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STABILITY OF THE CRITICAL STATIONARY OSCILLATIONS

NGUYEN VAN DINH
Institute of Mechanics, Hanoi Vietnam

In [1], a preliminary study on the so-called critical stationary oscillations has been proposed. In the present paper, some additional remarks on the second stability condition are of our interest. It will be shown that the compact form of the mentioned condition can be established for ordinary as well as for critical stationary oscillations.

1. The two parts of the resonance curve

Stationary oscillations of the quasilinear oscillating system examined in [1] are determined from the equations:

$$\begin{aligned} \{f\}_\theta &= \{A(\Delta,a)u + B(\Delta,a)v - E(\Delta,a)\}_\theta = 0, \\ \{g\}_\theta &= \{G(\Delta,a)u + H(\Delta,a)v - K(\Delta,a)\}_\theta = 0, \end{aligned} \tag{1.1}$$

where: f, g are polynomials of (Δ, a, u, v), linear relative to (u, v); the subscript "θ" indicates that u, v must be substituted by $u(\theta) = \sin\theta$, $v(\theta) = \cos\theta$; other notations have been explained in [1].

In the plane $R(\Delta, a)$, the resonance curve C is defined as the ensemble of representing points $I(\Delta, a)$ whose ordinate a is the amplitude of the stationary oscillations corresponding to the detuning parameter abscissa Δ.

In general, C consists of two parts: the ordinary C_1 and the critical C_2.

C_1 lies in the ordinary region $R_1 : D_0(\Delta, a) \neq 0$; it is given by the relationship:

$$W_1(\Delta,a) = \frac{D_1^2(\Delta,a) + D_2^2(\Delta,a)}{D_0^2(\Delta,a)} - 1 = 0. \tag{1.2}$$

To obtain (1.2) we have imposed the trigonometrical condition $u^2 + v^2 = 1$ on the expressions:

$$u(\Delta,a) = D_1(\Delta,a)/D_0(\Delta,a), \quad v(\Delta,a) = D_2(\Delta,a)/D_0(\Delta,a), \tag{1.3}$$

which are the solutions in R_1 of the equations (1.1) considered as the algebraic ones of two unknowns u, v (Δ, a play the role of parameters).

C_2 lies in the critical region (curve) $R_2 : D_0(\Delta, a) = 0$; it consists of critical representing points $I_*(\Delta_*, a_*)$. If the matrix $[D_0]$ is assumed to be of rank 1, C_2 is determined by the compatibility conditions (1.4) and the trigonometrical restrictions (1.5):

$$D_0(\Delta,a) = 0, \quad D_1(\Delta,a) = 0, \quad D_2(\Delta,a) = 0, \tag{1.4}$$

$$A^2 + B^2 \geq E^2, \quad G^2 + H^2 \geq K^2. \tag{1.5}$$

The conditions (1.4) assure the compatibility of the "linear" system(1.1) at I_* (they determine compatible points $I'(\Delta', a')$), while (1.5) assure the resolvability of the "trigonometrical" system (1.1) at I_* (they distinguishe critical representing points from compatible ones). Usually, instead of (1.2), the following relationship is used:

$$W(\Delta,a) = D_1^2(\Delta,a) + D_2^2(\Delta,a) - D_0^2(\Delta,a) = 0 \tag{1.6}$$

Obviously, (1.6) contains all the two parts C_1, C_2. However, (1.2) as well as (1.6) cannot be considered as the equation of the "whole" resonance curve $C = C_1 \cup C_2$: (1.2) does not contain C_2 while (1.6) contains 'in surplus" compatible non critical points (satisfying(1.4) but not (1.5)).

2. Singularity and trigonometrical restrictions

Critical points I_* satisfy the necessary conditions of singularity:

$$\left(\frac{\partial W}{\partial \Delta}\right)_* = 0, \qquad \left(\frac{\partial W}{\partial a}\right)_* = 0. \tag{2.1}$$

In practice, they are singular points of the curve $W = 0$. To determine the type of singularity of the critical point $I_*(\Delta_*, a_*)$, we use the development of $W(\Delta, a)$ in its vinicity:

$$W(\Delta - \Delta_*, a - a_*) = \frac{1}{2}\delta^2 W + \dots \tag{2.2}$$

where

$$\delta^2 W = A_{20}(a-a_*)^2 + 2A_{11}(a-a_*)(\Delta - \Delta_*) + A_{02}(\Delta - \Delta_*)^2,$$

$$A_{ij} = \left(\frac{\partial^{i+j} W}{\partial a^i \partial \Delta^j}\right)_* \quad (i, j = 0, 1, 2).$$

We shall base our discussion only on $\delta^2 W$, assuming that $\delta^2 W \not\equiv 0$ i.e. $A_{20}^2 + A_{11}^2 + A_{02}^2 \neq 0$. In this case, the singularity depends on the sign of the discriminant:

$$D_* = A_{11}^2 - A_{20}A_{02} \tag{2.3}$$

namely:
- if $D_* > 0$, I_* is a nodal point,
- if $D_* < 0$, I_* is an isolated point (as it will be shown below, in this case, the point of interest is a compatible non critical).

The case $D_* = 0$ requires a more detailed study.

It is noted that there exist certain liaisons between the singularity and the trigonometrical restrictions. Indeed, since rank $[D_0] = 1$, we can assume that $A_* \neq 0$. By (1) (2) we denote two trajectories in R_1 passing through I_*.

The expressions (1.3) are the solutions of the "algebraic" equations (1.1) in R_1. Therefore we have along (1) (2) an identity:

$$A(\Delta, a)D_1(\Delta, a) + B(\Delta, a)D_2(\Delta, a) - E(\Delta, a)D_0(\Delta, a) \equiv 0. \tag{2.4}$$

So, by differentiating along (1) (2), we obtain at I_*:

$$\left(A\frac{\partial D_1}{\partial \Delta} + B\frac{\partial D_2}{\partial \Delta} - E\frac{\partial D_0}{\partial \Delta}\right)_* + \left(A\frac{\partial D_1}{\partial a} + B\frac{\partial D_2}{\partial a} - E\frac{\partial D_0}{\partial a}\right)_* k_i = 0 \quad (i = 1, 2), \tag{2.5}$$

where k_i is the slope of the trajectory (i) at I_*.

Since $k_1 \neq k_2$, from (2.5), it follows:

$$\begin{aligned} \left(A\frac{\partial D_1}{\partial \Delta} + B\frac{\partial D_2}{\partial \Delta} - E\frac{\partial D_0}{\partial \Delta}\right)_* = 0 \quad \text{or} \quad \left(\frac{\partial D_1}{\partial \Delta}\right)_* = \frac{1}{A_*}\left(E\frac{\partial D_0}{\partial \Delta} - B\frac{\partial D_2}{\partial \Delta}\right)_*, \\ \left(A\frac{\partial D_1}{\partial a} + B\frac{\partial D_2}{\partial a} - E\frac{\partial D_0}{\partial a}\right)_* = 0 \quad \text{or} \quad \left(\frac{\partial D_1}{\partial a}\right)_* = \frac{1}{A_*}\left(E\frac{\partial D_0}{\partial a} - B\frac{\partial D_2}{\partial a}\right)_* . \end{aligned} \tag{2.6}$$

Using (2.6), the discriminant D_* can be transformed into:

$$D_* = \frac{4}{A_*^2}(A^2 + B^2 - E^2)_* \left(\frac{\partial D_2}{\partial \Delta}\frac{\partial D_0}{\partial a} - \frac{\partial D_2}{\partial a}\frac{\partial D_0}{\partial \Delta}\right)_*^2 . \tag{2.7}$$

This relation shows that:

- If $D_* > 0$, then $A_*^2 + B_*^2 > E_*^2$: the nodal point I_* satisfying the trigonometrical restrictions is a critical point.

- If $D_* < 0$, then $A_*^2 + B_*^2 < E_*^2$: the isolated point does not satisfy the trigonometrical restrictions; it is only a compatible non critical point and does not belong to C_2.

3. The dephases of stationary oscillations

For ordinary stationary oscillations i.e. on C_1, the dephases are given by (1.3):

$$\sin\theta = u(\theta) = u(\Delta, a), \quad \cos\theta = v(\theta) = v(\Delta, a). \tag{3.1}$$

Let $I_*(\Delta_*, a_*)$ be the critical point of interest. The corresponding dephases θ_* are determined - for instance - from the first "trigonometrical" equation of the system (1.1):

$$A_* \sin\theta + B_* \cos\theta - E_* = 0 \tag{3.2}$$

or, by the "algebraic" system:

$$A_* u + B_* v - E_* = 0 \tag{3.3a}$$
$$u^2 + v^2 = 1. \tag{3.3b}$$

The critical dephases can also be determined by (1.3) "at limit". In R_1, by suitable trajectories γ, we call those passing through I_* and admitting (at I_*), as slope, any root k of the equation:

$$A_{20}k^2 + 2A_{11}k + A_{02} = 0 \tag{3.4}$$

(each k corresponds to a "family" of trajectories of the same tangent at I_*; for simplicity, this family is called one trajectory). The quadratic equation (3.4) has just D_* as discriminant.

If $D_* > 0$ (nodal point) we have two suitable trajectories; if $D_* = 0$ - only one; if $D_* < 0$ (isolated non critical point), suitable trajectories do not exist.

At every point $I(\Delta, a)$ of γ, we have:

$$A(\Delta, a)u(\Delta, a) + B(\Delta, a)v(\Delta, a) - E(\Delta, a) = 0. \tag{3.5}$$

When I moves along γ and tends to I_*, by continuity, at I_*, we have:

$$A_* \lim u(\Delta, a) + B_* v(\Delta, a) - E_* = 0. \tag{3.6}$$

Suppose that the two limits of $u(\Delta, a)$, $v(\Delta, a)$ are given by:

$$\begin{aligned} \lim u(\Delta, a) &= \lim \left(\frac{D_1(\Delta, a)}{D_0(\Delta, a)}\right) = \left(\frac{dD_1}{d\Delta} \Big/ \frac{dD_0}{d\Delta}\right)_*, \\ \lim v(\Delta, a) &= \lim \left(\frac{D_2(\Delta, a)}{D_0(\Delta, a)}\right) = \left(\frac{dD_2}{d\Delta} \Big/ \frac{dD_0}{d\Delta}\right)_*, \end{aligned} \tag{3.7}$$

where

$$\left(\frac{dD_i}{d\Delta}\right)_* = \left(\frac{\partial D_i}{\partial a}\right)_* k + \left(\frac{\partial D_2}{\partial \Delta}\right)_* \quad (i = 0, 1, 2).$$

Regarding to (3.4), we can write:

$$\left(\lim u(\Delta, a)\right)^2 + \left(\lim v(\Delta, a)\right)^2 - 1 = \frac{1}{\left(\frac{dD_0}{d\Delta}\right)_*^2} \left\{A_{20}k^2 + 2A_{11}k + A_{02}\right\} = 0, \tag{3.8}$$

$$A_{ij} = \left(\frac{\partial^{i+j} W}{\partial a^i \partial \Delta^j}\right)_* = 2\left\{\left(\frac{\partial D_1}{\partial a}\right)^i \left(\frac{\partial D_1}{\partial \Delta}\right)^j + \left(\frac{\partial D_2}{\partial a}\right)^i \left(\frac{\partial D_2}{\partial \Delta}\right)^j - \left(\frac{\partial D_0}{\partial a}\right)^i \left(\frac{\partial D_0}{\partial \Delta}\right)^j\right\}_* \quad (i+j=2),$$

The expressions (3.6) (3.8) mean that $\lim u(\Delta, a)$, $\lim v(\Delta, a)$ satisfy (3.3a,b) i.e we have:

$$u_* = \lim u(\Delta, a), \quad v_* = \lim v(\Delta, a) \tag{3.9}$$

Remark. If $A_{20} = 0$, one root $k = \infty$.

4. The compact form of the second stability condition

The second (sufficient) condition for asymptotic stability is:

$$S_2 = \left\{\frac{\partial f}{\partial a}\left(\frac{\partial g}{\partial u}v - \frac{\partial g}{\partial v}u\right) - \frac{\partial g}{\partial a}\left(\frac{\partial f}{\partial u}v - \frac{\partial f}{\partial v}u\right)\right\}_\theta > 0. \tag{4.1}$$

On C_1, i.e. for ordinary stationary oscillations, regarding to (3.1), S_2 can directly be expressed through (Δ, a):

$$S_2 = \left\{\frac{\partial f}{\partial a}\left(\frac{\partial g}{\partial u}v - \frac{\partial g}{\partial v}u\right) - \frac{\partial g}{\partial a}\left(\frac{\partial f}{\partial u}v - \frac{\partial f}{\partial v}u\right)\right\}_{u(\Delta,a),v(\Delta,a)} > 0. \tag{4.2}$$

We remark again that (1.3) is the solution of the "algebraic" system (1.1) in R_1. So, in R_1, we have:

$$\begin{aligned} f(\Delta, a, u(\Delta, a), v(\Delta, a)) &\equiv 0, \\ g(\Delta, a, u(\Delta, a), v(\Delta, a)) &\equiv 0. \end{aligned} \tag{4.3}$$

Differentiating (4.3) relative to a, we obtain

$$\begin{aligned} \frac{\partial f}{\partial a} &\equiv -\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial a}\right), \\ \frac{\partial g}{\partial a} &\equiv -\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial a}\right). \end{aligned} \tag{4.4}$$

Using (4.4), on R_1, the expression inside the curly bracket of (4.1) can be written as (u, v substituted by (1.3)):

$$\frac{\partial f}{\partial a}\left(\frac{\partial g}{\partial u}v - \frac{\partial g}{\partial v}u\right) - \frac{\partial g}{\partial a}\left(\frac{\partial f}{\partial u}v - \frac{\partial f}{\partial v}u\right) = \frac{1}{2}D_0\frac{\partial W_1}{\partial a}. \tag{4.5}$$

Obviously, (4.5) is valid on C_1, too. Therefore, for ordinary stationary oscillations, the second stability condition takes the form:

$$S_2(\Delta, a) = \frac{1}{2}D_0 \cdot \frac{\partial W_1}{\partial a} > 0. \tag{4.6}$$

On R_1, we have $W(\Delta, a) = D_0^2 W_1(\Delta, a)$; so, on R_1:

$$\frac{1}{2}D_0\frac{\partial W_1}{\partial a} = \frac{1}{2D_0}\frac{\partial W}{\partial a} - \frac{1}{D_0^2}W\frac{\partial D_0}{\partial a} \tag{4.7}$$

On C_1, $D_0 \neq 0$, $W = 0$, we have:

$$S_2(\Delta, a) = \frac{1}{2D_0}\frac{\partial W}{\partial a} > 0. \tag{4.8}$$

The stability of ordinary stationary oscillations is doubtful:

- At ordinary singular point where $D_0 \neq 0$, $\frac{\partial W}{\partial a} = 0$ $\left(\frac{\partial W}{\partial \Delta} = 0 \text{ too}\right)$

- At the ends (of ordinary portions) having vertical tangents (parallel to the a-axis): along C_1, the slope is determined by

$$\frac{\partial W}{\partial a}\frac{da}{d\Delta} + \frac{\partial W}{\partial \Delta} = 0; \quad \text{if } \frac{da}{d\Delta} \to 0, \text{ then } \quad \frac{\partial W}{\partial a} \to 0.$$

For critical stationary oscillations, (u, v) must be substituted by (u_*, v_*) which are the limits (along suitable trajectories) of $u(\Delta, a)$, $v(\Delta, a)$ respectively. Therefore, we can write:

$$S_{2*} = \lim\left\{\frac{\partial f}{\partial a}\left(\frac{\partial g}{\partial u}v - \frac{\partial g}{\partial v}u\right) - \frac{\partial g}{\partial a}\left(\frac{\partial f}{\partial u}v - \frac{\partial f}{\partial v}u\right)\right\} > 0. \tag{4.9}$$

Regarding to (4.7), S_{2*} can be written as:

$$S_{2*} = \lim\left\{\frac{1}{2D_0}\frac{\partial W}{\partial a}\right\} - \lim\left\{\frac{1}{D_0^2}W\frac{\partial D_0}{\partial a}\right\} > 0, \tag{4.10}$$

$\frac{\partial D_0}{\partial a}$ is a polynomial, its limit is finite. Otherwise,

$$\lim\frac{W}{D_0^2} = \lim\left(\frac{D_1^2}{D_0^2} + \frac{D_2^2}{D_0^2} - 1\right) = u_*^2 + v_*^2 - 1 = 0.$$

Finally, we have:

$$S_{2*} = \lim\left\{\frac{1}{2D_0}\frac{\partial W}{\partial a}\right\} > 0 \tag{4.11}$$

or

$$S_{2*} = \lim\left\{\frac{\partial D_1}{\partial a}\left(\frac{D_1}{D_0}\right) + \frac{\partial D_2}{\partial a}\left(\frac{D_2}{D_0}\right) - \frac{\partial D_0}{\partial a}\right\} > 0, \tag{4.12}$$

$$S_{2*} = \frac{1}{2}\left(\frac{d}{d\Delta}\frac{\partial W}{\partial a}\right)_* \Big/ \left(\frac{dD_0}{d\Delta}\right)_* = \frac{1}{2}\left(A_{11} + A_{20}k\right)_* \Big/ \left(\frac{\partial D_0}{\partial \Delta} + \frac{\partial D_0}{\partial a}k\right)_* > 0. \tag{4.13}$$

5. Stability of the critical nodal point

Let us examine in detail the stability (in the sense of the second stability condition) of a critical nodal point $I_*(\Delta_*, a_*)$ - the intersection point of two ordinary portions (1) (2) - at which $D_* > 0$.

The portions (1) (2) are just two suitable trajectories; I_* represents two critical stationary oscillations with dephases θ_{*i} corresponding to two slopes k_i $(i = 1, 2)$ of the portions (1) (2) respectively.

We examine the point $I_*^{(1)}$ (I_* considered as an element of (1)) and suppose that the portion (1) is stable i.e. along (1) we have $S_2 > 0$. By continuity:

$$S_{2*}^{(1)} = \lim S_2 \geq 0. \tag{5.1}$$

However, S_{2*} cannot be equal to zero. Indeed, $S_{2*}^{(1)} = 0$ leads to;

$$A_{20}k + A_{11} = 0. \tag{5.2}$$

From (5.2) (3.4) it follows $D_* = 0$, in contradiction with the hypothesis on nodal point.

Thus $I_*^{(1)}$ is stable as the portion (1). In general, the stability character of a nodal critical point I_* is the same as that of ordinary portion considered as containing it.

Remark 1. If the tangent of the portion (1) at I_* is vertical, $S_{2*}^{(1)} = 0$, the stability is doubtful. Indeed, $k_1 = 0$, the first coefficient of the quadratic equation (3.4) is equal to zero:

$$A_{20} = 2\Big\{ \Big(\frac{\partial D_1}{\partial a}\Big)^2 + \Big(\frac{\partial D_2}{\partial a}\Big)^2 - \Big(\frac{\partial D_0}{\partial a}\Big)^2 \Big\}_* = 0. \tag{5.3}$$

If $\Big(\dfrac{\partial D_0}{\partial a}\Big)_* = 0$, then $\Big(\dfrac{\partial D_1}{\partial a}\Big)_* = \Big(\dfrac{\partial D_2}{\partial a}\Big)_* = 0$, therefore (from (4.12)) $S_{2*}^{(1)} = 0$

If $\Big(\dfrac{\partial D_0}{\partial a}\Big)_* \neq 0$, from (3.7) (3.9) we deduce:

$$u_* = \frac{\Big(\dfrac{\partial D_1}{\partial a}\Big)_*}{\Big(\dfrac{\partial D_0}{\partial a}\Big)_*}, \quad v_* = \frac{\Big(\dfrac{\partial D_2}{\partial a}\Big)_*}{\Big(\dfrac{\partial D_0}{\partial a}\Big)_*}$$

and. from (4.12):

$$S_{2*}^{(1)} = \frac{1}{\Big(\dfrac{\partial D_0}{\partial a}\Big)_*} \Big\{ \Big(\frac{\partial D_1}{\partial a}\Big)_*^2 + \Big(\frac{\partial D_2}{\partial a}\Big)_*^2 - \Big(\frac{\partial D_0}{\partial a}\Big)_*^2 \Big\} = \frac{A_{20}}{2\Big(\dfrac{\partial D_0}{\partial a}\Big)_*} = 0. \tag{5.4}$$

Remark 2. If $D_* = 0$, the stability is doubtful, too. Indeed, in this case, there exists only one suitable trajectory with $k = -a_{11}/A_{20}$. Therefore, the numerator of (4.13) vanishes and we have $S_{2*} = 0$.

Conclusions

The analysis presented above show that, for critical stationary oscillations, the compact form of the second stability condition is the limit of that of the ordinary stationary oscillation. The limit must be done on suitable trajectories. For critical nodal points, the stability character can directly be deduced from that of the ordinary portion considered as containing them.

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References

1. Nguyen Van Dinh. Stationary oscillations in critical case (in Vietnamese). Journal of Mechanics, NCNST of Vietnam, T.XVIII, No 2, 1996.

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ỔN ĐỊNH CỦA DAO ĐỘNG DỪNG TỚI HẠN

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