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ON A CLASS OF PROBLEMS ON UNSTEADY FLOW OF VISCOUS - PLASTIC FLUIDS IN PIPE - LINE

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SUMMARY. Combining the quasi - stationary principle with velocity profile properties of respective steady flow and Sliozkin - Targ's approximation we introduce a method to solve a class of problems on unsteady flow of viscous - plastic fluid in pipe - lines. Using this method we solve the problems on unsteady pressure flow in the horizontal cylindrical tube. We also compare the obtained results with those of Tiabin showed in [1].

1. INTRODUCTION

The model of viscous - plastic fluid (Svedov - Bingham's model) is used to solve many technical problems on pipe - line transport of fluids or slurries (fluid - solid mixtures), satisfying or basing on Svedov - Bingham's hypothesis about shear stress [2, 3, 6]

$$\tau = \tau_0 + \eta \frac{dv}{dn} \tag{1.1}$$

in which τ_0 is an untimate shear stress (yield stress).

A lot of problems on one - dimensional steady motion of viscous - plastic fluid in pipe - lines have been solved and showed in literatures. However, it is difficult to find complete solution of unsteady flow even by the approximate methods [1, 2, 3].

The Tiabin's solution of problem on unsteady flow in horizontal cylindrical tube was showed in [1] by using Sliozkin - Targ's method with approximation

$$\varphi(t) = \frac{1}{R - r_0} \int\limits_{r_0}^{R} \frac{\partial v}{\partial t} dr = \frac{1}{R - r_0} \int\limits_{r_0}^{R} \frac{\partial v}{\partial t}\Big|_{r=r_0} dr = \frac{\partial v}{\partial t}\Big|_{r=r_0} \tag{1.2}$$

In this paper we prove the first corollary of first average value theorem for the class of functions being the parabolas with their common symmetric axis. Using this proved corollary we introduce a method to be able to solve a class of problems on unsteady flow of viscous - plastic fluid with higher approximations. This method is the combination between quasi - stationary principle and Sliozkin - Targ's approximation

$$\varphi(t) = \frac{1}{R - r_0} \int\limits_{r_0}^{R} \frac{\partial v}{\partial t} dr \tag{1.3}$$

For illustrating this method we consider the unsteady flow of viscous - plastic fluid in the horizontal cylindrical tube. The obtained results (the laws of development of elastic core and velocity profile) are compared with those of Tiabin.

We close the paper with some comments about the introduced method and the obtained results, including an economized energy generated by the sublayer effect [4] in pipe - line hydrotransport.

2. FIRST COROLLARY OF AVERAGE VALUE THEOREM

Consider the function

$$f(x) = f_2(x) - f_1(x) \tag{2.1}$$

where $y = f_1(x)$ and $y = f_2(x)$ are two parabolas with the common symmetric axis and distance between their vertexs is c. Without loss of generality we may assume that

$$\begin{aligned} &y = f_1(x) = bx^2, \qquad (P_1)\\ \text{and}\quad &y = f_2(x) = ax^2 + c, \qquad (P_2) \end{aligned} \tag{2.2}$$

where $a \neq b \neq 0$ (Fig. 1).

Corollary 1. Assume that two parabolas (P_1) and (P_2) are crossed at the point $M_0(x_0, y_0)$, then there exists $\xi \in [0, x_0]$ such that

$$f(\xi) = \frac{1}{x_0}\int_0^{x_0} f(x)dx \tag{2.3}$$

and we always have

$$f(\xi) = \frac{2}{3}c = \frac{2}{3}[f_2(0) - f_1(0)] = \frac{2}{3}f(0). \tag{2.4}$$

Proof. The first average value theorem immediately give (2.3) (It is clear that $x_0 \neq 0$). From (2.1), (2.2) and (2.3) we have

$$f(\xi) = \frac{1}{x_0}\int_0^{x_0} f(x)dx = \frac{a-b}{3}x_0^2 + c.$$

Since (P_1) and (P_2) are croseed at $M_0(x_0, y_0)$, it follows $(a - b)x_0^2 = -c$. Note that $c = f_2(0) - f_1(0) = f(0)$ we obtain

$$f(\xi) = \frac{2}{3}c = \frac{2}{3}[f_2(0) - f_1(0)] = \frac{2}{3}f(0).$$

3. RECTILINEAR UNSTEADY PRESSURE MOTION OF VISCOUS-PLASTIC FLUID IN HORIZONTAL CYLINDRICAL TUBE

Consider the one-dimensional unsteady pressure flow generated by constant pressure gradient $\dfrac{\Delta p}{\ell}$ in horizontal cylindrical tube. Denote by R a radius of the tube and $Or\theta z$ a system of cylindrical coordinates, in which Oz coincides with the axis of the cylinder (Fig. 2). Throught the forthcoming, unless otherwise specified, we shall adopt the traditional, terminologies and notations.

a) Motion equations and their conditions

From the system of Henki - Iliušin's motion equations, equation of continuity and symmetry of flow we obtain the following motion equation [2]

$$\rho\frac{\partial v}{\partial t} = \eta\Big(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\Big) - \frac{\tau_0}{r} + \frac{\Delta p}{\ell}\,; \quad r_0 \leq r \leq R \tag{3.1}$$

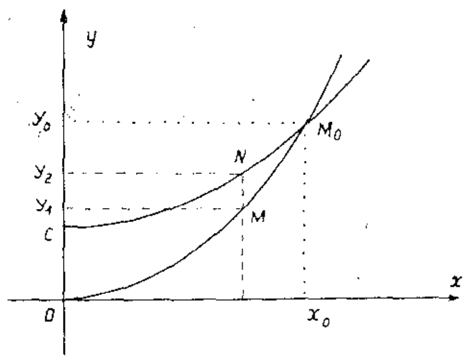

Fig. 1

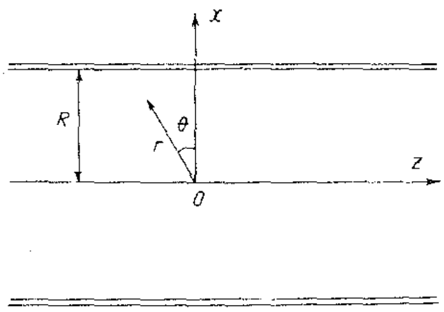

Fig. 2

The boundary conditions of this problem will be:

$$v(R,t)=0 \tag{3.2}$$

$$\left.\frac{\partial v(r,t)}{\partial r}\right|_{r=r_0}=0 \tag{3.3}$$

and initial condition will be

$$v(r,0)=0. \tag{3.4}$$

The velocity of flow in the region of elastic core is determined by

$$v_0(t)=v(r,t)\big|_{r=r_0};\quad r\le r_0. \tag{3.5}$$

The motion equation of elastic core will be in form:

$$\left.\frac{\partial v(r,t)}{\partial t}\right|_{r=r_0}=\frac{\Delta p}{\rho\ell}-\frac{2\tau_0}{\rho r_0}, \tag{3.6}$$

where $r_0=r_0(t)$ is the radius of elastic core, satisfying the condition

$$r_0(0)=R \tag{3.7}$$

and

$$\lim_{t\to\infty} r_0(t)=r_0(\infty)=\frac{2\ell}{\Delta p}\tau_0. \tag{3.8}$$

b) Method and results

Substituting $\partial v/\partial t$ in (3.1) by its average value in viscous - plastic region (the Sliozkin-Targ's approximation (1.3))

$$\varphi(t)=\frac{1}{R-r_0}\int_{r_0}^{R}\frac{\partial v}{\partial t}dr,$$

we obtain the following approximate equation

$$\rho\varphi=\eta\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)-\frac{\tau_0}{r}+\frac{\Delta p}{\ell}. \tag{3.9}$$

The solution of the equation (3.9) satisfying the boundary condition (3.2) and (3.3) is

$$v(r,t)=\frac{\Delta p}{4\ell\eta}(R^2-r^2)-\frac{\rho\varphi}{4\eta}(R^2-r^2)-\frac{\tau_0}{\eta}(R-r)+r_0\left(\frac{\Delta p}{2\ell\eta}r_0-\frac{\rho\varphi}{2\eta}r_0-\frac{\tau_0}{\eta}\right)\ln\frac{r}{R};\quad r_0\le r\le R \tag{3.10}$$

in which $r_0(t)$ and $\varphi = \varphi(t)$ are not determined yet.

For determining $\varphi(t)$ we use the quasi-stationary principle, i.e. we approximate velocity profile (3.10) by that of respective stationary steady flow, that is a semi-parabola with its vertex at $r = r_0$ [1]. Applying corollary 1 with $\frac{\partial v}{\partial t}$ playing the role of $f(x)$ and note that $c = \left.\frac{\partial u}{\partial t}\right|_{r=r_0}$, we obtain

$$\varphi(t) = \frac{2}{3}\left.\frac{\partial v}{\partial t}\right|_{r=r_0} \tag{3.11}$$

Combining (3.11) and the motion equation of elastic core (3.6) gives

$$\varphi(t) = \frac{2}{3}\left(\frac{\Delta p}{\rho \ell} - \frac{2\tau_0}{\rho r_0}\right) \tag{3.12}$$

Substituting (3.12) into (3.10), we obtain the approximate solution of considered problem

$$v(r,t) = \left(\frac{\Delta p}{12\ell\eta} + \frac{\tau_0}{3\eta r_0}\right)(R^2 - r^2) - \frac{\tau_0}{\eta}(R - r) + \frac{r_0}{3}\left(\frac{\Delta p}{2\ell\eta}r_0 - \frac{\tau_0}{\eta}\right)\ln\frac{r}{R}\ ; \quad r_0 \le r \le R \tag{3.13}$$

The velocity of flow in the elastic core is determined by (3.5) and will be

$$v_0(t) = v(r_0,t) = \left(\frac{\Delta p}{12\ell\eta} + \frac{\tau_0}{3\eta r_0}\right)(R^2 - r_0^2) - \frac{\tau_0}{\eta}(R - r_0) + \frac{r_0}{3}\left(\frac{\Delta p}{2\ell\eta}r_0 - \frac{\tau_0}{\eta}\right)\ln\frac{r_0}{r}\ , \quad r \le r_0 \tag{3.14}$$

From (3.6) and (3.13) we get the differential equation for determining the radius of elastic core:

$$\left[-\frac{\tau_0}{3\eta r_0^2}(R^2 - r_0^2) + \frac{\Delta p}{3\ell\eta}\left(\ln\frac{r_0}{R}\right)r_0 - \frac{\tau_0}{3\eta}\left(\ln\frac{r_0}{R}\right)\right]\frac{dr_0}{dt} = \frac{\Delta p}{\rho\ell} - \frac{2\tau_0}{\rho r_0}$$

After separating the variables, yields:

$$\left(r_0\ln\frac{r_0}{R} + \frac{\tau_0\ell}{\Delta p}\ln\frac{r_0}{R} + \frac{\tau_0\ell}{\Delta p} + \frac{R^2}{2r_0} + \frac{2\tau_0^2\left(\frac{\ell}{\Delta p}\right)^2\ln\frac{r_0}{R} + 2\tau_0^2\left(\frac{\ell}{\Delta p}\right)^2 - \frac{R}{2}}{r_0 - 2\tau_0\frac{\ell}{\Delta p}}\right)dr_0 = \frac{3\eta}{\rho}dt \tag{3.15}$$

Integrate (3.15) and note that

$$r_0^2 > r_0^2(\infty) = \left(\frac{2\tau_0\ell}{\Delta P}\right)^2,$$

we obtain the solution of equation (3.6) (or (3.15)) satisfying (3.7) (and (3.8)) as follows:

$$\begin{aligned}&\frac{1}{4}(R^2 - r_0^2) + \frac{1}{2}(R^2 + r_0 r_0(\infty) + r_0^2)\ln\frac{r_0}{R} + \frac{r_0^2(\infty)}{4}[(\ln R)^2 - (\ln r_0)^2] +\\ &+\frac{1}{2}[R^2 - r_0^2(\infty)]\ln\frac{R - r_0(\infty)}{r_0 - r_0(\infty)} + \frac{r_0^2(\infty)}{2}\left(\ln\frac{r_0}{R}\right)\ln[r_0 - r_0(\infty)] +\\ &+\frac{r_0^2(\infty)}{2}\sum_{n=1}^{\infty}\frac{r_0^n(\infty)}{n^2}\left(\frac{1}{R^n} - \frac{1}{r_0^n}\right) = \frac{3\eta}{\rho}t\end{aligned} \tag{3.16}$$

By determining $r_0(t)$ from (3.16) and substituting it into (3.13) and (3.14) we obtain $v = v(r,t)$ and $v_0 = v_0(t)$. The dischage of flow is determined as follows:

$$\begin{aligned}Q(t) &= \pi r_0^2 v_0(t) + 2\pi\int_{r_0}^{R} v(r,t)r\,dr \qquad (3.17)\\ &= \frac{\pi}{2}\left(\frac{\Delta p}{12\ell\eta} + \frac{\tau_0}{3\eta r_0}\right)(R^4 - r_0^4) - \frac{\pi\tau_0}{3\eta}(R^3 - r_0^3) - \frac{r_0}{6}\left(\frac{\Delta p}{2\ell\eta}r_0 - \frac{\tau_0}{\eta}\right)(R^2 - r_0^2).\end{aligned}$$

in which $r_0 = r_0(t)$ is determined from (3.16).

c) Discussion

With the approximation (1.2), Tiabin determined

$$\varphi(t) = \frac{\Delta p}{\rho \ell} - \frac{2\tau_0}{\rho r_0} \tag{3.18}$$

$$v_T(r,t) = \frac{\tau_0}{2\eta r_0}(R^2 - r^2) - \frac{\tau_0}{\eta}(R - r), \quad r_0 \le r \le R, \tag{3.19}$$

$$v_{0T}(t) = \frac{\tau_0}{2\eta r_0}(R^2 - r_0^2) - \frac{\tau_0}{\eta}(R - r_0), \quad r \le r_0 \tag{3.20}$$

and the law of developing elastic core is determined by formula:

$$\bar{t} = \frac{1}{6}\ln \bar{r}_0 + \frac{\bar{r}_0^2(\infty) + \bar{r}_0(\infty) - 2}{12}\ln\frac{\bar{r}_0 - \bar{r}_0(\infty)}{1 - \bar{r}_0(\infty)} + \frac{\bar{r}_0(\infty)(\bar{r}_0 - 1)}{12} \tag{3.21}$$

in which

$$\bar{t} = \frac{\eta}{\rho R^2}t, \quad \bar{r}_0(\infty) = \frac{r_0(\infty)}{R} = \frac{2\ell}{\Delta p}\cdot\frac{\tau_0}{R}, \quad \bar{r}_0 = \frac{r_0}{R}.$$

For convenience of comparison, the laws of developing the elastic cores and the velocity profiles in both of the cases are expressed in Fig. 3 and Fig. 4

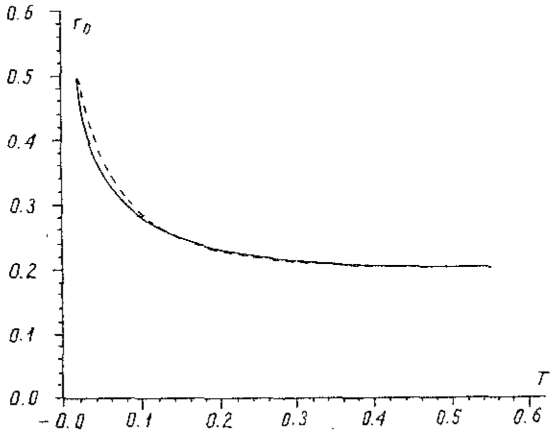

Fig. 3

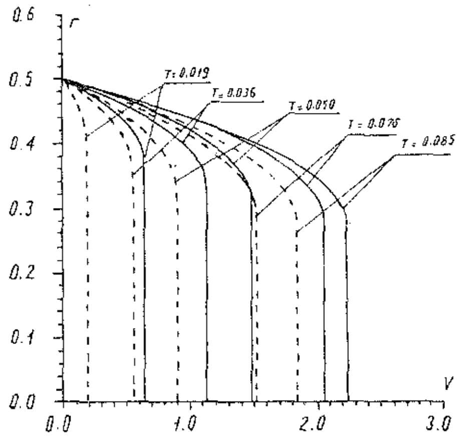

Fig. 4

Where - - - - - - - (dash lines) for Tiabin's results ((3.21); (3.19) & (3.20)),

———— (solid lines) for formulae (3.16); (3.13) & (3.14) and $T = \frac{3\eta}{\rho}t$.

From Fig. 3 it follows that, the elastic core determined by the formula (3.16) is changed faster at first and later on it's changed more slow than Tiabin's one. However, Fig. 4 shows that the velocity determined by (3.13) or (3.14) are always bigger than the respective velocity determined by (3.19) or (3.20).

When $t \to \infty$, the elastic cores and velocity profiles in both of the cases are tended to the those of respective steady flow, i.e. the obtained results (3.13), (3.14) and (3.17) will be identical to the well - known results of respective steady flow with $\tau_0 = \frac{\Delta p}{2\ell} r_0(\infty)$ [1, 6].

4. CONCLUDING REMARKS

1. The problems on unsteady flows of viscous - plastic fluid belong to the class of problems with mobile boundaries. As far as we know, there were some solutions of problems on unsteady flow in

the pipe - line with certain concrete boundary conditions or using the supplementary assumptions for the approximate function $\varphi(t)$ and the law of developing of elastic core [1, 2].

Note that, we always can write one condition at the shaded boundary expressing the motion equation of elastic core in form

$$\left.\frac{\partial v}{\partial t}\right|_{r=r_0} = f(r_0)$$

On the other hand, the velocity profiles of respective steady flows have been or can be determined. Therefore, combining the quasi - stationary principle and Sliozkin-Targ's method we can solve a class of problems on unsteady flows of viscous-plastic fluid in pipe-line with higher approximations.

2. The problem on unsteady flow with sublayer effect has two mobile boundary, hence, for solving it completely we need more a one equation, describing the law of development of sublayer $\delta = \delta(t)$ (or $R_1 = R_1(t)$ - see its respective steady flow [4]). According to Smoldurev and other, the viscosity coefficient in sublayer η_0 is substantially less than the coefficient of structural viscosity η of transport slurry (fluid - solid mixture), so the discharge of flow is increased considerably in the flow with the viscous sublayer effect near the wall. For the steady flow, this increased discharge showed in [4] is

$$\Delta Q = \frac{\pi \Delta p}{8\ell}(R^4 - R_1^4)\left(\frac{1}{\eta_0} - \frac{1}{\eta}\right) + \frac{\pi \Delta p}{6\ell\eta} r_0 (R^3 - R_1^3)$$

consequently a economized energy is the energy using to transport this mass (per unid of time).

3. For solving the problem on one - dimensional unsteady pressure flow of viscous - plastic fluid between two infinite planes we need not to use the quasi-stationary principle [5].

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VỀ MỘT LỚP CÁC BÀI TOÁN DÒNG CHẢY KHÔNG DỪNG CỦA CHẤT LỎNG NHỚT-DẺO TRONG ỐNG DẪN

Kết hợp nguyên lý tựa dừng với tính chất của profile vận tốc trong chuyển động dừng tương ứng và xấp xỉ Sliozkin-Targ chúng tôi đưa ra phương pháp giải lớp các bài toán chuyển động không dừng của chất lỏng nhớt-dẻo trong ống dẫn. Sử dụng phương pháp này chúng tôi giải bài toán chuyển động có áp trong ống hình trụ nằm ngang, các kết quả nhận được đã được so sánh với kết quả của Tiabin trình bày trong [1].