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ON THE DEPHASE ANGLE IN A VARIATIONAL SYSTEM OF THE EQUILIBRIUM REGIME

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SUMMARY. In the quasi-linear theory of vibrations, the initial variable x is transformed either to the variables (a, b) or to the amplitude-phase ones (r, θ) respectively by the formulae

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -\omega a\sin\omega t + \omega b\cos\omega t \tag{0.1}$$

$$x = r\cos(\omega t - \theta), \quad \dot{x} = -\omega r\sin(\omega t - \theta) \tag{0.2}$$

where ω is near the natural frequency of the system under consideration.

These two types of variables are equivalent if the oscillatory regime is concerned. However, for the equilibrium regime, the situation is more complicated.

In [1] (pp. 211-213), to study the stability of the equilibrium regime, the variables (r, θ) are transformed into the ones (a, b). The equilibrium regime corresponds to the couple of determinate values $a_0 = 0$, $b_0 = 0$ so that the signification of the perturbations $\delta a = a - a_0$, $\delta b = b - b_0$ is evident and the variational system can be easily established.

Other author [2] (pp. 617-624) has used the amplitude-phase variables (r, θ) for seeking the stability condition of the equilibrium regime which coresponds to the "zero" amplitude $r_0 = 0$. The dephase angle θ remains indeterminate. A certain constant (unknown) value θ_0 is assigned to the equilibrium regime. By this manner, the "classical" process of studying the stability by introducing the perturbations $\delta r = r - r_0$, $\delta\theta = \theta - \theta_0$ and the variational equations can be applied. As it will be seen below, the variational system obtained has an "anormal" form.

In the present paper, the signification of the mentioned dephase angle θ_0 will be discussed and another process for studying the stability of the equilibrium regime will be proposed.

§1. METHOD OF STUDYING THE STABILITY OF THE EQUILIBRIUM REGIME

Let us consider a quasi-linear parametrically-excited system described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{ - h\dot{x} + (\omega^2 - \omega_0^2)x - \gamma x^3 + 2px\cos 2\omega t\Big\} \tag{1.1}$$

where overdot denotes derivative with respect to time t; ω_0 is the natural frequency of the system considered; $2p > 0$ and 2ω are the intensity and the frequency of the parametric excitation, respectively, $\omega \approx \omega_0$, $h > 0$ is the damping linear coefficient; γ is the coefficient of the cubic non-linearity, $\varepsilon > 0$ is a small parameter.

In the phase plane $Ox\dot{x}$, the solution $x = 0$ (equilibrium regime) corresponds to the origine O.

Using the variables (a,b), in the first approximation, we obtain the averaged system:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{2\omega}\Big\{h\omega a + \big[(\omega^2-\omega_0^2)-p\big]b - \frac{3}{4}\gamma(a^2+b^2)b\Big\} \\ \dot{b} &= \frac{\varepsilon}{2\omega}\Big\{\big[(\omega^2-\omega_0^2)+p\big]a - h\omega b - \frac{3}{4}\gamma(a^2+b^2)a\Big\} \end{aligned} \tag{1.2}$$

The equilibrium regime corresponds to the solution $a_0 = b_0 = 0$ which is assumed to be isolated i.e. the following condition is imposed:

$$p^2 \neq h^2\omega^2 + (\omega^2-\omega_0^2)^2 \tag{1.3}$$

Introducing the pertubations $\delta a = a - a_0$, $\delta b = b - b_0$ we can easily establish:
- the variational system:

$$\begin{aligned} (\delta a)^{\cdot} &= -\frac{\varepsilon}{2\omega}\Big\{(h\omega)\delta a + \big[(\omega^2-\omega_0^2)-p\big]\delta b\Big\} \\ (\delta b)^{\cdot} &= \frac{\varepsilon}{2\omega}\Big\{\big[(\omega^2-\omega_0^2)+p\big]\delta a - (h\omega)\delta b\Big\} \end{aligned} \tag{1.4}$$

which coincides with the linear part of the averaged system (1.2) - then, the characteristic equation:

$$\begin{vmatrix} -\varepsilon\frac{h}{2}-\rho & -\frac{\varepsilon}{2\omega}\big[(\omega^2-\omega_0^2)-p\big] \\ \frac{\varepsilon}{2\omega}\big[(\omega^2-\omega_0^2)+p\big] & -\varepsilon\frac{h}{2}-\rho \end{vmatrix} = \rho^2+\varepsilon h\rho+\frac{\varepsilon^2}{4\omega^2}\Big\{h^2\omega^2+(\omega^2-\omega_0^2)^2-p^2\Big\} = 0 \tag{1.5}$$

Since $h > 0$, the stability condition is:

$$p^2 < h^2\omega^2 + (\omega^2-\omega_0^2)^2 \tag{1.6}$$

If the second type of variables - the amplitude-phase variables $(r, 0)$ is used, in the first approximation, the averaged system is:

$$\begin{aligned} \dot{r} &= -\frac{\varepsilon r}{2\omega}\Big\{h\omega + p\sin 2\theta\Big\} \\ r\dot{\theta} &= \frac{\varepsilon r}{2\omega}\Big\{-\frac{3}{4}\gamma r^2 + (\omega^2-\omega_0^2) + p\cos 2\theta\Big\} \end{aligned} \tag{1.7}$$

It is noted that the system (1.7) can be transformed into (1.2) by formulae:

$$a = r\cos\theta, \quad b = r\sin\theta \tag{1,8}$$

The equilibrium regime corresponds now to the "zero" amplitude $r_0 = 0$. The dephase angle θ in an indeterminate value.

Assuming that the equilibrium regime corrersponds to a certain constant (unknown) value θ_0 and following the classical process of studying the stability we introduce the perturbations $\delta r = r - r_0$, $\delta\theta = \theta - \theta_0$ and obtain the variational system:

$$(\delta r)^{\cdot} = -\frac{\varepsilon}{2\omega}\Big\{h\omega + p\sin 2\theta_0\Big\}\delta r \tag{1.9a}$$

$$0 = \frac{\varepsilon}{2\omega}\Big\{(\omega^2-\omega_0^2) + p\cos 2\theta_0\Big\}\delta r \tag{1.9b}$$

Since $\delta r \neq 0$, the second equation (1.9b) leads to the trigonometrical equation:

$$(\omega^2-\omega_0^2) + p\cos 2\theta_0 = 0 \tag{1.10}$$

from which, the unknown value θ_0 can be calculated

$$\cos 2\theta_0 = -\frac{\omega^2 - \omega_0^2}{p} \tag{1.11a}$$

$$\sin 2\theta_0 = \pm\frac{1}{p}\sqrt{p^2 - (\omega^2 - \omega_0^2)^2} \tag{1.11b}$$

Substituting (1.11b) into (1.9a), we obtain:

$$(\delta r)^{\cdot} = -\frac{\varepsilon}{2\omega}\Big\{h\omega \pm \sqrt{p^2 - (\omega^2 - \omega_0^2)^2}\Big\}\delta r \tag{1.12}$$

The solution $r_0 = 0$ of (1.7) (equilibrium regime) is stable if:

$$\mathrm{Re}\Big\{-\frac{1}{2\omega}\big[h\omega \pm \sqrt{p^2 - (\omega^2 - \omega_0^2)^2}\,\big]\Big\} < 0 \tag{1.13}$$

This inequality coincides with (1.6).

It is noted that, in (1.11)- (1.13), θ_0 and δr may be either real or complex.

§2. THE DEPHASE ANGLE OF THE VARIATIONAL SYSTEM IN AMPLITUDE-PHASE VARIABLES

Let us remark that, in the second method presented above, the dephase angle θ_0 is introduced without interpretation. Otherwise, the variational system (1.9) has an "anormal" form: the perturbation $\delta\theta$ does not present in the system, the second equation (1.9b) cannot be considered as a differential equation. Some questions are arisen: Is it "logical" to assigne any constant dephase angle θ_0 to the equilibrium regime? What pictures of motion can be obtained from the variational system (1.9)? Are these pictures analogous with those corresponding to center, nodal, focus or saddle points? To obtain a necessary answer, let us return to the variables (a, b). Doing this, we have replaced the phase plane $Ox\dot{x}$ by the plane Oab.

1 - If $p^2 > (\omega^2 - \omega_0^2)$, there are two separated real characteristic values:

$$\rho_{1,2} = \frac{\varepsilon}{2\omega}\Big\{-h\omega \pm \sqrt{p^2 - (\omega^2 - \omega_0^2)^2}\Big\} \tag{2.1}$$

corresponding to two families of characteristic motions:

$$a_j = A_j e^{\rho_j t}, \qquad b_j = \xi_j A_j e^{\rho_j t} \tag{2.2a}$$

$$\xi_j = \frac{b_j}{a_j} = \pm\sqrt{\frac{p + (\omega^2 - \omega_0^2)}{p - (\omega^2 - \omega_0^2)}} \quad (j = 1, 2) \tag{2.2b}$$

where A_j are constants, depending on initial (suitable) conditions.

In the Oab the origin O is either an unstable saddle point if $p^2 > h^2\omega^2 + (\omega^2 - \omega_0^2)^2$ (fig.1) or a stable nodal one if $p^2 < h^2\omega^2 + (\omega^2 - \omega_0^2)^2$ (fig. 2). In both cases, there exist two straight trajectories $\Delta_{1,2}$ whose slopes $\theta_{1,2}$ are given by formulae:

$$\mathrm{tg}\theta_j = \xi_j \tag{2.3}$$

According to (1.8), $\theta_{1,2}$ are just the dephase angles of characteristic motions.

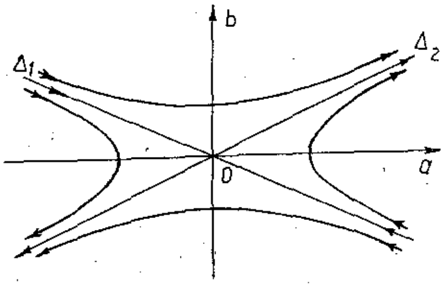

Fig. 1

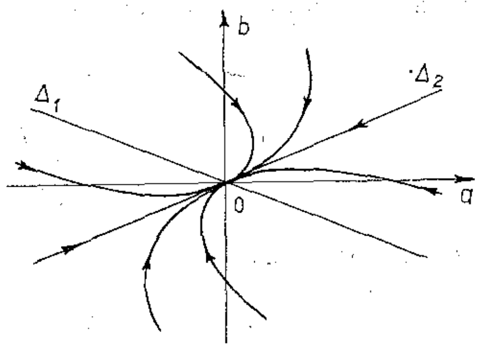

Fig. 2

2 - If $p^2 = (\omega^2 - \omega_0^2)^2$, the characteristic equation (1.5) admits a double real solution:

$$\rho = -\frac{\varepsilon h}{2} \tag{2.4}$$

The origine O becomes a stable nodal point with only one straight trajectory which coincides either with Ob if $p = \omega^2 - \omega_0^2$ (Fig. 3) or with Oa if $p = \omega_0^2 - \omega^2$ (fig. 4).

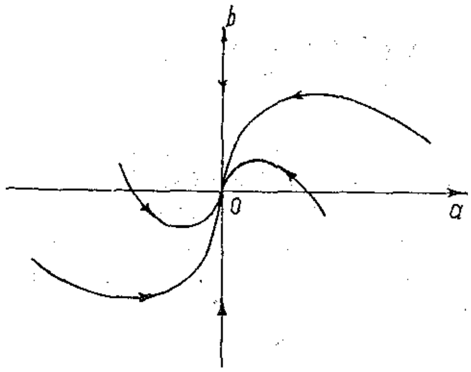

Fig. 3

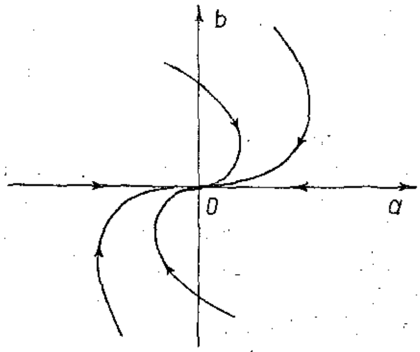

Fig. 4

3 - If $p^2 < (\omega^2 - \omega_0^2)^2$, there are two conjugate complex characteristic values:

$$\rho_{1,2} = -\frac{\varepsilon h}{2} \pm i\frac{\varepsilon}{2\omega}\sqrt{(\omega^2 - \omega_0^2)^2 - p^2} \tag{2.5}$$

The origine O is a stable focus point whose spirals turn round O either in the direction from Oa to Ob if $p < \omega^2 - \omega_0^2$ (fig. 5) or in the inverse one if $p < \omega_0^2 - \omega^2$ (fig. 6).

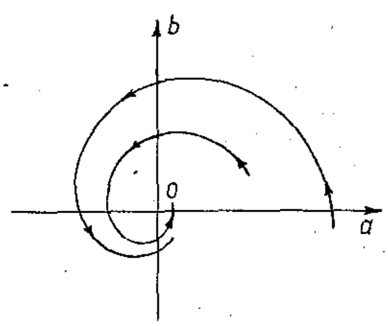

Fig. 5

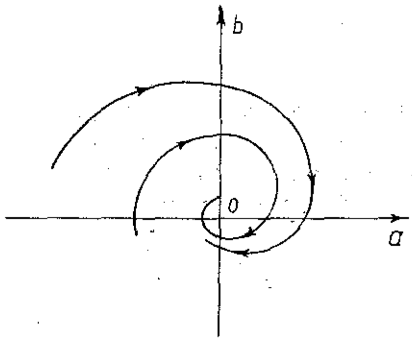

Fig. 6

In the last case, the straight trajectories are complex since the ratio $\frac{b_j}{a_j}$ becomes immaginary.

$$\mathrm{tg}\theta_j = \frac{b_j}{a_j} = \xi_j = \mp i \frac{\sqrt{(\omega^2-\omega_0^2)^2 - p^2}}{(\omega^2-\omega_0^2) - p} = \\ = \begin{cases} \mp i \sqrt{\dfrac{(\omega^2-\omega_0^2)+p}{(\omega^2-\omega_0^2)-p}} & \text{if} \quad p < \omega^2 - \omega_0^2 \\ \pm i \sqrt{\dfrac{(\omega_0^2-\omega^2)-p}{(\omega_0^2-\omega^2)+p}} & \text{if} \quad p < \omega_0^2 - \omega^2 \end{cases} \tag{2.6}$$

We compare now the values of θ_j with those of θ_0 determined by the second equation of the variational system (1.9). For the first and third cases, according to (2.2b) and (2.6), respectively, it is easy to prove that:

$$\cos 2\theta_j = \frac{1-\mathrm{tg}^2\theta_j}{1+\mathrm{tg}^2\theta_j} = \frac{1-\xi_j^2}{1+\xi_j^2} = -\frac{\omega^2-\omega_0^2}{p} = \cos 2\theta_0 \tag{2.7}$$

For the second case, $p^2 = (\omega^2-\omega_0^2)^2$, we obtain:

$$\cos 2\theta_0 = \mp 1 \text{ or } \theta_0 = \begin{cases} \dfrac{\pi}{2} + k\pi & \text{if} \quad p = \omega^2 - \omega_0^2 \\ k\pi & \text{if} \quad p = \omega_0^2 - \omega^2 \\ & (k = 0, \pm 1, \pm 2, \dots) \end{cases} \tag{2.8}$$

i.e. the straight trajectory is either Ob or Oa.

Thus, the dephase angles θ_0 of the variational system in amplitude-phase variables coincide with those of characteristic motions. Consequently, after substituting $\sin 2\theta_0$ by formula (1.11b), the first equation of the mentioned variational system becomes the differential one, governing the amplitude variation of characteristic motions.

§3. ANOTHER PROCESS FOR STUDYING THE STABILITY OF THE EQUILIBRIUM REGIME

The analyses presented permit us to propose another process for studying the stability of the equilibrium regime.

First, as in [2], we establish the averaged system (1.7) whose trivial solution $r_0 = 0$ corresponds to the equilibrium regime.

To study the stability of this regime, the motion with small amplitude must be considered. Hence, the term of hight power of r - the term $\frac{3}{4}\gamma r^3$ - can be neglected and since $r \neq 0$, the second equation of the system (1.7) can be divided by r. We obtain thus the following equations:

$$\dot{r} = -\frac{\varepsilon r}{2\omega}\Big\{h\omega + p\sin 2\theta\Big\} \tag{3.1a}$$

$$\dot{\theta} = \frac{\varepsilon}{2\omega}\Big\{(\omega^2-\omega_0^2) + p\cos 2\theta\Big\} \tag{3.1b}$$

which play the role of the variational system.

Secondly, we consider the characteristic motions, defined as those corresponding to constant dephase angles. Let $\dot{\theta} = 0$, the equation (3.1b) leads to the trigonometrical equation (1.10) giving us θ_0.

Then, substituting θ_0 into (3.1a), we obtain the differential equation governing the amplitude variation of characteristic motions:

$$\dot{r} = -\frac{\varepsilon r}{2\omega}\Big\{h\omega + p\sin 2\theta_0\Big\} \tag{3.2}$$

Finally, the stability condition is obtained:

$$\mathrm{Re}\Big\{-\frac{\varepsilon r}{2\omega}\big[h\omega + p\sin 2\theta_0'\big]\Big\} < 0 \tag{3.3}$$

The process proposed can be applied for more general oscillatory system. For instance, let us consider the system described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \omega t) \tag{3.4}$$

$$\begin{aligned} f(x,\dot{x},\omega t) &= x\Big\{A + \sum_{j=1}^{N}\big[B_j\cos j\omega t + C_j\sin j\omega t\big]\Big\} \\ &\quad + \dot{x}\Big\{D + \sum_{j=1}^{N}\big[E_j\cos j\omega t + F_j\sin j\omega t\big]\Big\} + \ldots \end{aligned} \tag{3.5}$$

where $f(0,0,\omega t) \equiv 0$; $f(x,\dot{x},\omega t)$ is 2π - periodic function with respect to ωt; A, B_j, C_j, D, E_j, F_j are constants; $(\ldots)$ denotes the terms of high power with respect to x, $\dot{x}$.

Using the transformation (0.2), in the first approximation, the averaged system in amplitude phase variables is of the form:

$$\begin{aligned} \dot{r} &= -\frac{\varepsilon r}{2\omega}\Big\{-D\omega + \frac{B-\omega F}{2}\sin 2\theta + \frac{C+\omega E}{2}\cos 2\theta\Big\} + (\ldots) \\ r\dot{\theta} &= \frac{\varepsilon r}{2\omega}\Big\{A - \frac{C+\omega E}{2}\sin 2\theta + \frac{B-\omega F}{2}\cos 2\theta\Big\} + (\ldots) \end{aligned} \tag{3.6}$$

where $B(C,E,F)$ is $B_2(C_2,E_2,F_2)$; $(\ldots)$ denotes the terms of high power with respect to r.

The equilibrium regime corresponds to the solution $r_0 = 0$. To study the stability of this regime, we use the "simplified" system:

$$\dot{r} = -\frac{\varepsilon r}{2\omega}\Big\{-D\omega + \frac{B-\omega F}{2}\sin 2\theta + \frac{C+\omega E}{2}\cos 2\theta\Big\} \tag{3.7a}$$

$$\dot{\theta} = \frac{\varepsilon}{2\omega}\Big\{A - \frac{C+\omega E}{2}\sin 2\theta + \frac{B-\omega F}{2}\cos 2\theta\Big\} \tag{3.7b}$$

Setting the right-hand side of the differential equation (3.7b) equal to zero, we obtain a trigonometrical equation giving us the constant dephase angles of characteristic motions:

$$\frac{C+\omega E}{2}\sin 2\theta - \frac{b-\omega F}{2}\cos 2\theta = A \tag{3.8}$$

Let:

$$\frac{B-\omega F}{2}\sin 2\theta + \frac{C+\omega E}{2}\cos 2\theta = \Delta \tag{3.9}$$

Squaring and adding (3.8) and (3.9), we obtain:

$$\Delta = \pm\Big[\Big(\frac{C+\omega E}{2}\Big)^2 + \Big(\frac{B-\omega F}{2}\Big)^2 - A^2\Big]^{1/2} \tag{3.10}$$

Hence, the equation (3.7a) takes the form:

$$\dot{r} = -\frac{\varepsilon r}{2\omega}\Big\{ - D\omega \pm \Big[\Big(\frac{C+\omega E}{2}\Big)^2 + \Big(\frac{B-\omega F}{2}\Big)^2 - A^2\Big]^{1/2}\Big\} \tag{3.11}$$

Finally, the stability condition is found:

$$\mathrm{Re}\Big\{ - D\omega \pm \Big[\Big(\frac{C+\omega E}{2}\Big)^2 + \Big(\frac{B-\omega F}{2}\Big)^2 - A^2\Big]^{1/2}\Big\} > 0 \tag{3.12}$$

Remark. The method presented can be applied if:

$$\Big(\frac{B-\omega F}{2}\Big)^2 + \Big(\frac{C+\omega E}{2}\Big)^2 \neq 0 \tag{3.13}$$

The existence of constant solutions θ_* (real or immaginary simple or double) of the differential equation (3.7b) is affirmed by the analysis analogous of that, realized in §2 on the variational system in Decartesian variables.

In the case where (3.13) is not satisfied i.e. if $B = \omega F$, $C = -\omega E$, we have:

$$\dot{r} = \frac{\varepsilon D}{2} r\,, \qquad \dot{\theta} = \frac{\varepsilon A}{2\omega} \tag{3.14}$$

The oscillating system considered, in the first approximation, becomes autonomous the dephase $\theta = \dfrac{\varepsilon A}{2\omega} t +$ const and the stability condition is given by the inequality $D < 0$.

CONCLUSION

The variational system (1.9) of the equilibrium regime in amplitude-phase variables has been examined. It has been shown that the values θ_0, determined by the second equation (1.9b) of the mentioned system are just those of the dephase angles of characteristic motions. Consequently, substituting θ by θ_0, the first equation (1.9a) becomes the differential one, governing the amplitude variation of these motions. Based on the results obtained, another process for studying the stability of the equilibrium regime has been proposed.

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VỀ GÓC LỆCH PHA Ở MỘT HỆ PHƯƠNG TRÌNH BIẾN PHÂN CỦA CHẾ ĐỘ CÂN BẰNG

Trong [2], các biến biên độ - pha được sử dụng để khảo sát ổn định của chế độ cân bằng, xem như một chế độ dao động với biên độ $r_0 = 0$ và với góc lệch pha hằng, tùy ý θ_*. Bài báo này xét một thí dụ đơn giản, sử dụng các biến Đề các và phương pháp trung bình ở xấp xỉ thứ nhất cho thấy θ_* là góc lệch pha của chuyển động gọi là đặc trưng. Nhận xét này dẫn đến trình tự mới để khảo sát ổn định của chế độ cân bằng khi sử dụng các biến biên độ - pha; trình tự này không ràng buộc chế độ cân bằng với góc lệch pha nào.