:

$$(\varphi_2')_{21}-\frac{K_1}{1-\lambda a_2}(\varphi_2)_{21}=\frac{1+\lambda a_2}{1-\lambda a_2}(\psi_2')_{21}+\frac{K_1}{1-\lambda a_2}(\psi_2)_{11} \tag{2.17}$$

If $1-\lambda a_2\neq 0$ then a solution of eq. (2.17) is:

$$(\varphi_2)_{21}=e^{\frac{K_1}{1-\lambda a_2}(a_2t-\ell_2)}\cdot\int\limits_{\ell_2}^{(a_2t-\ell_2)}e^{-\frac{K_1}{1-\lambda a_2}\tau}\cdot\left[\frac{1+\lambda a_2}{1-\lambda a_2}\psi_2'(\tau)+\frac{K_1}{1-\lambda a_2}\psi_2(t)\right]d\tau \tag{2.18}$$

If $1-\lambda a_2=0$ we have:

$$(\varphi_2)_{21}=-\frac{2}{K_1}(\psi_2')_{11}-(\psi_2)_{11} \tag{2.19}$$

So the wave function $(\varphi_2')_{21}=(\varphi_2'(a_2t-\ell_2))_{21}$ is known from eq. (2.14) the function $(\psi_2')_{21}$ is determined. Replacing this result into eq. (2.15) the wave function $(\psi_1')_{21}$ can be found. Doing similary we can determine the wave functions at the i^{th} period of first bar. We have:

$$(\psi_2'')_{2i}=(\varphi_2'')_{2i}+\frac{1}{A}[2(\varphi_1')_{2i}+B(\varphi_2')_{2i}+C(\psi_2')_{2i}]\cdot[(-\varphi_2')_{2i}+(\psi_2')_{2i}]^{1/2} \tag{2.20}$$

$$(\psi_1')_{2i}=(\varphi_1')_{2i}+\frac{1}{\alpha}[(-\varphi_2')_{2i}+(\psi_2')_{2i}] \tag{2.21}$$

$$(\varphi_1')_{2i}=(\psi_1')_{2(i-1)} \tag{2.22}$$

$$(\varphi_2')_{2i}-\frac{K_1}{1-\lambda a_2}(\varphi_2)_{2i}=\frac{1+\lambda a_2}{1-\lambda a_2}(\psi_2')_{1i}+\frac{K_1}{1-\lambda a_2}(\psi_2)_{1i} \tag{2.23}$$

If $1-\lambda a_2\neq 0$ then solution of eq. (2.23) is:

$$(\varphi_2)_{2i}=(\varphi_2(a_2t-\ell_2))_{2i}= \tag{2.24}$$

$$=e^{\frac{K_1}{1-\lambda a_2}(a_2t-\ell_2)}\cdot\left\{\int\limits_{[\ell_2+a_2(i-1)T_1]}^{(a_2t-\ell_2)}e^{-\frac{K_1}{1-\lambda a_2}\tau}\cdot\left[\frac{1+\lambda a_2}{1-\lambda a_2}\psi_2'(\tau)+\frac{K_1}{1-\lambda a_2}\psi_2(\tau)\right]d\tau+C_{2i}\right\}$$

where

$$C_{2i}=\varphi_2[\ell_2+a_2(i-1)T_1-0]\cdot e^{-\frac{K_1}{1-\lambda a_2}[\ell_2+a_2(i-1)T_1]}$$

If $1-\lambda a_2=0$ then

$$(\varphi_2)_{2i}=-\frac{2}{K_1}(\psi_2')_{1i}-(\psi_2)_{1i} \tag{2.25}$$

So that the wave functions $(\psi_2')_{2i}$ and $(\psi_1')_{2i}$ are determined. Now we determine the wave functions is odd part of the second period of the second bar. From conditions (1.3), (1.4) and (1.5) we have:

$$(\psi_2'')_2=(\varphi_2'')_2+\frac{1}{A}[2(\varphi_1')_2+B(\varphi_2')_2+C(\psi_2')_2]\cdot[(-\varphi_2')_2+(\psi_2')_2]^{1/3} \tag{2.26}$$

$$(\psi_1')_2=(\varphi_1')_2+\frac{1}{\alpha}[(-\varphi_2')_2+(\psi_2')_2] \tag{2.27}$$

$$(\varphi_1')_2=(\psi_1')_{2i}$$

$$(\varphi_2')_2-\frac{K_1}{1-\lambda a_2}(\varphi_2)_2=\frac{1+\lambda a_2}{1-\lambda a_2}(\psi_2')_1+\frac{K_1}{1-\lambda a_2}(\psi_2)_1 \tag{2.28}$$

If $1-\lambda a_2 \neq 0$ then

$$\left(\varphi_2\right)_2 = e^{\frac{K_1}{1-\lambda a_2}(a_2t-\ell_2)} \cdot \left\{ \int_{\ell_2+a_2iT_1}^{a_2t-\ell_2} e^{-\frac{K_1}{1-\lambda a_2}\tau} \cdot \left[\frac{1+\lambda a_2}{1-\lambda a_2}\psi_2'(\tau) + \frac{K_1}{1-\lambda a_2}\psi_2(\tau)\right] d\tau + C_2 \right\} \tag{2.29}$$

where

$$C_2 = \varphi_2(\ell_2 + a_2iT_1 - 0)e^{-\frac{K_1}{1-\lambda a_2}(\ell_2+a_2iT_1)}$$

If $1-\lambda a_2 = 0$ then

$$\left(\varphi_2\right)_2 = -\frac{2}{K_1}\left(\psi_2'\right)_1 - \left(\psi_2\right)_1 \tag{2.30}$$

So that the wave functions $\left(\psi_2'\right)_2$ and $\left(\psi_1'\right)_2$ are determined. If the shocks of two bare are still not finished yet in second period of second bar, the next periods are studied is the same method as above mentioned. Let $(\ )_{pn}$ be the wave function in n^{th} period of first and in p^{th} period of second bar. In interval $(p-1)T_2 + (n-1)T_1 < t < (p-1)T_2 + nT_1$ with $n = 1, 2, \dots,$ the problem is studied as following:

From conditions (1.3) and (1.4) we have:

$$\left(\psi_2''\right)_{pn} = \left(\varphi_2''\right)_{pn} + \frac{1}{A}\left[2\left(\varphi_1'\right)_{pn} + B\left(\varphi_2'\right)_{pn} + C\left(\psi_2'\right)_{pn}\right] \cdot \left[\left(-\varphi_2'\right)_{pn} + \left(\psi_2'\right)_{pn}\right]^{1/3} \tag{2.31}$$

$$\left(\psi_1'\right)_{pn} = \left(\varphi_1'\right)_{pn} + \frac{1}{\alpha}\left[\left(-\varphi_2'\right)_{pn} + \left(\psi_2'\right)_{pn}\right] \tag{2.32}$$

$$\left(\varphi_1'\right)_{pn} = \left(\psi_1'\right)_{p(n-1)} \tag{2.33}$$

By simillar way mentioned above, and from condition (1.5) we obtain:

If $1-\lambda a_2 \neq 0$ then

$$\left(\varphi_2\right)_{pn} = e^{\frac{K_1}{1-\lambda a_2}(a_2t-\ell_2)} \cdot \left\{ \int_{a_2[(2p-3)\frac{T_2}{2}+(n-1)T_1]}^{a_2t-\ell_2} e^{-\frac{K_1}{1-\lambda a_2}\tau} \cdot \left[\frac{1+\lambda a_2}{1-\lambda a_2}\psi_2'(\tau) + \frac{K_1}{1-\lambda a_2}\psi_2(\tau)\right] d\tau + C_{pn} \right\} \tag{2.34}$$

where $p \geq 2$, and

$$C_{pn} = \varphi_2\left[a_2(2p-3)\frac{T_2}{2} + a_2(n-1)T_1 - 0\right] \cdot e^{-\frac{K_1}{1-\lambda a_2}[a_2(2p-3)\frac{T_2}{2}+a_2(n-1)T_1]}.$$

If $1-\lambda a_2 = 0$ then

$$\left(\varphi_2\right)_{pn} = -\frac{2}{K_1}\left(\psi_2'\right)_{(p-1)n} - \left(\psi_2\right)_{(p-1)n} \tag{2.35}$$

So that the wave functions $\left(\psi_2'\right)_{pn}$ and $\left(\psi_1\right)_{pn}$ are determined. Now we are studying this problem in final odd part of the p^{th} period of the second bar, or $(p-1)T_2 + iT_1 < t < (p-1)T_2 + iT_1 + qT_1 = pT_2$. Let $(\ )_p$ be wave function determined in odd part of p^{th} period of the second bar. From condition (1.3) and (1.4) we have

$$\left(\psi_2''\right)_p = \left(\varphi_2''\right)_p + \frac{1}{A}\left[2\left(\varphi_1'\right)_p + B\left(\varphi_2'\right)_p + C\left(\psi_2'\right)_2\right] \cdot \left[\left(-\varphi_2'\right)_p + \left(\psi_2'\right)_p\right]^{1/3} \tag{2.36}$$

$$\left(\psi_1'\right)_p = \left(\varphi_1'\right)_p + \frac{1}{\alpha}\left[\left(-\varphi_2'\right)_p + \left(\psi_2'\right)_p\right] \tag{2.37}$$

$$\left(\varphi_1'\right)_p = \left(\psi_1'\right)_{pi} \tag{2.38}$$

Doing similarly, from eq. (1.5) we get:
If $1 - \lambda a_2 \neq 0$ then

$$\left(\varphi_2\right)_p = e^{\frac{K_1}{1-\lambda a_2}(a_2 t - \ell_2)} \cdot \left\{ \int\limits_{a_2[(2p-3)\frac{T_2}{2}+iT_1]}^{a_2 t - \ell_2} e^{-\frac{K_1}{1-\lambda a_2}\tau} \cdot \left[\frac{1+\lambda a_2}{1-\lambda a_2}\psi_2'(\tau) + \frac{K_1}{1-\lambda a_2}\psi_2(\tau)\right] d\tau + C_p \right\} \tag{2.39}$$

where

$$C_p = \varphi_2\left[a_2(2p-3)\frac{T_2}{2} + ia_2T_1 - 0\right] \cdot e^{-\frac{K_1}{1-\lambda a_2}[a_2(2p-3)\frac{T_2}{2} + a_2 i T_1]}.$$

If $1 - \lambda a_2 = 0$ then

$$\left(\varphi_2\right)_p = -\frac{2}{K_1}\left(\psi_2'\right)_{(p-1)} - \left(\psi_2\right)_{(p-1)} \tag{2.40}$$

So that the wave functions $\left(\psi_2'\right)_p$ and $\left(\psi_1'\right)_p$ are determined. Impact-pressing force F between two bars is determined by the following expression: $\left(F\right)_{pn} = E_2F_2\left[\left(-\varphi_2'\right)_{pn} + \left(\psi_2'\right)_{pn}\right]$ and $\left(F\right)_p = E_2F_2\left[\left(-\varphi_2'\right)_p + \left(\psi_2'\right)_p\right]$. Impact time determined by the following expression $\left(F\right)_{pn} = 0$, or $\left(F\right)_p = 0$. So we can determine the wave functions $\varphi_1'(a_1 t - x_1)$, $\psi_1'(a_1 t + x_1)$, $\varphi_2'(a_2 t - x_2)$ and $\psi_2'(a_2 t + x_2)$ at each section of two bars in impact time, and whence stress, velocity in each section of bars can be found.

§3. CONCLUSION

In this paper the authors have studied the problem of longitudinal shock of two spherical end elastic bars with visco-elastic resistance force. The wave function, stress, velocity in each section of bars, impact-pressing force between two bars and impact time are given. The considered model can be applied for pile driving on visco-elastic soil.

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VA CHẠM DỌC CỦA HAI THANH ĐẦU HÌNH CẦU VỚI LỰC CẢN ĐÀN NHỚT

Trong bài báo này các tác giả xét bài toán va chạm dọc của hai thanh đàn hồi đầu hình cầu với đầu kia của thanh thứ hai gặp lực cản đàn nhớt. Đã xác định được hàm sóng, từ đó xác định được ứng suất, vận tốc tại mỗi thiết diện của thanh, lực nén va chạm giữa hai thanh và thời gian va chạm.