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SENSITIVITY ANALYSIS FOR A PROBLEM OF OPTIMAL STRUCTURE DESIGN

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SUMMARY. On the basis of theory of structure sensitivity analysis, the gradient-projection method and the finite element method, a detailed and effective algorithm of determination of the sensitivity-vector for the optimal structure design problem is proposed. Numerical results illustrating the program corresponding to this algorithm have been given for some plane frames.

§1. PROBLEM OF OPTIMAL STRUCTURE DESIGN

The problem of optimal elastic structure design here is formulated as follows [1]:

Determine design variable vector $b \in R^k$ to minimize objective function $\psi_0(z, \xi, b)$ (it can be the weight of the structure) satisfying the following state equations and function constraints:

1. The equations of equilibrium for a structure (static and dynamic):

$$\begin{aligned} &h(z,b) = 0 \text{ . and} \\ &K(b)y = \xi M(b)y \end{aligned} \tag{1.1}$$

2. Function constraints:

$$\psi_j(z, \xi, b) \geq 0, \quad j = 1, m$$

(they can be constraints on displacement, stress or natural frequency and design variables). where

$K(b)$ - stiffness matrix of the system.

$M(b)$ - mass matrix

$z \in R^n$ - displacement vector

$\xi \in R^n$ - natural frequency of vibration of system

$y \in R^n$ - eigenvector

z, ξ, y all they are state variables.

Note that, if the structure is a truss-system of m elements, then the weight function has the form:

$$\psi_0 = \sum_{i=1}^{m} \varrho_i L_i A_i$$

where L_i - the length of i-th truss element

A_i - its cross area

ϱ_i - material density of i-element

The design variables b (they can be L_i or A_i) are chosen and changed by a designer, but state variables z, ξ, y or stresses of a structure depend on equilibrium conditions and correlations between displacements and stresses. Therefore a designer can not change state variables directly.

There are many methods for solving the above problem [2, 3, 4]. For almost iterative optimal structure methods it requires to know gradient values, which are received in result of sensitivity analysis. The essence of this analysis is presented in the following section.

§2. SENSITIVITY ANALYSIS OF STRUCTURE

Here the influence of project variation is investigated by approximating nonlinear function of the problem with linear expression corresponding to considering variables.

The change of the objective function $\psi_0(z,\xi,b)$ and of the function constrains $\psi_j(z,\xi,b)$, $j = \overline{1,m}$ corresponding to small changes of variables can be written in the form:

$$\begin{aligned}\partial\psi_0[z_0,\xi_0,b_0] &= \frac{\partial\psi_0}{\partial z}[z_0,\xi_0,b_0]\delta z + \frac{\partial\psi_0}{\partial\xi}[z_0,\xi_0,b_0]\delta\xi + \frac{\partial\psi_0}{\partial b}[z_0,\xi_0,b_0]\delta b,\\ \partial\psi_j[z_0,\xi_0,b_0] &= \frac{\partial\psi_j}{\partial z}[z_0,\xi_0,b_0]\delta z + \frac{\partial\psi_j}{\partial\xi}[z_0,\xi_0,b_0]\delta\xi + \frac{\partial\psi_j}{\partial b}[z_0,\xi_o,b_0]\delta b.\end{aligned} \tag{2.1}$$

Because equation of equilibrium $h(z_0,b_0) = 0$ is also true in the case of increasing displacement z and design variable b with a small value, we have:

$$h(z_0+\delta z, b_0+\delta b) = 0$$

it follows:

$$\frac{\partial h(z_0,b_0)}{\partial z}\delta z + \frac{\partial h(z_0,b_0)}{\partial b}\delta b = 0 \tag{2.2}$$

This equation can be considered as the condition to determine δz as a function of δb. Then with the notation

$$J = \frac{\partial h_0(z_0,b_0)}{\partial z}$$

equation (2.2) has the form:

$$J\delta z = -\frac{\partial h}{\partial b}\delta b \tag{2.3}$$

On the other hand, if we consider column vectors λ^i as a solution of conjugate equation:

$$J^T\lambda^i = \frac{\partial\psi_i^T}{\partial z} \qquad 0 \le i \le m \tag{2.4}$$

from (2.3) and (2.4) we have a relation between δb and δz:

$$-\lambda^{iT}\frac{\partial h}{\partial b}\delta b = \frac{\partial\psi_i}{\partial z}\delta z \tag{2.5}$$

Similarly, a variation $\delta\xi$ depends on δb [1] as follows:

$$\delta\xi = \ell^{\xi T}\delta b$$

where

$$\ell^{\xi T} = [\frac{\partial}{\partial b}\{y^T K(b) y\} - \xi\frac{\partial}{\partial b}\{y^T M(b) y\}]\delta b \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.1) we get equations

$$\begin{aligned}\delta\psi_0 &= \ell^{0T}\delta b\\ \delta\psi &= \ell^T\delta b\end{aligned}$$

where vector ℓ_0 and column-vector ℓ_i of matrix ℓ at point z_0, ξ_0, b_0 according to formula

$$\ell^i = \frac{\partial \psi_i^T}{\partial b} - \frac{\partial h^T}{\partial b}\lambda^i + \frac{\partial \psi_i}{\partial \xi}\ell^\xi. \tag{2.7}$$

Components of the vector ℓ_i are called sensitivity coefficients of the constraint-function ψ_i corresponding to design variable b. These vectors give derivatives of the object function and constraint-function (ℓ_j^i is derivative of ψ_i with respect to j-th design variable).

These components ℓ_j^i are needed for designers because they present the influence of design variable changes on object function or constraint-function. If $\ell_j^i > 0$ then increasing b_j follows increasing ψ_i. If $\ell_j^i < 0$ then increasing b_j follows decreasing ψ_i. Moreover, order of a value of the different sensitivity coefficients l_j^i informs designer that what design variable has great or small influence on ψ_i.

Remark that an elastic structure, best of all, is simulated by the finite element methods. When equation (1.1) has the linear form:

$$h(b,z) = K(b)z - S(b) = 0 \tag{2.8}$$

$S(b)$ - matrix of external loads and the Jacobian of this equation is expressed as:

$$J = \frac{\partial h}{\partial z} = K$$

Because K is a symmetric matrix, equation (2.4) takes the form:

$$K\lambda^i = \frac{\partial \psi_i^T}{\partial z} \tag{2.9}$$

This equation has the same form as the equation (2.8) does. The difficulty in obtaining ℓ^i lies in derivation $\partial h(b,z)/\partial b$ or $(\partial K(b)z - S(b))/\partial b$. In this work it is formed by derivation with respect to design variables for each element stiffness matrix and sum the results by an algorithm of the finite element method.

§3. CONNECTION BETWEEN SENSITIVITY VECTOR ℓ AND SOLUTION OF OPTIMAL PROBLEM

Now the optimal problem (1.1) can be reduced to the following problem [1]: To find δb for minimum $\delta\psi_0 = \ell^{0T}\delta b$, satisfying conditions:

1. $\delta b^T W \delta b \le \beta^2$, where β - small parameter; W - weight matrix
2. linear constraint

$$\delta\tilde{\psi} = \frac{\partial \psi}{\partial b}\delta b = \ell^T \delta b = \begin{cases} = \Delta\psi_j, & \text{when } j = 1, ..., n \\ \le \Delta\psi_j, & \text{when } j > n; \quad \psi_j(b_0) \ge -\varepsilon \end{cases}$$

where $\Delta\psi_j = -\psi_j(b_0)$ is the limit of the change of the constraint-functions for ε-active constraint [1]. On the basic theorem of Kunatake about the existence of a solution of nonlinear programming problems it is shown in [1] that there exists a vector-multiplier μ and a scalar $\gamma \ge 0$ satisfying the following equations:

$$\ell_0 + \ell\mu + 2\gamma W\delta b = 0$$

$$\mu_i(\delta\psi_i - \Delta\psi_i) = 0 \qquad i \ge n$$

$$\gamma(\delta b^T W \delta b - \beta^2) = 0$$

From these equations we have formulae determining $\delta b, \mu, \gamma$ by the sensitivity vector,

$$\delta b = -\frac{1}{2}\delta b_1 + \delta b_2 \tag{3.1}$$

where

$$\delta b_1 = W^{-1}[\ell_0 + \ell\mu_1], \qquad \delta b_2 = -W^{-1}\ell\mu_2$$

μ_1 and μ_2 are solutions of the equations

$$M_{\psi\psi}\mu_1 = -M_{\psi\psi_0}, \qquad M_{\psi\psi}\mu_2 = -\Delta\tilde{\psi}$$

where

$$M_{\psi\psi} = \ell^T W^{-1}\ell, \qquad M_{\psi\psi_0} = \ell^T W^{-1}\ell_0.$$

If all μ_i corresponding to $\psi_i(b_0)$ - are positive when b_0 is a solution satisfying Kunatake condition. If exist some $\mu_i < 0$ corresponding to $\psi_i(b_0) \geq -\epsilon$ for $i > 0$, then one can receive a better solution by excluding the corresponding constraint $\psi_i(b)$ and it is necessary to calculate $\ell, M_{\psi\psi_0}, M_{\psi\psi}$, once more and the process will be stopped when all $\mu_i > 0$ and δb will be calculated by (3.1).

Thus, clearly that the sensitivity vector of the constraint-functions ψ_i corresponding to design variables has important role in calculating variables δb in the optimum design of the structures.

§4. ALGORITHM FOR DETERMINATION OF SENSITIVITY VECTORS NUMERICAL EXAMPLES AND CONCLUSION

As the result of above investigation, the effective algorithm for obtaining sensitivity vectors is presented, and consists of the following steps:

1. Chose engineering design variable b_0 for project.
2. Solve equilibrium equations:

$$\begin{aligned} h(z) &= K(b)z - S(b) = 0 \qquad \text{or :} \\ K(b)y &= \xi M(b)y, \end{aligned}$$

finding z, y, ξ, corresponding to b_0.

3. Check if the obtained values z, y, ξ, satisfy the conditions of the constraints. If they don't, then establish corresponding constraint-function vector $[\psi]$.
4. Solve equation $K\lambda_i = \partial\psi_i^T/\partial z$ to find λ_i.
5. Calculate derivatives $\partial h/\partial b$, $\partial\psi_i/\partial b$, $\partial\psi_i/\partial\xi$ and ℓ_i corresponding to constraint ψ_i by formula (8). Two numerical examples are given, which illustrate the algorithm and its effectiveness.

Example 1. Consider the truss system of ten elements (see fig. 1.) with the parameters: $E = 10^7 N/m^2$, $\varrho = 0.1 N/m^3$, the critical displacement $z = 2m$, critical stress $\sigma_c = \pm 25.10^3 N/m^2$. This system is subjected to external loads : $P_1 = 10^5 N$, $P_2 = 10^5 N$, $P_3 = 15.10^3 N$, $P_4 = 13.10^3 N$.

The results (see table 1) show that the sensitivity vectors ℓ_1, ℓ_2 are corresponding to displacements in the direction of the axis Oy at the points 1 and 4 (their values are greater than critical displacements). Vectors ℓ_3, ℓ_4, ℓ_5 are corresponding to the stresses in the elements 5, 6, 7 (there the stresses are greater than critical value). The column vector ℓ_3 shows that, in order to decrease the stress in the element 5 one must increase the cross areas of the elements 3-7, 9 and decrease the cross area of the element 1, 2, 8 and 10.

Note that, if the external loads P_3 and P_4 are absent, our results for sensitivity vectors coincide with the results given in [1], that shows exactness of the program.

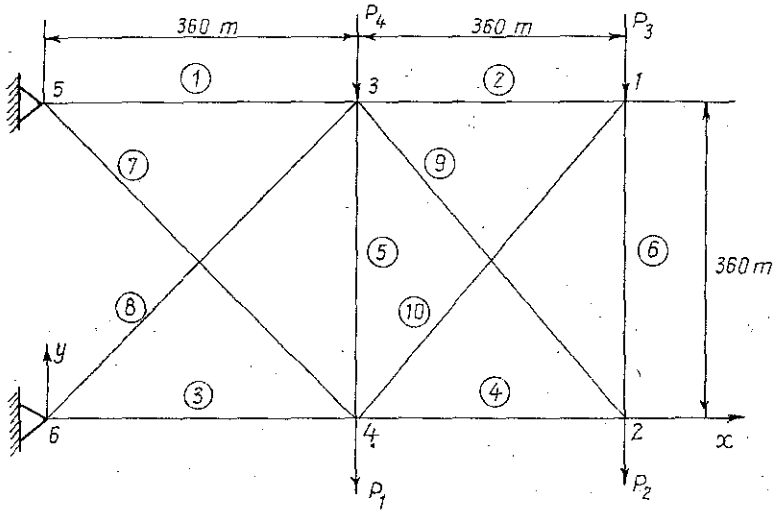

Fig. 1

Table 1

N	Area	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
1	28.8	-.00673	-.00069	.00974	-.00200	-.00111
2	.2	-.08581	.01289	.05038	.26849	.02056
3	23.6	-.00525	-.00357	-.01143	.00235	.00130
4	15.4	-.00454	-.00018	-.00071	-.00379	-.00029
5	.2	-.05090	-.59515	-1.32544	.47776	-.94925
6	.2	-.23030	-.12960	-.50639	.94390	-.20671
7	3.6	-.06593	-.27170	-.87009	.17876	-.45728
8	21.0	-.00862	-.00199	.02800	-.00575	-.00318
9	21.8	-.00641	-.00026	-.00100	-.00535	-.00041
10	.2	-.24270	.03647	.14249	.75941	.05816

Example 2. Consider the truss system of 29 elements (see fig. 2.) with the parameters: $E = 10^7 N/m^2$, $\varrho = 0.1 N/m^3$, the critical displacement $z = 0.026m$, the critical stress $\sigma_c = 26.10^3 N/m^2$. This system is subjected to external loads: $P_1 = 500N$, $P_2 = 1000N$, $P_3 = 1200N$, $P_4 = 1400N$, $P_5 = 1600N$.

The results (see table 2) show that sensitivity vectors ℓ_1 are corresponding to the displacement in the direction of the axis Ox at the points 16 (these values are greater than critical displacement). The vectors ℓ_2, ℓ_3, ℓ_4, ℓ_5 are corresponding to the stresses in the elements 1, 4, 5, 10 (there stresses are greater than critical value). Column vector ℓ_2 shows that, in order to decrease the stress in element 1 one must increase the cross areas of the elements 1, 3-5, 9, 10, 11, 13, 14, 16, 19, 20 and decrease the cross area of the element 2, 6.

In the tables 1 and 2 N is an index of the beam, the second column is the cross area of the beam. To decrease the stress or displacement of the beam corresponding ℓ_i need to be interested in the value of the column vector component ℓ_i. If the value of vector component ℓ_i is negative then the cross area of corresponding beam must be increased. If the value of vector component ℓ_i is positive then cross area of corresponding beam must be decreased. The oder of value ℓ_i in the tables informs us about the change level of the cross area.

Table 2

N	Area	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
1	.250	.62782	-1.00190	.51501	-.00001	.00001
2	.150	.10769	.76114	.76115	.00000	.00000
3	.150	.10333	-.73033	-.73032	.00000	.00000
4	.250	.62242	-.51058	.95028	-.00001	.00001
5	.250	.36414	-.00001	.00001	.13223	.00001
6	.200	.01757	.12316	.12316	.00000	.00000
7	.150	.08749	.00000	.00000	.00000	.00000
8	.150	.08749	.00000	.00000	.00000	.00000
9	.200	.02166	-.15181	-.15181	.00000	.00000
10	.250	.36414	-.00001	.00001	-.00001	.13222
11	.250	.41645	-.00001	.00001	-.00001	.00001
12	.200	.01084	.00000	.00000	.00000	.00000
13	.150	.23867	-.00001	.00001	-.00001	.00001
14	.150	.23867	-.00001	.00001	-.00001	.00001
15	.200	.02073	.00000	.00000	.00000	.00000
16	.250	.41645	-.00001	.00001	-.00001	.00001
17	.250	.12610	.00000	.00000	.00000	.00000
18	.200	.00639	.00000	.0000	.00000	.00000
19	.150	.16181	-.00001	.00000	.00000	.00000
20	.150	.16181	-.00001	.00000	.00000	.00000
21	.200	.01836	.00000	.00000	.00000	.00000
22	.250	.12610	.00000	.00000	.00000	.00000
23	.250	.02015	.00000	.00000	.00000	.00000
24	.200	.00006	.00000	.00000	.00000	.00000
25	.150	.08606	.00000	.00000	.00000	.00000
26	.150	.08606	.00000	.00000	.00000	.00000
27	.200	.01398	.00000	.00000	.00000	.00000
28	.250	.02015	.00000	.00000	.00000	.00000
29	.200	.01591	.00000	.00000	.00000	.00000

CONCLUSION

Sensitivity analysis have a great role in solving the optimal design problem. With the help of above mentioned algorithm and program we obtain the sensitivity vectors for solving optimal design problems.

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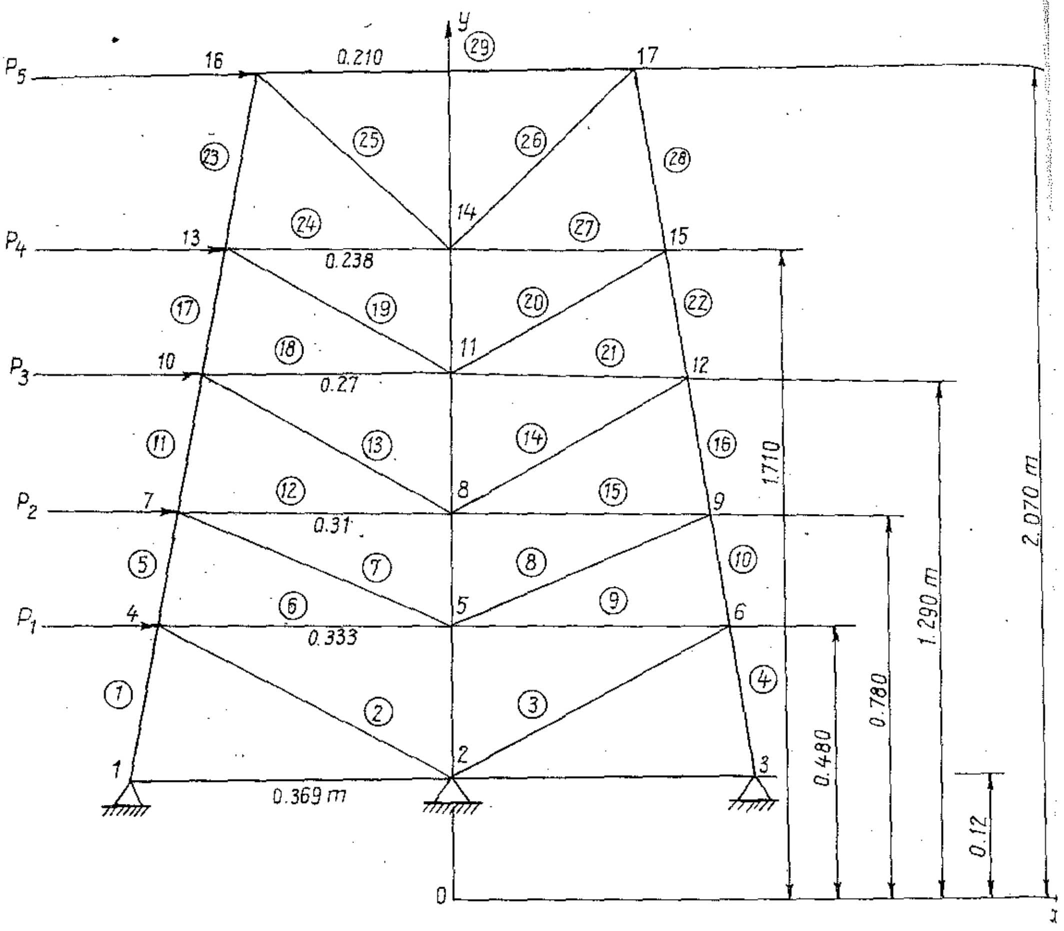

Fig. 2

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PHÂN TÍCH ĐỘ NHẠY CẢM TRONG BÀI TOÁN THIẾT KẾ TỐI ƯU

Trên cơ sở lý thuyết phân tích độ nhậy cảm của cấu trúc, phương pháp chiếu gradient và phương pháp phần tử hữu hạn, một thuật toán chi tiết và hiệu quả để xác định véc tơ nhậy cảm trong bài toán thiết kế tối ưu đã được đưa ra. Các kết quả số minh họa cho chương trình tương ứng với thuật toán trên đã được thực hiện với một số dàn phẳng.