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PROBABILISTIC CRITERION FOR GAUSIAN EQUIVALENT LINEARIZATION

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ABSTRACT. Within the scope of Gaussian equivalent linearization, a new probabilistic criterion for determining the coefficients of the linearized equivalent equation is proposed to treat stationary response of non-linear systems under zero mean Gaussian random excitation. Application to the Duffing oscillator subjected to white noise is presented that shows significant improvement over corresponding accuracy of the classical Gaussian equivalent linearization for both weak and strong non-linearities.

1. INTRODUCTION

In recent years, there has been an extensive investigation into the response of non-linear stochastic systems due to fact that many excitations of engineering interest are basically random in nature. Since all real engineering systems are, more or less, non-linear and for those systems the exact solutions are known only for a number of special cases, it is necessary to develop approximate techniques to determine the response statistics of non-linear systems under random excitation. One of the known approximate techniques is the Gaussian equivalent linearization which was first proposed by Caughey [3] and has been developed by many authors, see e.g. [2, 4, 7, 8, 10, 11, 13]. The well known classical version of the Gaussian equivalent linearization consists of replacing a given non-linear equation by a linear one for which the coefficients of linearization are found from a mean square error criterion and then evaluated by an assumption about the Gaussianity of the original non-linear equation response. It has been shown that the Gaussian equivalent linearization is presently the simplest tool widely used for analysis of non-linear stochastic problems, however, the major limitation of this method is seemingly that its accuracy decreases as the non-linearity increases and it can lead to unacceptable errors in the second moments [1, 6].

To improve the accuracy of this excellent technique a new criterion for determining the coefficients of linearization is proposed in the paper. The criterion is based on the probabilistic approach to the approximate solutions of stochastic equations. The proposed method is then applied to an oscillator with non-linear stiffness under a zero mean Gaussian white noise. It is obtained that the method yields a significant improvement over the corresponding accuracy of the classical Gaussian equivalent linearization for both weak and strong non-linearities.

2. GAUSSIAN EQUIVALENT LINEARIZATION (GEL)

To describe the basic idea of GEL we consider the non-linear stochastic equation

$$\ddot{z} + 2h\dot{z} + \omega_0^2 z + \varepsilon g(z, \dot{z}) = f(t) \tag{2.1}$$

where a dot denotes time differentiation, h, ω_0, ε are positive constants, g is a non-linear function which can be expanded into a polynomial series form. The excitation $f(t)$ is a zero mean Gaussian stationary process with the correlation function and spectral density given, respectively, by

$$R_f(\tau) = \langle f(t)f(t+\tau)\rangle, \quad S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_f(\tau)e^{i\omega\tau}d\tau, \tag{2.2}$$

where $\langle\ \rangle$ denotes the expectation. For the sake of simplicity, we restrict to the case of stationary response of equation (2.1), if it exists.

Denote

$$e(z) = \ddot{z} + 2h\dot{z} + \omega_0^2 z + \varepsilon g(z,\dot{z}) - f(t). \tag{2.3}$$

Equation (2.1) yields

$$e(z) = 0. \tag{2.4}$$

Following the GEL method, we introduce new linear terms in the expression of e

$$e(z) = \ddot{z} + (2h+\mu)\dot{z} + (\omega_0^2+\lambda)z + \varepsilon g(z,\dot{z}) - \mu\dot{z} - \lambda z - f(t) \tag{2.5}$$

Let $x(t)$ be a stationary solution of the linearized equation

$$\ddot{x} + (2h+\mu)\dot{x} + (\omega_0^2+\lambda)x - f(t) = 0. \tag{2.6}$$

Using (2.6) one gets from (2.5)

$$e(x) = \varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x. \tag{2.7}$$

If $x(t)$ is such a solution for which $e(x) = 0$, it is evident that $x(t)$ is also an exact solution for the original non-linear equation (2.1). However, it is seen that it is not possible and $e(x)$ is an equation error which is different from zero. Thus, the problem reduces to the linearized equation (2.6) where the coefficients of linearization μ, λ are to be found from an optimal criterion. There are some criteria for determining the coefficients μ, λ (see e.g. [12]). The most extensively used criterion is the mean square error criterion which requires that the mean square of equation error be minimum

$$\langle e^2(x)\rangle = \langle(\varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x)^2\rangle \to \min_{\mu,\lambda}. \tag{2.8}$$

Thus, from

$$\frac{\partial}{\partial\mu}\langle e^2(x)\rangle = 0, \qquad \frac{\partial}{\partial\lambda}\langle e^2(x)\rangle = 0$$

it follows

$$\mu = \varepsilon\,\frac{\langle g\dot{x}\rangle}{\langle\dot{x}^2\rangle}, \qquad \lambda = \varepsilon\,\frac{\langle gx\rangle}{\langle x^2\rangle}. \tag{2.9}$$

Since the process $x(t)$ is a solution of the linearized equation (2.6) under Gaussian process excitation, one gets that $x(t)$ and $\dot{x}(t)$ are Gaussian or normal processes. Hence, all higher moments $\langle gx\rangle$, $\langle g\dot{x}\rangle$ can be expressed in terms of second moments $\langle x^2\rangle$, $\langle\dot{x}^2\rangle$ and the relation (2.9) results in two algebraic equations for 4 unknowns μ, λ, $\langle x^2\rangle$ and $\langle\dot{x}^2\rangle$. To close the system (2.9) two other equations for second moments $\langle x^2\rangle$, $\langle\dot{x}^2\rangle$ can be derived from (2.6)

$$\begin{aligned}\langle\dot{x}^2\rangle &= \int_{-\infty}^{\infty}\frac{\omega^2 S_f(\omega)d\omega}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2}\\ \langle x^2\rangle &= \int_{-\infty}^{\infty}\frac{S_f(\omega)d\omega}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2}\end{aligned} \tag{2.10}$$

So, the classical version of GEL as described above supposes that the minimization of the equation error in mean square sense may give a minimization of the solution error. However, it has been shown by many authors that in the case of major non-linearity the solution error may be unacceptable for the second moments. One of the possible reasons is expected to obtain by substituting (2.9) into (2.8)

$$\langle e^2(x)\rangle_{\min} = \varepsilon^2\Big(\langle g^2\rangle - \frac{\langle g\dot{x}\rangle^2}{\langle \dot{x}^2\rangle}\,\frac{\langle gx\rangle^2}{\langle x^2\rangle}\Big) \tag{2.11}$$

It is seen from (2.11) that when ε becomes large $\langle e^2\rangle_{\min}$ may become large too.

3. PROBABILISTIC APPROACH TO GEL

An alternative approach to GEL is the probabilistic one which requires that

$$P\Big\{-\rho \le e(x) \le \rho\Big\} \to \max_{\mu,\lambda} \tag{3.1}$$

where $P\{E\}$ is the probability of an event E, and ρ is a small positive constant. denote $p(y,\mu,\lambda)$ the probability density function of the stationary random process $e(x)$. One has

$$P\Big\{-\rho \le e(x) \le \rho\Big\} = \int_{-\rho}^{\rho} p(y,\mu,\lambda)dy \approx 2\rho p(0,\mu,\lambda). \tag{3.2}$$

So, instead of (3.1) one can require

$$p(0,\mu,\lambda) \to \max_{\mu,\lambda}. \tag{3.3}$$

For further analysis one can use an approximate analytic representation of the probability density function of a random process [1, 5, 9]. An alternative possibility is to express the unknown even probability density $p(y)$ approximately by a truncated Gram-Charlier series

$$p(y,\mu,\lambda) = \frac{e^{-y^2/2\langle e^2\rangle}}{\sqrt{2\pi\langle e^2\rangle}}\Big\{1 + \sum_{k=2}^{N} \frac{1}{(2k)!}\,\frac{b_{2k}}{\langle e^2\rangle^k} H_{2k}\Big(\frac{y}{\sqrt{\langle e^2\rangle}}\Big)\Big\} \tag{3.4}$$

where b_{2k} are quasi - moments of order $2k$, $H_{2k}(x)$ are Hermite polynomials which satisfy the differentiation law

$$\frac{d}{dx}H_n(x) = nH_{n-1}(x) \tag{3.5}$$

and the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x). \tag{3.6}$$

Substituting (3.4) into (3.3) yields the following criterion for determining the coefficients of linearization μ, λ

$$p(0,\mu,\lambda) = \frac{1}{\sqrt{2\pi\langle e^2\rangle}}\Big\{1 + \sum_{k=2}^{N} \frac{1}{(2k)!}\,\frac{b_{2k}}{\langle e^2\rangle^k} H_{2k}(0)\Big\} \to \max_{\mu,\lambda}. \tag{3.7}$$

In the case the summation in (3.4) has only one term, i.e. when the equation error process e is supposed to be a Gaussian one, the criterion (3.7) reduces to

$$\frac{1}{\sqrt{2\pi\langle e^2\rangle}} \to \max_{\mu,\lambda}. \tag{3.8}$$

Hence, one obtains again the mean square error criterion (2.8). In the paper, next extension step to the classical GEL, is investigated in detail for the case $N = 2$, so the criterion (3.7) yields

$$p(0,\mu,\lambda) = \frac{1}{\sqrt{2\pi\langle e^2\rangle}}\Big\{1 + \frac{1}{8}\Big(\frac{\langle e^4\rangle}{\langle e^2\rangle^2} - 3\Big)\Big\} \rightarrow \max_{\mu,\lambda} \tag{3.9}$$

since $b_4 = \langle e^4\rangle - 3\langle e^2\rangle^2$, $H_4(x) = x^4 - 6x^2 + 3$. It is seen from (3.9) that the equation error fourth moment $\langle e^4\rangle$ is involved in the procedure determining the linearized coefficients it μ and λ.

4. CLOSE EQUATIONS FOR RESPONSE SECOND MOMENTS

In this section four equations used to determine four unknowns $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$, μ and λ are derived in explicit form. Two of them were obtained already in (1.10), another two can be obtained from

$$\frac{\partial}{\partial\mu}p(0,\mu,\lambda) = 0, \quad \frac{\partial}{\partial\lambda}p(0,\mu,\lambda) = 0 \tag{4.1}$$

Using (3.9) we get

$$\frac{\partial}{\partial s}p(0,\mu,\lambda) = \frac{1}{16\sqrt{2\pi\langle e^2\rangle^2}}\Big\{2\langle e^2\rangle\frac{\partial}{\partial s}\langle e^4\rangle - 5(\langle e^4\rangle + \langle e^2\rangle^2)\frac{\partial}{\partial s}\langle e^2\rangle\Big\} \tag{4.2}$$

where $s = \mu, \lambda$. Thus, it follows from (4.1) and (4.2)

$$2\langle e^2\rangle\frac{\partial}{\partial s}\langle e^4\rangle - 5(\langle e^4\rangle + \langle e^2\rangle^2)\frac{\partial}{\partial s}\langle e^2\rangle = 0; \quad s = \mu, \lambda \tag{4.3}$$

Further, using (2.7) gives

$$\begin{aligned}
\langle e^2\rangle &= \varepsilon^2\langle g^2\rangle + \mu^2\langle \dot{x}^2\rangle - 2\varepsilon\mu\langle g\dot{x}\rangle + \lambda^2\langle x^2\rangle - 2\varepsilon\lambda\langle gx\rangle \\
\langle e^4\rangle &= \varepsilon^4\langle g^4\rangle + \mu^4\langle \dot{x}^4\rangle - 4\varepsilon\mu^3\langle g\dot{x}^3\rangle + 6\varepsilon^2\mu^2\langle g^2\dot{x}^2\rangle - 4\varepsilon^3\mu\langle g^3\dot{x}\rangle \\
&\quad + 12\varepsilon^3\mu\lambda\langle g^2x\dot{x}\rangle - 12\varepsilon\mu\lambda^2\langle gx^2\dot{x}\rangle - 12\varepsilon\mu^2\lambda\langle gx\dot{x}\rangle + 6\mu^2\lambda^2\langle x^2\dot{x}^2\rangle \\
&\quad + 4\lambda\mu^3\langle x\dot{x}^3\rangle + \lambda^4\langle x^4\rangle - 4\varepsilon\lambda^3\langle gx^3\rangle + 6\varepsilon^2\lambda^2\langle g^2x^2\rangle - 4\varepsilon^3\lambda\langle g^3x\rangle
\end{aligned} \tag{4.4}$$

The explicit expression for (4.3) can be determined easily by substituting (4.4) into (4.3). Hence, there are 4 equations (2.10), (4.3) for 4 unknowns $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$, μ and λ. The problem of existence and uniqueness of solutions for the system of equations (2.10), (4.3) requires a further investigation.

5. DUFFING OSCILLATOR

As an illustration of the technique proposed consider a single degree of freedom system with linear damping and non-linear spring, the Duffing oscillator, which has been applied to model many mechanical systems. The equation of motion of such a system is given by

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \varepsilon\gamma x^3 = f(t) \tag{5.1}$$

here $f(t)$ is a Gaussian white noise excitation for which

$$\langle f(t)f(t+\tau)\rangle = \sigma^2\delta(\tau), \quad S_f(\omega) = \frac{\sigma^2}{2\pi}, \tag{5.2}$$

where $\delta(\tau)$ is the Dirac delta function.

It is easy to see that the equations (4.3) are satisfied for $\mu = 0$ and then reduce to the following equation for λ:

$$\begin{aligned}\ell(\lambda) = {} & 8(\langle x^2\rangle\lambda^2 - 6\varepsilon\gamma\langle x^2\rangle^2\lambda + 15\varepsilon^2\gamma^2\langle x^2\rangle^3)(3\langle x^2\rangle^2\lambda^3 - 45\varepsilon\gamma x^2\rangle^3\lambda^2 \\ & + 315\varepsilon^2\gamma^2\langle x^2\rangle^4\lambda - 945\varepsilon^3\gamma^3\langle x^2\rangle^5) - 10(\langle x^2\rangle\lambda - 3\varepsilon\gamma\langle x^2\rangle^2)[3\langle x^2\rangle^2(\lambda^4 \\ & - 20\varepsilon\gamma\langle x^2\rangle\lambda^3 + 210\varepsilon^3\gamma^2\langle x^2\rangle^2\lambda^2 - 1260\varepsilon^3\gamma^3\langle x^2\rangle^3\lambda + 3465\varepsilon^4\gamma^4\langle x^2\rangle^4) \\ & + \langle x^2\rangle^2(\lambda^4 - 12\varepsilon\gamma\langle x^2\rangle\lambda^3 + 66\varepsilon^2\gamma^2\langle x^2\rangle^2\lambda^2 - 180\varepsilon^3\gamma^3\langle x^2\rangle^3\lambda + 225\varepsilon^4\gamma^4\langle x^2\rangle^4)] = 0.\end{aligned} \tag{5.3}$$

Thus, the corresponding linearized equation (2.6) is

$$\ddot{x} + 2h\dot{x} + (\omega_0^2 + \lambda)x = f(t). \tag{5.4}$$

The second equation for λ and $\langle x^2\rangle$ is obtained from (5.4) and noting (5.2)

$$\langle x^2\rangle = \frac{\sigma^2}{4h(\omega_0^2 + \lambda)} \tag{5.5}$$

From (5.3) one gets

$$\begin{aligned}\ell(0) &= 205200\langle x^2\rangle^8 > 0 \\ \ell(\lambda \to +\infty) &\approx -16\langle x^2\rangle^3\lambda^5 < 0.\end{aligned} \tag{5.6}$$

Hence, there exists in the interval $(0, +\infty)$ λ_* such that

$$\begin{aligned}\ell(\lambda_*) > 0 \quad \text{for} \quad \lambda < \lambda_*, \\ \ell(\lambda_*) = 0, \quad \text{and} \quad \ell(\lambda_*) < 0 \quad \text{for} \quad \lambda > \lambda_*.\end{aligned} \tag{5.7}$$

Inequalities (5.7) indicate that the function $p(0,\lambda)$ has a local maximum at the value $\lambda = \lambda_*$ for the Duffing oscillator. The results obtained by the procedure proposed (equations (5.3) and (5.5)) $\langle x^2\rangle_2$ are compared in Table 1 which the values $\omega = 1$, $\gamma = 1$, $\sigma^2 = 4h$, and for different values of ε. In addition, the results obtained by the classical GEL technique $\langle x^2\rangle_1$ are also shown. Obviously, the solutions $\langle x^2\rangle_2$ are much closer to the exact solutions $\langle x^2\rangle_e$, than the solutions $\langle x^2\rangle_1$.

Table 1. Approximate mean square of displacement, Duffing equation (5.1), (5.2)

N	ε	$\langle x^2\rangle_e$	$\langle x^2\rangle_1$	Error %	$\langle x^2\rangle_2$	Error %
1	0.1	0.8176	0.8054	-1.49	0.8250	0.90
2	1.0	0.4679	0.4343	-7.19	0.4615	-1.36
3	10.0	0.1889	0.1667	-11.8	0.1802	-4.61
4	100.0	0.0650	0.0561	-13.6	0.0610	-6.15

7. CONCLUSIONS

The main question inherent in Gaussian equivalent linearization is how the coefficients of the linearized equation are found. Instead of the well-known mean square criterion, a probabilistic criterion has been proposed to determine these coefficients. Further, a truncated Gram-Charlier series with two terms is used to express approximately the even probability density function of equation error. It is obtained that the technique proposed is a quite general and simple as within the scope of Gaussian equivalent linearization. Application to Duffing oscillator shows significant improvement over the corresponding accuracy of the classical GEL.

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TIÊU CHUẨN XÁC SUẤT ĐỐI VỚI TUYẾN TÍNH HÓA TƯƠNG ĐƯƠNG GAUSS

Trong khuôn khổ của phương pháp tuyến tính hóa tương đương Gauss, các tác giả đã đề nghị một tiêu chuẩn xác suất mới để xác định các hệ số tuyến tính hóa tương đương nhằm tính toán đáp ứng dừng của hệ cơ học phi tuyến chịu kích động ngẫu nhiên dừng chuẩn. Kết quả áp dụng cho hệ Duffing cho kết quả tốt hơn so với phương pháp tuyến tính hóa kinh điển.