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PRINCIPLE OF COMPATIBILITY APPLIED TO DETERMINATION OF REACTION FORCES OF KINEMATICS PAIRS

DO SANH
Hanoi University of Technology

ABSTRACT. The aim of investigation is to present the principle of compatibility and to apply it for determining the reaction forces of kinematics pairs in mechanism. For this purpose the system is released from the given constrains. This resulted to increase the number of coordinates of the system. In order that the freed system realizes the motion of the given system, the coordinates of the freed system must satisfy some relations called the constraint equations. The reaction forces of the formed constraints are just the reaction forces, which are of interest to us.

1. INTRODUCTION

As known, in order to write the equations of motion of a constrained mechanical system, it is possible to apply the principle of compatibility [1, 2]. The reaction forces of constrains acting on the system are determined by a closing set of algebraic equations. By mean of this, the motion of the system is described by differential equations, but that not by algebraic - differential equations.

However there is difficulty of the calculation of Christofell symbols (of three indices) of first kind [1, 2].

In this paper it is presented the method which allows to surmount this difficulty. By means of mentioned advantage this method is applied fruitfully to determination of reaction forces of kinematics pairs in mechanisms.

2. THE PRINCIPLE OF COMPATIBILITY AND EQUATIONS OF MOTION OF A CONSTRAINED MECHANICAL SYSTEM

Let us consider a mechanical system of N particles, the coordinates and masses of which are denoted by x_k, m_k respectively $(k = \overline{1, 3N})$, where: $m_{3k} = m_{3k-1} = m_{3k-2}$.

Let further the forces acting on the particle M_k be x_k and the constraint equations be of the form

$$\sum b_{\alpha k}\ddot{x}_k + b_\alpha = 0; \quad \alpha = \overline{1, s}$$

which are written in the form:

$$\mathbf{b}\ddot{\mathbf{x}} + \mathbf{b}_0 = 0 \tag{2.1}$$

where $\ddot{\mathbf{x}}$ is the $3N \times 1$ matrix of accelerations; $\mathbf{b}$ - the $s \times 3N$ matrix; $\mathbf{b}_0$ - the $s \times 1$ matrix. The elements of two last matrices are the functions of coordinates and velocities of particles.

The constraints (2.1) impose the restrictions on the positions, velocities and accelerations of particles of the considered system.

As known, the equations of motion of the free system are of the form:

$$\mathbf{M\ddot{x}} = \mathbf{X} \tag{2.2}$$

where $\mathbf{M}$ is the $3N \times 3N$ diagonal matrix of the elements m_k $(k = \overline{1, 3N})$, $\mathbf{X}(t, \mathbf{x}, \dot{\mathbf{x}})$ - the $3N \times 1$ matrix of applied forces.

Equations (2.2) do not satisfy the constraint equations (2.1). By following the principle of compatibility [1, 2] the equations of motion of the system with constraints (2.1) must be written in the form:

$$\mathbf{M\ddot{X}} = \mathbf{X}(t, \mathbf{x}, \dot{\mathbf{x}}) + \mathbf{X}^* \tag{2.3}$$

where $\mathbf{X}^*$ is the $3N \times 1$ matrix of unknown reaction forces generated by the constraints (2.1).

Equations (2.3) will describe the motion of the considered system if and only if the matrix $\mathbf{X}^*$ satisfied the following equation:

$$\mathbf{BX}^* + \mathbf{B}_0 = 0 \tag{2.4}$$

where $\mathbf{B}$ is the $s \times 3N$ matrix of the form:

$$\mathbf{B} = \mathbf{bM}^{-1}; \qquad \mathbf{B}_0 = \mathbf{b}_0 + \mathbf{BX}; \qquad \mathbf{M}^{-1}\mathbf{M} = \mathbf{E}$$

$\mathbf{E}$ is the unit matrix.

This means that the action of the constraints imposed on the system can be replaced by the reaction forces acting on particles of the system.

In such a manner we obtain a set of s equations of $3N$ unknowns X_k^* $(k = \overline{1, 3N};\ s < 3N)$. For supplementing to the equations (2.4) we concern to the condition of ideality of constraints (2.1), that is:

$$\mathbf{X}^{*T}\delta\ddot{\mathbf{x}} = 0 \tag{2.5}$$

where $\delta\ddot{\mathbf{x}}$ is the matrix of virtual displacements, $\mathbf{X}^{*T}$ - the transpose of the matrix $\mathbf{X}^*$

It is easy to prove that the variational equation (2.5) is equivalent to $(3N - s)$ equations:

$$\mathbf{dX}^* = 0 \tag{2.6}$$

where: $\mathbf{d}(\mathbf{x}, \dot{\mathbf{x}})$ is the $(3N - s) \times 3N$ matrix, the elements of which are the coefficients in the term of the expressions of accelerations represented through independent accelerations.

In the results we get $3N$ algebraic equations (2.4), (2.6) of $3N$ unknowns X_k^* $(k = \overline{1, 3N})$. Solving these equations we obtain:

$$X_k^*(t, x_k, \dot{x}_k) \tag{2.7}$$

By substituting (2.7) into (2.3) we have:

$$\mathbf{M\ddot{x}} = \mathbf{X}(t, \mathbf{x}, \dot{\mathbf{x}}) + \mathbf{X}^*(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{2.8}$$

which describes the motion of the system with constraints (2.1).

Integrating the equations (2.8) with the given initial conditions we get: $x_k = x_k(t)$ and substituting into (2.7) we obtain:

$$X_k^* = X_k^*(t, x_k(t), \dot{x}_k(t)) \equiv X_k^*(t)$$

which are just the reaction forces of the constraints (2.1) acting on the considered system.

Let us consider now a mechanical system with holonomic coordinates q_i $(i = \overline{1, n})$. The constraint equations in this case take the form:

$$\mathbf{b\ddot{q}} + \mathbf{b}_0 = 0 \tag{2.9}$$

where $\mathbf{b}$ is an $s \times n$ matrix, but $\mathbf{b}_0$ - a $s \times 1$ matrix; The elements of these matrices are functions of coordinates and velocities, i.e. $\mathbf{b} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})$; $\mathbf{b}_0 = \mathbf{b}_0(\mathbf{q}, \dot{\mathbf{q}})$.

Suppose that the considered system has the matrix of inertia $\mathbf{A}$, which is an $n \times n$ positive define symmetric matrix. The elements of this matrix depend only on generalized coordinates, i.e. $\mathbf{A} = \mathbf{A}(\mathbf{q})$, where $\mathbf{q}$ is an $n \times 1$ matrix of generalized coordinates, i.e.

$$\mathbf{q}^T = \|q_1 q_2 \dots q_n\|. \tag{2.10}$$

The expression of the kinetic energy of the system is of the form:

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A} \dot{\mathbf{q}} \tag{2.11}$$

where $\dot{\mathbf{q}}$ is the $n \times 1$ matrix of generalized velocities, but $\dot{\mathbf{q}}^T$ is the transpose of the matrix $\dot{\mathbf{q}}$, i.e.:

$$\dot{\mathbf{q}}^T = \|\dot{q}_1 \dot{q}_2 \dots \dot{q}_n\| \tag{2.12}$$

Denote the generalized forces of applied forces by $Q_i(t, q_j, \dot{q}_j)$ and $\mathbf{Q}$ be the column matrix of these forces, i.e.

$$\mathbf{Q}^T = \|Q_1 Q_2 \dots Q_n\| \tag{2.13}$$

As known, the motion of the freed system, i.e. of the system without constraints (2.9), can be described by equations [3].

$$\mathbf{A}\ddot{\mathbf{q}} - \mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{2.14}$$

where $\mathbf{G}$ is the $n \times 1$ matrix, which has the form:

$$\mathbf{G} = \dot{\mathbf{q}}^* \mathbf{G}_0 \dot{\mathbf{q}}_0 \tag{2.15}$$

but $\dot{\mathbf{q}}^*$ is the $n \times n$ diagonal matrix, the principle diagonal of which are of the form (2.12), i.e.

$$\dot{\mathbf{q}}^* = \left\| \begin{matrix} \dot{q}_1 & \dot{q}_2 & \dots & \dot{q}_n & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dot{q}_1 & \dot{q}_2 & \dots & \dot{q}_n & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \dot{q}_1 & \dot{q}_2 & \dots & \dot{q}_n \end{matrix} \right\| \tag{2.16}$$

$\dot{\mathbf{q}}_0^T$ is the $1 \times n^2$ matrix consisting of n lines of (2.12) placed in series of a line, i.e.

$$\dot{\mathbf{q}}_0^T = \|\dot{q}_1 \dot{q}_2 \dots \dot{q}_n \dot{q}_1 \dot{q}_2 \dots \dot{q}_n \dots \dot{q}_1 \dot{q}_2 \dots \dot{q}_n\| \tag{2.17}$$

$\mathbf{G}_0$ is the $n^2 \times n^2$ matrix, which is a cage diagonal matrix, consisting of square cages of $n \times n$ dimensions, i.e.

$$\mathbf{G}_0 = \left\| \begin{matrix} \mathbf{G}_{01} & & & & \\ & \mathbf{G}_{02} & & & \\ & & \mathbf{G}_{03} & & \\ & & & \ddots & \\ & & & & \mathbf{G}_{0N} \end{matrix} \right\| \tag{2.18}$$

The elements of the matrix $\mathbf{G}_{0k}$ $(k = \overline{1, n})$ are calculated by means of the elements of the matrix of inertia.

Evidently, the equations (2.14) do not satisfy the constraints (2.9). In accordance with the principle of compatibility the equations of motion of the system with constraints (2.9) must be written in the form [2]

$$\mathbf{A}\ddot{\mathbf{q}} - \mathbf{G} = \mathbf{Q} + \mathbf{Q}^* \tag{2.19}$$

where $\mathbf{Q}^*$ is the matrix of reaction forces of constraints (2.9) acting on the considered system.

By the principle of compatibility the Q_i^* $(i = \overline{1, n})$ are determined by the equations:

$$\mathbf{B}\mathbf{Q}^* + \mathbf{B}_0 = 0 \tag{2.20}$$

where $\mathbf{B}$ is the $s \times n$ matrix of the form

$$\mathbf{B} = \mathbf{b}\mathbf{A}^{-1} \tag{2.21}$$

$\mathbf{A}^{-1}$ is the inverse of the matrix of inertia $\mathbf{A}$ and $\mathbf{B}_0$ is the $s \times 1$ matrix of the form:

$$\mathbf{B}_0 = \mathbf{B}(\mathbf{Q} + \mathbf{G}) + \mathbf{b}_0 \tag{2.22}$$

We obtain s algebraic equations (2.22) containing n unknowns Q_i^* $(i = \overline{1, n}, s < n)$

It is necessary to supplement $(n \times s)$ equations to the equations (2.22). To do this let us concern the condition of ideality of the constraints (2.9) which give us:

$$\mathbf{D}\mathbf{Q}^* = 0 \tag{2.23}$$

where $\mathbf{D}$ is the $(n - s) \times n$ matrix, the elements of which are the coefficients in the expressions of the generalized accelerations written in the form of independent generalized accelerations by solving the equations (2.9).

By means of (2.20) and (2.23) we get a closing set of equations, which allow to find the reaction forces $Q_i^*(t, q_j, \dot{q}_j)$ and substituting them into (2.9) we have:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{G} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) + \mathbf{Q}^*(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{2.24}$$

Equations (2.24) describe the motion of the system with the constraints (2.9)

3. APPLICATION OF THE PRINCIPLE OF COMPATIBILITY FOR DETERMINING THE REACTION FORCES OF KINEMATICS PAIRS

The determination of reaction forces at the kinematics pairs of mechanism plays an important role in dynamics of machines. Usually, this problem is realized after the step of determining the motion of the system. By applying the principle of compatibility the reaction forces of constraints can be determined independently with respect to the construction of equations of motion. The discovery of the class of reaction forces will help us deep to know the action of the given constraints. This is useful in many technical application. such as, in the control theory, in the investigation of stability of motion of a multibody system... Of course, the reaction forces depend on the occurred motion of the considered system, i.e. on the solution of the differential equation (2.24) with the given initial conditions. In other words, the reaction forces acting on the given system are the functions of the time, i.e. $Q_i^* = Q_i^*(t, q_i(t), \dot{q}_i(t)) = Q_i^*(t)$.

Let us apply now the mentioned method for computing the reaction forces of kinematics pairs. For this purpose let us release (partially or wholly) the given constraints from the considered system and replace the action of released constraints by the reaction forces. In other words the considered system is then assumed to be without constraints and generating reactions, which are of interest to us. The number of degrees of freedom of the freed system, will be increased. Corresponding to this it is necessary to introduce some new coordinates being equal the increased number of degrees of freedom. In order that the freed system is equivalent to the given system, the generalized coordinates must satisfy some relation called the constraint equations. In other words, the given constrained mechanical system is treated as the released system realizing the constraints just established. For illustration let us consider the following example.

EXAMPLE. A pendulum is suspended from a slider as shown in Fig. 1. The slider has the mass of m_0 sliding on horizontal ground without friction the radius of inertia of the pendulum about its center of mass C ($OC = a$) is of ρ i.e. $I_c = m\rho^2$, where I_c is the moment of inertia of the pendulum about C. Assume that there is the force $\vec{F}$ acting on the slider in horizontal direction. Determine the reaction force generating by the ground acting on the slider

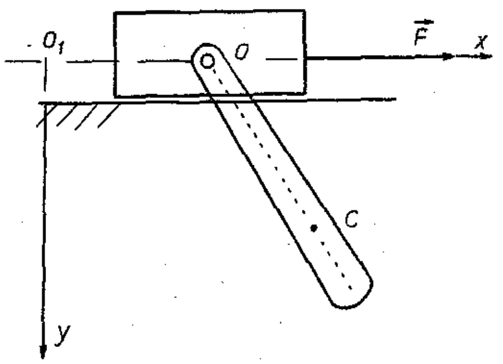

Fig. 1

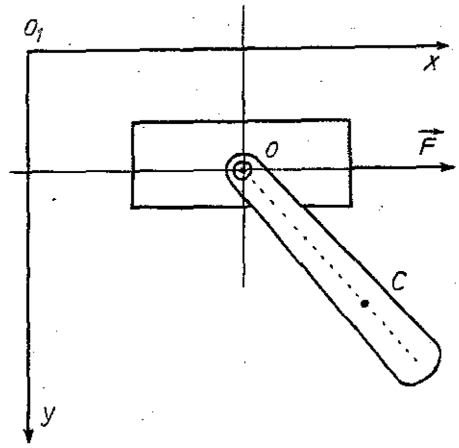

Fig. 2

To determine the reaction force between the slider and the ground let us separate the slider from the ground and choose the generalized coordinates to be x, y and φ (Fig. 2).

The matrix of inertia of the system released from the ground will be:

$$\mathbf{A} = \begin{Vmatrix} M & 0 & mac_\varphi \\ 0 & M & -mas_\varphi \\ mac_\varphi & -mas_\varphi & m\rho^2 \end{Vmatrix} \tag{3.1}$$

The inverse of the matrix A is:

$$\mathbf{A}^{-1} = \begin{Vmatrix} \dfrac{\rho^2 - Ka^2 s_\varphi^2}{M(\rho^2 - Ka^2)} & \dfrac{-Ka^2 s_\varphi c_\varphi}{M(\rho^2 - Ka^2)} & \dfrac{-ac_\varphi}{M(\rho^2 - Ka^2)} \\ \dfrac{-Ka^2 s_\varphi c_\varphi}{M(\rho^2 - Ka^2)} & \dfrac{\rho^2 - Ka^2 c_\varphi^2}{M(\rho^2 - Ka^2)} & \dfrac{Kas_\varphi}{M(\rho^2 - Ka^2)} \\ \dfrac{-ac_\varphi}{M(\rho^2 - ka^2)} & \dfrac{kas_\varphi}{M(\rho^2 - Ka^2)} & \dfrac{1}{M(\rho^2 - Ka^2)} \end{Vmatrix} \tag{3.2}$$

where the following symbols are used: $c_\varphi = \cos\varphi$; $s_\varphi = \sin\varphi$ and $M = m_0 + m$; $K = m/M$. In general, the force $\vec{F}$ acting on the slider is a function of coordinates x and φ, $F = F(t, x, \varphi)$.

The generalized forces of applied forces are of the form:

$$\mathbf{Q}^T = \|F \quad Mg \quad -mgas_\varphi\|. \tag{3.3}$$

For purpose of determining the reaction force generating by the ground acting on the slider the freed system must realize the constraint:

$$y = 0. \tag{3.4}$$

To write the equations (2.20) and (2.23) let us calculate the following matrices

The 9×9 matrix $\mathbf{G}_0$ is of the form:

$$\mathbf{G}_0 = \begin{Vmatrix} \mathbf{G}_{01} & 0 & 0 \\ 0 & \mathbf{G}_{02} & 0 \\ 0 & 0 & \mathbf{G}_{03} \end{Vmatrix} \tag{3.5}$$

where the 3×3 square matrices are written as follows:

$$\mathbf{G}_{01} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & mas_\varphi \end{Vmatrix}, \quad \mathbf{G}_{02} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & mac_\varphi \end{Vmatrix}, \quad \mathbf{G}_{03} = \begin{Vmatrix} 0 & 0 & -\frac{1}{2}mas_\varphi \\ 0 & 0 & -\frac{1}{2}mac_\varphi \\ -\frac{1}{2}mas_\varphi & -\frac{1}{2}mac_\varphi & 0 \end{Vmatrix}. \tag{3.6}$$

The matrices (2.16) and (2.17) are of the form now:

$$\dot{\mathbf{q}}_* = \begin{Vmatrix} \dot{x} & \dot{y} & \dot{\varphi} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{x} & \dot{y} & \dot{\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dot{x} & \dot{y} & \dot{\varphi} \end{Vmatrix}, \quad \dot{\mathbf{q}}_0^T = \| \, \dot{x} \; \dot{y} \; \dot{\varphi} \; \dot{x} \; \dot{y} \; \dot{\varphi} \; \dot{x} \; \dot{y} \; \dot{\varphi} \, \|. \tag{3.7}$$

By following the formulae (2.15) we obtain the 1×3 matrix $\mathbf{G}$:

$$\mathbf{G}^T = \begin{Vmatrix} mas_\varphi \dot{\varphi}^2 & mac_\varphi \dot{\varphi}^2 & 0 \end{Vmatrix}. \tag{3.8}$$

According to the constraint equation (3.4) we write:

$$\mathbf{b} = \|0 \; 1 \; 0\| \; ; \quad \mathbf{b}_0 = 0. \tag{3.9}$$

By the formulae (2.21) and (2.22) we obtain the 1×3 matrix $\mathbf{B}$:

$$\mathbf{B} = \mathbf{b}\mathbf{A}^{-1} = \begin{Vmatrix} \frac{-ka^2 s_\varphi c_\varphi}{M(\rho^2 - Ka^2)} & \frac{\rho^2 - Ka^2 c_\varphi^2}{M(\rho^2 - Ka^2)} & \frac{as_\varphi}{M(\rho^2 - Ka^2)} \end{Vmatrix} \tag{3.10}$$

and the 1×1 matrix $\mathbf{B}_0$:

$$\mathbf{B}_0 = \mathbf{B}(\mathbf{Q} + \mathbf{G}) + \mathbf{b}_0 = \begin{Vmatrix} \frac{g(\rho^2 - Ka^2 c_\varphi^2)}{\rho^2 - Ka^2} - mga^2 s_\varphi^2 + mac_\varphi(\rho^2 - Ka^2)\dot{\varphi}^2 + \frac{1}{2}\frac{FKa^2 s_\varphi^2}{2M(\rho^2 - Ka^2)} \end{Vmatrix}$$

The 2×3 matrix D in accordance with the constraint (3.4) is written as follows:

$$\mathbf{D} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{3.11}$$

Equations of determining the reaction forces (2.20) and (2.23) will be now:

$$R_x = 0; \quad R_\varphi = 0; \tag{3.12}$$

$$\frac{\rho^2 - Ka^2 c_\varphi^2}{M(\rho^2 - Ka^2)} R_y - mga^2 s_\varphi^2 + \frac{g(\rho^2 - Ka^2 c_\varphi^2)}{\rho^2 - Ka^2} + mac_\varphi(\rho^2 - Ka^2)\dot{\varphi}^2 + \frac{1}{2}\frac{FKa^2 s_\varphi^2}{M(\rho^2 - Ka^2 0} = 0.$$

The reaction forces generated by the ground acting on the slider is equal then:

$$R_y = g\Big(\frac{ma^2 s_\varphi^2}{\rho^2 - Ka^2 c_\varphi^2} - m\Big) + \frac{FKa^2 s_\varphi^2}{2(\rho^2 - Ka^2)} - \frac{mac_\varphi}{\rho^2 - Ka^2 c_\varphi^2}(\rho^2 - Ka^2)\dot{\varphi}^2. \tag{3.13}$$

By means of this expression it is possible to know deep the structure of the reaction force

CONCLUSION

The above presented method is an useful tool for calculating the reaction forces of kinematics pairs. Especially, the finding of the structure of reaction forces allows to investigate qualitatively many technical problems.

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ỨNG DỤNG NGUYÊN LÝ PHÙ HỢP ĐỂ XÁC ĐỊNH PHẢN LỰC KHỚP ĐỘNG

Mục đích của việc khảo sát là trình bày nguyên lý phù hợp và ứng dụng nó để xác định các phản lực khớp động của cơ cấu. Với mục đích đó, hệ được giải phóng khỏi các liên kết đã cho. Điều này sẽ làm tăng tọa độ của hệ. Để hệ tự do có chuyển động giống như hệ đã cho, các tọa độ của hệ tự do phải thỏa mãn một số ràng buộc gọi là các phương trình liên kết. Các phản lực liên kết được tạo thành từ các liên kết chính là các phản lực mà chúng ta quan tâm.