# REDUCED IMPEDANCE OF BRANCH COMPONENT WITH HYPERSTATIC INTERFACE

NGUYEN THAC SI
*Hanoi University of Mining and Geology*

## 1. Introduction

An analytical technique proposed by Berman [1] allows an exact representation of a branch component within the model of the main structure by considering the impedance matrix of the component and its inverse.

This paper has its genesis in an early study [3] in which reduced impedance of a statically determinate branch component was found by using component modes. It is intended to apply the method to the case of hyperstatic interface.

## 2. Definition

Consider a structural system composed of the main component "$M$" and a branch component "$k$" as illustrated in Fig. 1.

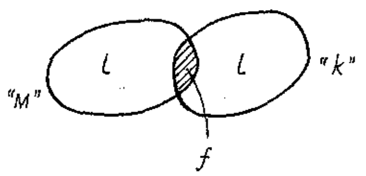


*Fig. 1*

It is convenient to rearrange and partition the elements of the two impedance matrices in the following way:

$$Z_M = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell f} \\ Z_{f\ell} & Z_{ff} \end{bmatrix} \quad \text{and} \quad Z_k = \begin{bmatrix} Z^k_{ff} & Z^k_{f\ell} \\ Z^k_{\ell f} & Z^k_{\ell\ell} \end{bmatrix},$$

where $f$ refers to interface coordinates and $\ell$ - to non-interface coordinates.

The impedance of the system, then, may be formed by superimposing these matrices in the form:

$$Z_s = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell f} & 0 \\ Z_{f\ell} & Z_{ff} + Z^k_{ff} & Z^k_{f\ell} \\ 0 & Z^k_{\ell f} & Z^k_{\ell\ell} \end{bmatrix}.$$

If a valid model of component "$k$" could be formed using only the interface coordinates, the impedance of the system could be written as

$$Z_s = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell f} \\ Z_{f\ell} & Z_{f\ell} + \tilde{Z}^k_{ff} \end{bmatrix},$$

where $\tilde{Z}^k_{ff}$ is called reduced impedance of component "$k$".

## 3. Reduced Impedance

Consider a branch component the interface of which is assumed hyperstatic. Its displacement may be expressed by the vector

$$q(t) = \begin{bmatrix} q_f \\ q_\ell \end{bmatrix} = \begin{bmatrix} q_r \\ q_c \\ q_\ell \end{bmatrix} \tag{3.1}$$

in which $q_r$ are rigid-body displacement on the connection interface and $q_c$ the remainder of displacements on the connection interface.

The equation of motion of the component is

$$M\ddot{q} + Kq = F, \tag{3.2}$$

where the mass, stiffness and force matrices are

$$M = \begin{bmatrix} M_{rr} & M_{rc} & M_{r\ell} \\ M_{cr} & M_{cc} & M_{c\ell} \\ M_{\ell r} & M_{\ell c} & M_{\ell\ell} \end{bmatrix} \; ; \quad K = \begin{bmatrix} K_{rr} & K_{rc} & K_{r\ell} \\ K_{cr} & K_{cc} & K_{c\ell} \\ K_{\ell r} & K_{\ell c} & K_{\ell\ell} \end{bmatrix} \; ; \quad F = \begin{bmatrix} F_r \\ F_c \\ 0 \end{bmatrix}.$$

The displacement of any point is found by superimposing the motion excited by the main component through the interface and the elastic motion relative to the latter, so that

$$\begin{bmatrix} q_r \\ q_c \\ q_\ell \end{bmatrix} = [\,\phi_r \quad \phi_c \quad \phi_p\,] \begin{bmatrix} q_r \\ q_c \\ y_p \end{bmatrix}, \tag{3.3}$$

$\phi_r$ is rigid-mode matrix which can be considered as having resulted from arbitrary displacement of each of the statically determinate constraints $q_r$. It can be partitioned as

$$\phi_r = \begin{bmatrix} I \\ \phi_{cr} \\ \phi_{\ell r} \end{bmatrix},$$

and it must satisfy the basic condition

$$K\phi_r = 0, \tag{3.4}$$

$\phi_c$ is the constraint-mode matrix, which is produced by giving each redundant constraint $q_c$ an arbitrary displacement while keeping all other constraints fixed:

$$\phi_c = \begin{bmatrix} 0 \\ I \\ \phi_{\ell c} \end{bmatrix},$$

$\phi_p$ is the natural mode matrix, which is formed by fixed-constraint natural modes of vibration of the component:

$$\phi_p = \begin{bmatrix} 0 \\ 0 \\ \phi_{\ell p} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ x_\ell^{(1)} & x_\ell^{(2)} & \dots & x_\ell^{(n)} \end{bmatrix},$$

where $x_\ell^{(i)}$ are eigenvectors of the equation

$$[K_{\ell\ell} - \lambda M_{\ell\ell}]x_\ell = 0,$$

and the corresponding eigenvalues will be designated by

$$\omega_1, \omega_2, \ldots, \omega_n.$$

Replacing $q$ by (3.3) in (3.2) then multiplying by $[\phi_r\ \phi_c\ \phi_p]^T$ the equation (3.2) becomes

$$\begin{bmatrix} m_{rr} & L_{cr}^T & L_{pr}^T \\ L_{cr} & m_{cc} & L_{pc}^T \\ L_{pr} & L_{pc} & m_{pp} \end{bmatrix} \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_c \\ \ddot{y}_p \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{cc} & 0 \\ 0 & 0 & k_{pp} \end{bmatrix} \begin{bmatrix} q_r \\ q_c \\ y_p \end{bmatrix} = \begin{bmatrix} F_r + \phi_{cr}^T F_c \\ F_c \\ 0 \end{bmatrix}, \tag{3.5}$$

where

$$\begin{aligned} m_{rr} &= \phi_r^T M \phi_r, & L_{cr} &= \phi_c^T M \phi_r, & k_{cc} &= \phi_c^T K \phi_c, \\ m_{cc} &= \phi_c^T M \phi_c, & L_{pr} &= \phi_p^T M \phi_r, & k_{pp} &= \phi_p^T K \phi_p, \\ m_{pp} &= \phi_p^T M \phi_p, & L_{pc} &= \phi_p^T M \phi_c. \end{aligned}$$

The equation (3.5) can be written in the form:

$$m_{rr}\ddot{q}_r + L_{cr}^T \ddot{q}_c + L_{pr}^T \ddot{y}_p = F_r + \phi_{cr}^T F_c, \tag{3.6}$$

$$L_{cr}\ddot{q}_r + m_{cc}\ddot{q}_c + L_{pc}^T \ddot{y}_p + k_{cc} q_c = F_c, \tag{3.7}$$

$$L_{pr}\ddot{q}_r + L_{pc}\ddot{q}_c + m_{pp}\ddot{y}_p + k_{pp} y_p = 0. \tag{3.8}$$

For a harmonic excitation

$$\begin{aligned} F_r &= \overline{F}_r e^{j\omega t}, \\ F_c &= \overline{F}_c e^{j\omega t}, \end{aligned} \tag{3.9}$$

a steady-state solution is assumed of the form:

$$\begin{aligned} q_r &= \overline{q}_r e^{j\omega t}, \\ q_c &= \overline{q}_c e^{j\omega t}, \\ y_p &= \overline{y}_p e^{j\omega t}. \end{aligned} \tag{3.10}$$

Substitution of (3.10) into (3.8) yields:

$$-\omega^2 L_{pr}\overline{q}_r - \omega^2 L_{pc}\overline{q}_c - \omega^2 m_{pp}\overline{y}_p + k_{pp}\overline{y}_p = 0$$

or, alternatively

$$-\omega^2 L_{ir}\overline{q}_r - \omega^2 L_{ic}\overline{q}_c - \omega^2 m_{ii}\overline{y}_i + k_{ii}\overline{y}_i = 0 \quad (i = 1, 2, \ldots, n).$$

From this and taking into account $k_{ii} = m_{ii}\omega_i^2$ we have:

$$\overline{y}_i = \frac{\omega^2}{(\omega_i^2 - \omega^2) m_{ii}} (L_{ir}\overline{q}_r + L_{ic}\overline{q}_c). \tag{3.11}$$

Introducing the notations:

$$\begin{aligned} \frac{1}{m_{ii}} L_{ir}^T L_{ir} &= M_{rr,i}, & \frac{1}{m_{ii}} L_{ic}^T L_{ic} &= M_{cc,i}, \\ \frac{1}{m_{ii}} L_{ic}^T L_{ir} &= M_{cr,i}, & \frac{1}{m_{ii}} L_{ir}^T L_{ic} &= M_{rc,i}, \end{aligned} \tag{3.12}$$

and substituting (3.10) into (3.7), taking into account (3.11) we obtain

$$\overline{F}_c = -\omega^2 \big[L_{cr} + \sum_{i=1}^{n} \frac{\omega^2}{\omega_i^2 - \omega^2} M_{cr,i}\big] \overline{q}_r - \omega^2 \big[m_{cc} - \frac{k_{cc}}{\omega^2} + \sum_{i=1}^{n} \frac{\omega^2}{\omega_i^2 - \omega^2} M_{cc,i}\big] \overline{q}_c. \tag{3.13}$$

Similarly, substitution of (3.10) into (3.6), taking into account (3.11) and (3.13) yields:

$$\begin{aligned} \overline{F}_r = &-\omega^2 \big[m_{rr} - \phi_{cr}^T L_{cr} + \sum_{i=1}^{n} \frac{\omega^2}{\omega_i^2 - \omega^2} (M_{rr,i} - \phi_{cr}^T M_{cr,i})\big] \overline{q}_r \\ &- \omega^2 \big[L_{cr}^T - \phi_{cr}^T m_{cc} + \phi_{cr}^T \frac{k_{cc}}{\omega^2} + \sum_{i=1}^{n} \frac{\omega^2}{\omega_i^2 - \omega^2} (M_{cr,i} - \phi_{cr}^T M_{cc,i})\big] \overline{q}_c. \end{aligned} \tag{3.14}$$

The equations (3.13) and (3.14) can be rewritten in the form:

$$\begin{bmatrix} \overline{F}_r \\ \overline{F}_c \end{bmatrix} = \begin{bmatrix} Z_{rr} & Z_{rc} \\ Z_{cr} & Z_{cc} \end{bmatrix} \begin{bmatrix} \overline{q}_r \\ \overline{q}_c \end{bmatrix}. \tag{3.15}$$

According to the definition of reduced impedance, it follows from (3.15) that

$$\tilde{Z}_{ff}^k = \begin{bmatrix} Z_{rr} & Z_{rc} \\ Z_{cr} & Z_{cc} \end{bmatrix},$$

the elements of which are defined in (3.13) and (3.14).

## 4. Example

Consider a simple rod as a branch component which is assumed to interface a main component at both its ends and to be submitted to a longitudinal motion $u(x)$ (Fig. 2).

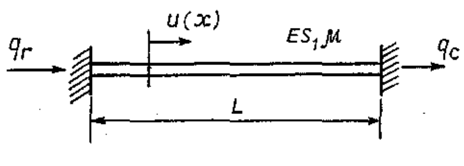


*Fig. 2*

The equation of motion of the rod is:

$$ESu'' - \mu\ddot{u} = 0.$$

From which we have the modal frequencies

$$\omega_i = \frac{i\pi}{L}\sqrt{\frac{ES}{\mu}} \ ,$$

and the corresponding mode shapes and the generalized masses:

$$\varphi_i(x) = C \sin \frac{i\pi}{L} x,$$

$$m_{ii} = \int_0^L \mu \varphi_i^2 dx = \frac{1}{2} C^2 \mu L.$$

The rigid-body mode is given as

$$\phi_r(x) = 1,$$

from which we obtain the rigid-body mass

$$m_{rr} = \int_0^L \mu \phi_r \phi_r dx = \mu L.$$

The constraint mode takes the form:

$$\phi_c(x) = \frac{x}{L} \ ,$$

from which it follows that

$$m_{cc} = \int_0^L \mu \phi_c \phi_c dx = \frac{1}{3} \mu L,$$

$$k_{cc} = \int_0^L ES \phi_c' \phi_c' dx = \frac{ES}{L} \ ,$$

$$L_{cr} = \int_0^L \mu \phi_c \phi_r dx = \frac{1}{2} \mu L,$$

$$\phi_{cr} = 1,$$

and

$$L_{ir} = \int_0^L \mu \varphi_i \phi_r dx = C \mu L \frac{1 - \cos i\pi}{i\pi} \ ,$$

$$L_{ic} = \int_0^L \mu \varphi_i \phi_c dx = -C \mu L \frac{\cos i\pi}{i\pi} \ ,$$

$$M_{rr,i} = 2\mu L \left( \frac{1 - \cos i\pi}{i\pi} \right)^2,$$

$$M_{cc,i} = 2\mu L \frac{1}{i^2 \pi^2} \ ,$$

$$M_{rc,i} = M_{cr,i} = 2\mu L \frac{1 - \cos i\pi}{i^2 \pi^2} \ .$$

Finally, we obtain the elements of reduced impedance

$$Z_{rr} = -\mu L\omega^2\Big[\frac{1}{2} + 2\sum_{i=1}^{\infty}\frac{\omega^2}{\omega_i^2-\omega^2}\,\frac{1-\cos i\pi}{i^2\pi^2}\Big],$$

$$Z_{rc} = -\mu L\omega^2\Big[\frac{1}{6} - 2\sum_{i=1}^{\infty}\frac{\omega^2}{\omega_i^2-\omega^2}\,\frac{2\cos i\pi}{i^2\pi^2}\Big] - \frac{ES}{L},$$

$$Z_{cc} = -\mu L\omega^2\Big[\frac{1}{3} - 2\sum_{i=1}^{\infty}\frac{\omega^2}{\omega_i^2-\omega^2}\,\frac{1}{i^2\pi^2}\Big] + \frac{ES}{L},$$

$$Z_{cr} = -\mu L\omega^2\Big[\frac{1}{2} + 2\sum_{i=1}^{\infty}\frac{\omega^2}{\omega_i^2-\omega^2}\,\frac{1-\cos i\pi}{i^2\pi^2}\Big].$$

## 5. Conclusions

A method has been developed for determining the reduced impedance of a branch component attached hyperstatically to the main component by using modal synthesis techniques. As an illustration of the method a simple longitudinal model was analyzed as a branch component.

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## TRỞ KHÁNG THU GỌN NHÁNH CÓ TIẾP NỐI SIÊU TĨNH

Một phương pháp xác định trở kháng thu gọn của nhánh có tiếp nối siêu tĩnh với cấu trúc chính đã được trình bày. Phương pháp dựa trên kỹ thuật tổng hợp các dạng thành phần. Một mô hình nhánh đơn giản dưới dạng thanh đã được khảo sát như một ví dụ.