

# SIMPLE RESONANCE IN NONLINEAR TORSIONAL VIBRATION SYSTEMS OF THREE VARIABLE GENERALIZED MASSES

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## 1. Introduction

Torsional vibrations of transmission systems are often encountered in mechanical engineering. Torsional vibration analysis of linear systems has drawn the attention of investigators [1-4]. However, such an attention to nonlinear systems has not been paid properly.

In this paper our objective is to examine simple resonance solutions for a torsional vibration system of three variable generalized masses by using the small parameter method [5].

## 2. Differential equations of vibration

Let us consider a nonlinear torsional vibration system of three variable generalized masses as shown in Fig. 1

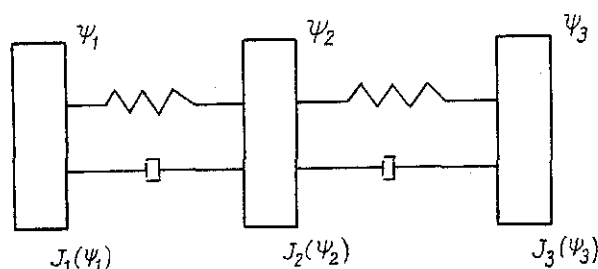


Fig. 1

In technical practice, it is often the case that  $J_1 = \text{const}$  while  $J_2$  and  $J_3$  depend on  $\psi_i$  so that

$$J_i = J_{0i} + J_{1i} \cos 2\psi_i,$$

where  $J_{0i}$  and  $J_{1i}$  are constants.

For more explicitness, we consider the case when  $\dot{\psi}_1 = \Omega = \text{const}$  and the angles  $\psi_2$  and  $\psi_3$  can be written in the form

$$\psi_i = \Omega t + \varphi_i, \quad (i = 2, 3). \tag{2.1}$$

Further, it is assumed that the damping coefficients  $b_i^{(1)}, b_i^{(3)}$  and nonlinearly elastic coefficients

$c_i^{(3)}$  are small, while reduced moments of inertia are written in the form:

$$J_i = J_{0i} \left( 1 + \frac{J_{1i}}{J_{0i}} \cos 2\psi_i \right) = J_{0i} (1 + \varepsilon \bar{J}_{1i} \cos 2\psi_i), \quad (i = 2, 3),$$

where  $\varepsilon$  is a small parameter.

Taking into consideration all assumptions, the vibration differential equation of the system is given under the form [6, 8]

$$\underline{M} \ddot{\bar{q}} + \underline{C} \dot{\bar{q}} = \varepsilon \bar{\phi}, \quad (2.2)$$

where

$$\underline{M} = \begin{bmatrix} J_{02} & 0 \\ 0 & J_{03} \end{bmatrix}, \quad \bar{q} = \begin{bmatrix} \varphi_2 \\ \varphi_3 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} c_1^{(1)} + c_2^{(1)} & -c_2^{(1)} \\ -c_2^{(1)} & c_2^{(1)} \end{bmatrix}, \quad \bar{\phi} = \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix}. \quad (2.3)$$

The functions  $\phi_2, \phi_3$  have the following expressions [8]

$$\begin{aligned} \phi_2 = & -\frac{1}{2} \Omega^2 \bar{J}_{2,2} - \bar{J}_{2,2} \left( \varphi_2 \ddot{\varphi}_2 + \frac{1}{2} \dot{\varphi}_2^2 \right) - \bar{J}_{2,22} \left( \frac{1}{2} \varphi_2^2 \ddot{\varphi}_2 + \frac{1}{2} \varphi_2 \dot{\varphi}_2^2 + \Omega \varphi_2 \dot{\varphi}_2 \right) \\ & - \bar{J}_{2,222} \left( \frac{1}{2} \Omega \varphi_2^2 \dot{\varphi}_2 + \frac{1}{4} \Omega^2 \varphi_2^2 \right) - \frac{1}{12} \Omega^2 \bar{J}_{2,2222} \varphi_2^3 - c_1^{(3)} \varphi_2^3 \\ & - c_2^{(3)} (\varphi_2 - \varphi_3)^3 - b_1^{(3)} \dot{\varphi}_2^3 - b_2^{(3)} (\dot{\varphi}_2 - \dot{\varphi}_3)^3 - b_1^{(1)} \dot{\varphi}_2 \\ & - b_2^{(1)} (\dot{\varphi}_2 - \dot{\varphi}_3) - \Omega \bar{J}_{2,2} \dot{\varphi}_2 - \frac{1}{2} \Omega^2 \bar{J}_{2,22} \varphi_2, \end{aligned} \quad (2.4.a)$$

$$\begin{aligned} \phi_3 = & -\frac{1}{2} \Omega^2 \bar{J}_{3,3} - \bar{J}_{3,3} \left( \varphi_3 \ddot{\varphi}_3 + \frac{1}{2} \dot{\varphi}_3^2 \right) - \bar{J}_{3,33} \left( \frac{1}{2} \varphi_3^2 \ddot{\varphi}_3 + \frac{1}{2} \varphi_3 \dot{\varphi}_3^2 + \Omega \varphi_3 \dot{\varphi}_3 \right) \\ & - \bar{J}_{3,333} \left( \frac{1}{2} \Omega \varphi_3^2 \dot{\varphi}_3 + \frac{1}{4} \Omega^2 \varphi_3^2 \right) - \frac{1}{12} \Omega^2 \bar{J}_{3,3333} \varphi_3^3 + c_2^{(3)} (\varphi_2 - \varphi_3)^3 \\ & + b_2^{(3)} (\dot{\varphi}_2 - \dot{\varphi}_3)^3 + b_2^{(1)} (\dot{\varphi}_2 - \dot{\varphi}_3) - \Omega \bar{J}_{3,3} \dot{\varphi}_3 - \frac{1}{2} \Omega^2 \bar{J}_{3,33} \varphi_3. \end{aligned} \quad (2.4.b)$$

In order to have an equation in modal coordinates we make a coordinate transformation relating physical coordinates  $\bar{q}$  to the modal ones  $\bar{p}$

$$\bar{q} = \underline{V} \bar{p} \quad (2.5)$$

in which  $\underline{V}$  is the modal matrix. The matrix  $\underline{V}$  has the form [8]

$$\underline{V} = \begin{bmatrix} 1 & 1 \\ v_2^{(1)} & v_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{c_1^{(1)} + c_2^{(1)} - J_{02} \omega_1^2}{c_2^{(1)}} & \frac{c_1^{(1)} + c_2^{(1)} - J_{02} \omega_2^2}{c_2^{(1)}} \end{bmatrix}, \quad (2.6)$$

where

$$\omega_{1,2}^2 = \frac{c_2^{(1)} J_{02} + J_{03} (c_1^{(1)} + c_2^{(1)}) \pm \sqrt{[c_2^{(1)} J_{02} + J_{03} (c_1^{(1)} + c_2^{(1)})]^2 - 4c_1^{(1)} c_2^{(1)} J_{02} J_{03}}}{2J_{02} J_{03}}$$

Substituting the expression (2.5) into equation (2.2), we obtain the vibration differential equations in the modal coordinates

$$p_i'' + \lambda_i p_i = \varepsilon F_i \quad (2.7)$$

in which

$$\begin{aligned}
F_i = & \sum_j a_{ij} \cos 2m\tau p_j + \sum_j (\bar{a}_{ij} \sin 2m\tau + \bar{a}_{ij}^0) p_j' + \sum_{j,k} b_{ijk} \sin 2m\tau p_j p_k \\
& + \sum_{j,k} \bar{b}_{ijk} \sin 2m\tau p_j' p_k' + \sum_{j,k} \bar{c}_{ijk} \sin 2m\tau p_j p_k'' + \sum_{j,k} c_{ijk} \cos 2m\tau p_j p_k \\
& + \sum_{j,k,\ell} (d_{ijk\ell} \cos 2m\tau + d_{ijk\ell}^0) p_j p_k p_\ell + \sum_{j,k,\ell} \bar{d}_{ijk\ell}^0 p_j' p_k' p_\ell' \\
& + \sum_{j,k,\ell} (e_{ijk\ell} \sin 2m\tau p_j p_k p_\ell') + \sum_{j,k,\ell} \bar{e}_{ijk\ell} \cos 2m\tau p_j p_k p_\ell'' \\
& + \sum_{j,k,\ell} f_{ijk\ell} \cos 2m\tau p_j p_k' p_\ell' + a_i \sin 2m\tau.
\end{aligned} \tag{2.8}$$

The indices  $j, k, \ell$  are taken for the values 1 and 2 and the coefficients in (2.8) are determined in [8].

### 3. Analysis of simple resonance

It is supposed that

$$\lambda_1 = n^2(1 - \varepsilon\alpha),$$

where  $n$  is an integer. By introducing the notations:

$$\Lambda_1 = n^2, \quad \Lambda_2 = \lambda_2,$$

the equation (2.7) can be rewritten under the form:

$$p_i'' + \Lambda_i p_i = \varepsilon(F_i + n^2 \alpha \delta_i^1 p_i). \tag{3.1}$$

In the first approximation we shall find the solution of (3.1) in the form:

$$p_i^{(0)} = \delta_i^1 (R \cos n\tau + S \sin n\tau). \tag{3.2}$$

According to [5] the condition for the existence of periodic solutions of period  $2\pi$  implies that:

$$\int_0^{2\pi} (F_1 + n^2 p_1^{(0)}) \begin{pmatrix} \cos n\tau \\ \sin n\tau \end{pmatrix} d\tau = 0. \tag{3.3}$$

From this we obtain the amplitude-frequency equation as follows:

$$\begin{aligned}
& \left( n^2 \alpha + \frac{3d_{1111}^0 A^2}{4} \right) \begin{pmatrix} R \\ S \end{pmatrix} + \frac{a_{11} - n\bar{a}_{11}}{2} \begin{pmatrix} R \\ -S \end{pmatrix} \delta_m^n + \frac{d_{1111}}{2} \begin{pmatrix} R^3 \\ -S^3 \end{pmatrix} \delta_m^n + \\
& \frac{n^2 \bar{e}_{1111}}{2} \begin{pmatrix} -R^3 \\ S^3 \end{pmatrix} \delta_m^n + \frac{ne_{1111}}{4} \begin{pmatrix} -R^3 + RS^2 \\ S^3 - SR^2 \end{pmatrix} \delta_m^n + \frac{n^2}{2} \begin{pmatrix} RS^2 \\ -SR^2 \end{pmatrix} f_{1111} \delta_m^n + \\
& a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_m^{n/2} + \frac{b_{111}}{4} \begin{pmatrix} 2RS \\ A^2 + 2S^2 \end{pmatrix} \delta_m^{n/2} + \frac{n^2 \bar{b}_{111}}{4} \begin{pmatrix} -2RS \\ 3R^2 + S^2 \end{pmatrix} \delta_m^{n/2} + \\
& \frac{nc_{111}}{4} \begin{pmatrix} 2RS \\ S^2 - R^2 \end{pmatrix} \delta_m^{n/2} - \frac{n^2 \bar{c}_{111}}{4} \begin{pmatrix} 2RS \\ 3S^2 + R^2 \end{pmatrix} \delta_m^{n/2} + \\
& \frac{b_{111}}{4} \begin{pmatrix} 2RS \\ R^2 - S^2 \end{pmatrix} \delta_m^{3n/2} + \frac{n^2 \bar{b}_{111}}{4} \begin{pmatrix} -2RS \\ -R^2 + S^2 \end{pmatrix} \delta_m^{3n/2} + \\
& \frac{nc_{111}}{4} \begin{pmatrix} 2RS \\ R^2 - S^2 \end{pmatrix} \delta_m^{3n/2} - \frac{n^2 \bar{c}_{111}}{4} \begin{pmatrix} 2RS \\ R^2 - S^2 \end{pmatrix} \delta_m^{3n/2} + \\
& \frac{d_{1111} - ne_{1111} - n^2 \bar{e}_{1111}}{8} \begin{pmatrix} R^3 - 3RS^2 \\ S^3 - 3SR^2 \end{pmatrix} \delta_m^{2n} - \frac{n^2 f_{1111}}{8} A^2 \begin{pmatrix} R \\ S \end{pmatrix} \delta_m^{2n} + \\
& n\bar{a}_{11}^0 \begin{pmatrix} S \\ -R \end{pmatrix} + \frac{3n^3 \bar{d}_{1111}^0 A^2}{4} \begin{pmatrix} S \\ -R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}
\end{aligned}$$

The equation (3.4) shows that the following resonance cases might occur:

- Fundamental resonance:  $\omega_2 = \Omega$ , ( $m = n$ ).
- Superharmonic resonance:  $\omega_2 = \frac{\omega}{2}$ , ( $m = 2n$ ).
- Super-subharmonic resonance:  $\omega_2 = \frac{2\Omega}{3}$ , ( $m = \frac{3n}{2}$ ).
- Subharmonic resonance of  $2^{nd}$  - kind:  $\omega_2 = 2\Omega$ , ( $m = \frac{n}{2}$ )

Consider now the super-subharmonic resonance for  $\omega_2 = \frac{2\Omega}{3}$ . The equation (3.4) yield

$$n^2 \alpha = -\frac{3fA^2}{4} \pm \sqrt{\frac{c^2 A^2}{144} - \left( nb + \frac{3n^3 g A^2}{4} \right)^2} \tag{3.5}$$

in which

$$c = b_{111}; \quad b = \bar{a}_{11}^0; \quad a = a_{11}; \quad e = d_{1111}; \quad f = d_{1111}^0; \quad g = \bar{d}_{1111}^0; \quad d = c_{111}.$$

We can find the stability condition for this case with the help of the condition established in [5].

If the positive sign or the negative sign is chosen before the square root, the stability condition will be respectively:

$$\frac{d\alpha}{dA} < 0 \quad \text{or} \quad \frac{d\alpha}{dA} > 0. \tag{3.6}$$

For the solution  $A = 0$ , it becomes

$$b < 0, \quad n^2 b^2 > -n^2 \alpha^2. \tag{3.7}$$

From here we conclude that the solution  $A = 0$  is always stable.

After similar calculations for other cases we obtain the results presented in the table below

Resonance Case	Expression of Amplitude
$\omega_2 = \Omega \ (m = n)$	$n^2 \alpha = -\frac{3fA^2}{4} \pm \frac{2a+eA^2}{2(4a+eA^2)} \sqrt{(4a+eA^2)^2 - (4nb+3n^3gA^2)^2}$
$\omega_2 = \frac{2\Omega}{3} \ (m = \frac{3n}{2})$	$n^2 \alpha = -\frac{3fA^2}{4} \pm \sqrt{\frac{e^2A^2}{4} - \left(nb + \frac{3n^3gA^2}{4}\right)^2}$
$\omega_2 = 2\Omega \ (m = \frac{n}{2})$	$n^2 \alpha = -\frac{3fA^2}{4}$ $\pm \frac{4a_1+9cA^2}{A(4a_1+3cA^2)} \sqrt{\left(a_1 + \frac{3cA^2}{4}\right)^2 - \left(nb + \frac{3n^3gA^2}{4}\right)^2} A^2$
$\omega_2 = \frac{\Omega}{2} \ (m = 2n)$	$n^2 \alpha = -\left(\frac{3e}{64} + \frac{3f}{4}\right) A^2 \pm \sqrt{\frac{e^2A^4}{64^2} - \left(nb + \frac{3n^3gA^2}{4}\right)^2}$

In Fig. 2 the amplitude - frequency curve is plotted for the case  $m = 3n/2$ . The heavy line corresponds to the stable vibration and the dashed line - to the unstable one.

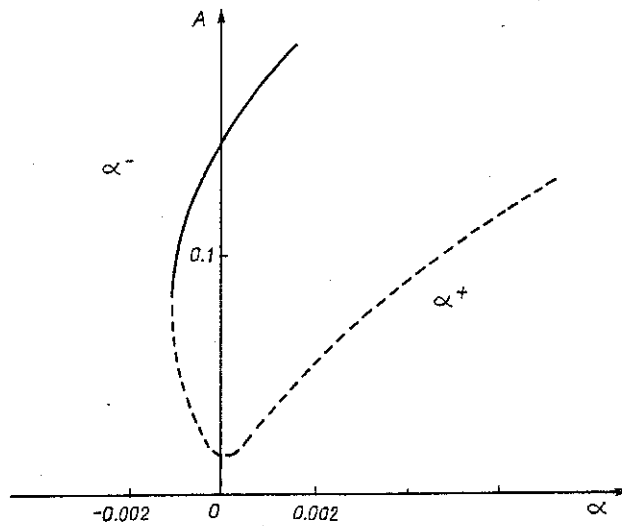
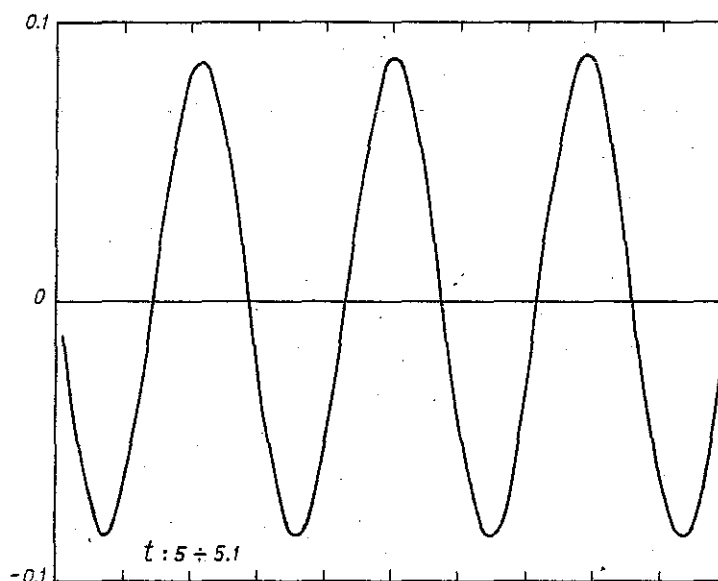


Fig. 2

In Fig. 3 the result from computer simulation is presented for  $\alpha = -10^3$ . Comparing this result with the one obtained in Fig. 2:  $A \approx 0.087$  for  $\alpha = -10^{-3}$ , it is seen that these results are coincident.



*Fig. 3*

#### 4. Conclusion

The calculation of nonlinear parametrical torsional vibration presents a quite complicated problem. In this paper an amplitude-frequency equation for a nonlinear torsional vibration system of three variable generalized masses was set up. The result of analytical investigation perfectly coincides with the ones obtained from computer simulation. The combined resonance vibration will be given in an other publication.

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### VỀ CÁC CỘNG HƯỞNG ĐƠN CỦA HỆ DAO ĐỘNG XOẮN PHI TUYẾN CÓ BA KHỐI LƯỢNG THU GỌN BIẾN ĐỔI

Trong bài báo này đã thiết lập được phương trình biên độ - tần số cho một hệ dao động xoắn phi tuyến có ba khối lượng thu gọn biến đổi. Các kết quả thu được bằng phương pháp giải tích phù hợp với các kết quả thu được bằng mô phỏng số.