Tạp chí Cơ học Journal of Mechanics, NCNST of Vietnam T. XIX, 1997, No 1 (27–33)

THE VANDERPOL'S SYSTEM UNDER EXTERNAL AND PARAMETRIC EXCITATIONS

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The Vanderpol's system subjected to external and parametric excitations has been studied [1, 2, 3]. In the present paper, we shall consider the case when these two excitations simultaneously act on the system of interest: the first excitation is external and in the fundamental resonance (order 1) and the second one is parametric and in the subharmonic resonance of order 1/2. Critical singular points will be used to classify different forms of the resonance curve [4].

1. System under consideration - Ordinary and critical stationary oscillations

Let us consider a quasilinear oscillating system described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{\Delta x + \alpha(1 - 4x^2)\dot{x} + 2px\cos 2\omega t + e\cos(\omega t + \sigma)\Big\}, \tag{1.1}$$

where : x is an oscillatory variable; overdots denote the differentiation with respect to time t; $\varepsilon > 0$ is a small parameter; $\alpha > 0$ is the coefficient characterizing the self-excitation; (e, ω) $(2p, 2\omega)$ are intensities, frequencies of the external and parametric excitations, respectively; $e > 0$, $p > 0$; σ $(0 \le \sigma < 2\pi)$ is the dephase between two excitations; $\varepsilon\Delta = \varepsilon(\omega^2 - 1)$ is the detuning parameter (1 - own frequency)

Introducing slowly varying amplitude and phase (a, θ) by means of formulae:

$$x = a\cos\psi, \quad \dot{x} = -\omega a\sin\psi, \quad \psi = \omega t + \theta, \tag{1.2}$$

and using the averaging method, we obtain for a and θ the averaged differential equations:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{2\omega}f_0, \quad f_0 = \alpha\omega a(a^2 - 1) + pa\sin 2\theta + e\sin(\theta - \sigma), \\ a\dot{\theta} &= -\frac{\varepsilon}{2\omega}g_0, \quad g_0 = \Delta a + pa\cos 2\theta + e\cos(\theta - \sigma). \end{aligned} \tag{1.3}$$

The constant amplitude and phase (a, θ) of the stationary oscillations will be determined from the equations:

$$f_0 = 0, \quad g_0 = 0 \tag{1.4}$$

or by their equivalents:

$$\begin{aligned} f &= f_0\cos\theta - g_0\sin\theta = (p - \Delta)a\sin\theta + \alpha\omega a(a^2 - 1)\cos\theta - e\sin\sigma = 0, \\ g &= f_0\sin\theta + g_0\cos\theta = \alpha\omega a(a^2 - 1)\sin\theta - (p + \Delta)a\cos\theta + e\cos\sigma = 0. \end{aligned} \tag{1.5}$$

By $\overline{D}_0$, $\overline{D}_1$, $\overline{D}_2$ and D_0, D_1, D_2 we denote following determinants:

$$\overline{D}_0 = a^2 D_0, \quad D_0 = \begin{vmatrix} p - \Delta & \alpha\omega(a^2-1) \\ \alpha\omega(a^2-1) & p+\Delta \end{vmatrix}, \tag{1.6}$$

$$\overline{D}_1 = aeD_1, \quad D_1 = \begin{vmatrix} \sin\sigma & \alpha\omega(a^2-1), \\ -\cos\sigma & p+\Delta \end{vmatrix}, \quad \overline{D}_2 = aeD_2, \quad D_2 = \begin{vmatrix} p-\Delta & \sin\sigma \\ \alpha\omega(a^2-1) & -\cos\sigma \end{vmatrix}.$$

In the ordinary region where

$$D_0 \neq 0 \tag{1.7}$$

from (1.5), we can calculate $(\sin\theta, \cos\theta)$ and the ordinary part C_1 of the resonance curve C is given by:

$$W_1(\Delta, a^2) = \frac{e^2(D_1^2 + D_2^2)}{a^2 D_0^2} - 1 = 0. \tag{1.8}$$

The critical region is characterized by the equality:

$$D_0 = 0. \tag{1.9}$$

It is the resonance curve C_0 of the Vanderpol's system subjected only to the parametric excitation $2px\cos 2\omega t$ $(e = 0)$.

To determine the critical part C_2 of the resonance curve, we have to solve the system:

$$D_0 = 0, \quad D_1 = 0, \quad D_2 = 0 \tag{1.10}$$

under the restrictions:

$$\begin{aligned} a^2\{(p-\Delta)^2 + \alpha^2\omega^2(a^2-1)^2\} &\geq e^2\sin^2\sigma, \\ a^2\{\alpha^2\omega^2(a^2-1)^2 + (p+\Delta)^2\} &\geq e^2\cos^2\sigma. \end{aligned} \tag{1.11}$$

From (1.10) we obtain a (compatible) point I_* of coordinates

$$\Delta_* = p\cos 2\sigma, \quad a_*^2 = 1 - \frac{p\sin 2\sigma}{\alpha\sqrt{1 + p\cos 2\sigma}} \tag{1.12}$$

for which, the restrictions (1.11) lead to an unique inequality:

$$a_*^2 \geq \frac{e^2}{4p^2} \cdot \tag{1.13}$$

Thus, if (1.13) is satisfied, the critical part C_2 consists of an unique point I_*.

By rejecting those points satisfying (1.10) but not (1.11), the whole resonance curve C $(C_1 + I_*)$ can be found from the relationship:

$$W(\Delta, a^2) = e^2(D_1^2 + D_2^2) - a^2 D_0^2 = 0. \tag{1.14}$$

I_* is a nodal point if:

$$D > 0, \quad \text{where} \quad D = \left(\frac{\partial^2 W}{\partial\Delta\partial a^2}\right)^2 - \left(\frac{\partial^2 W}{\partial\Delta^2}\right)\left(\frac{\partial^2 W}{\partial(a^2)^2}\right) \cdot \tag{1.15}$$

If $D < 0$, I_* is an isolated point and does not belong to the resonance curve.

2. Different forms of the resonance curve

The equality (1.9) can be written as:

$$a^2 = 1 \pm \frac{1}{\alpha}\sqrt{\frac{p^2 - \Delta^2}{1 + \Delta}}\,.$$

Thus, the critical region C_0 is a closed curve - an "oval" of center $(\Delta = 0,\ a_0^2 = 1)$.

If $p^2 < \alpha^2 - \dfrac{\alpha^4}{4}$, C_0 lies above the abscissa - axis Δ. For very small values e, the resonance curve C consists of two branches: the lower C' and the upper C''. The lower branch C' corresponds to very small values a^2. The upper branch C'' consists of two loops, lying respectively "inside" and "outside" C_0. These two loops are joined at the critical nodal point I_*.

Increasing e, C'' becomes larger. At certain values e_j, C' joins to C'' at ordinary singular point J. As e exceeds e_j, J disappears.

If $e > e_* = 4p^2 a_*^2$, I_* becomes an isolated point, the "inside" loop will either disappear or change into an closed branch.

If $p^2 \geq \alpha^2 - \dfrac{\alpha^4}{4}$, the abscissa - axis Δ intersects C_0.

In Fig. 1, for fixed values ($\sigma = 0$; $\alpha = 0.1$; $p = 0.05$) the resonance curves (0)–(5) correspond to $e = 0$; 0.015; 0.0177; 0.05; 0.1; 0.12 respectively. The curve (0) represents the critical region C_0. The resonance curve (1) consists of two branches C' and C''. For $e \approx 0.0177$, C' joints to C'' at an ordinary singular point J. Increasing e, J disappears and the resonance curve will be of form (3) corresponding to $e = 0.05$. When e reaches the value $e = 0.1$, the "inside" loop is reduced to the returning point I_*. Increasing e further I_* becomes an isolated point, the resonance curve takes the form(5) corresponding to $e = 0.12$.

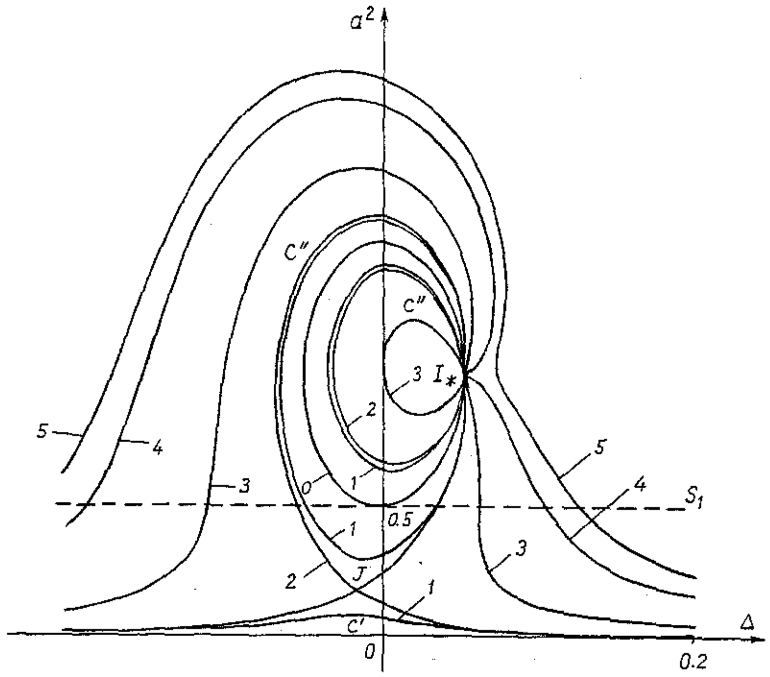

Fig. 1

In Fig. 2, for fixed values ($\sigma = \frac{\pi}{4}$; $\alpha = 0.1; p = 0.05$) the resonance curves (0)–(7) are plotted for $e = 0$; 0.04; 0,0483; 0.05; 0.0516; 0.055; 0.0648; 0.98 respectively. There are ordinary singular points when $e \approx 0.0483$ or $e \approx 0.0516$ (curve (2) and (4)) and new lower loops for $e = 0.055$; 0.0648. I_* is an isolated point for $e = 0.08$.

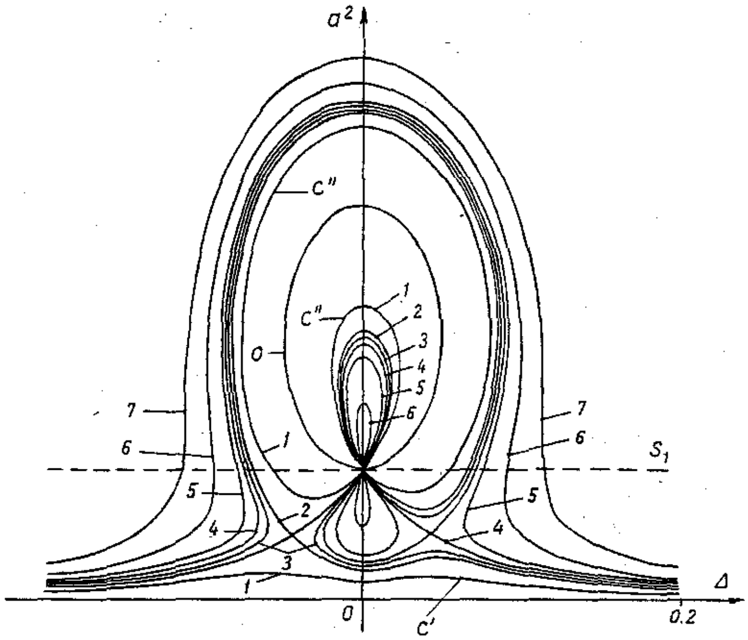

Fig. 2

In Fig. 3, for fixed values $\alpha = 0.1$; $p = 0.12$, the curve (0) corresponds to $e = 0$ and the resonance curves (1)–(4) correspond to $e = 0.06$ and $\sigma = 0$; $\sigma = \frac{\pi}{12}$; $\sigma = \frac{\pi}{6}$; $\sigma = \frac{\pi}{4}$ respectively.

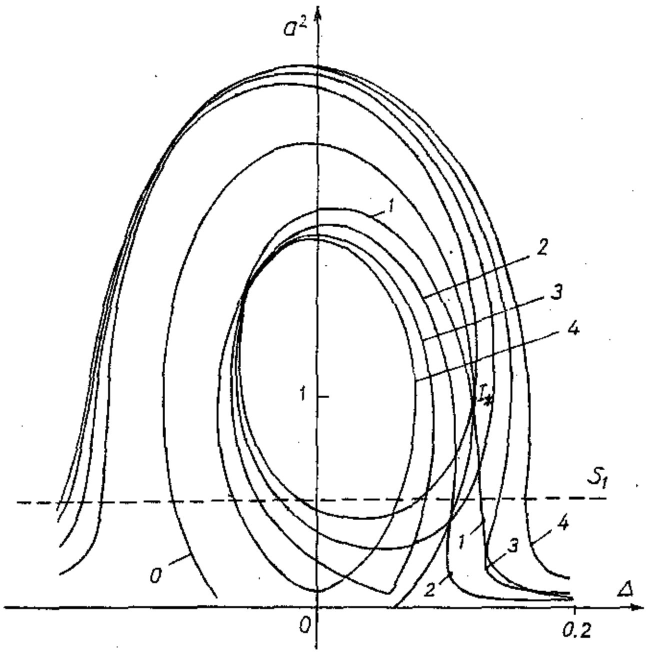

Fig. 3

The resonance curves (1) and (2) have nodal points; the resonance curves (3) and (4) have "inside" closed branches.

When σ varies, the critical singular point I_* moves along C_0. This can easily be seen in Fig. 3 as well in Fig. 4; the latter has been drawn for $\alpha = 0.1$, $p = 0.05$, $e = 0.05$ and respectively for $\sigma = 0$ (1), $\sigma = \frac{\pi}{4}$ (2), $\sigma = \frac{\pi}{2}$, $\sigma = \frac{3\pi}{4}$.

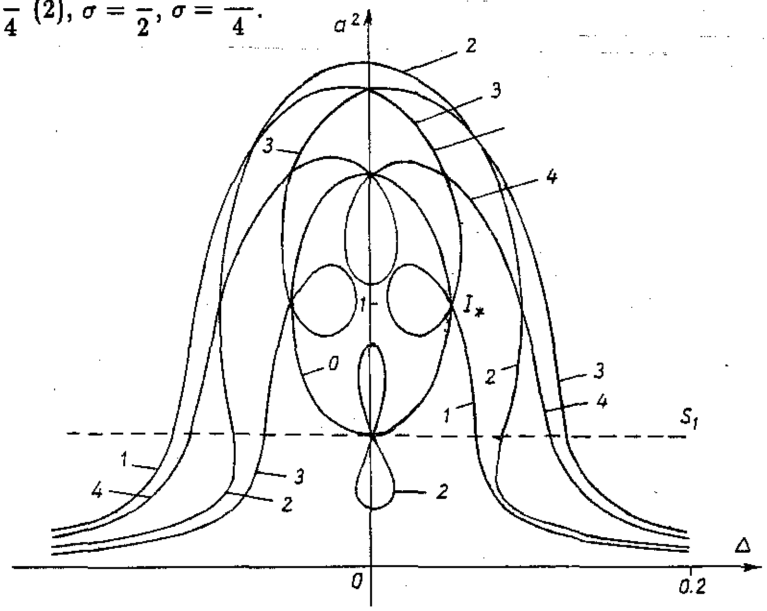

Fig. 4

3. System with cubic non-linearity

The results obtained can be generalized for the system with cubic non-linearity. In this case, the governing differential equation is of the form:

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{\Delta x - \frac{4}{3}\gamma x^3 + \alpha(1 - 4x^2)\dot{x} + 2px\cos 2\omega t + e\cos(\omega t + \sigma)\Big\}, \tag{3.1}$$

where $\frac{4}{3}\gamma$ is the coefficient of the cubic nonlinearity.

The averaged differential equations become:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{2\omega} f_0 = -\frac{\varepsilon}{2\omega}\Big\{\alpha\omega a(a^2 - 1) + pa\sin 2\theta + e\sin(\theta - \sigma)\Big\}, \\ a\dot{\theta} &= -\frac{\varepsilon}{2\omega} g_0 = -\frac{\varepsilon}{2\omega}\Big\{ - (\gamma a^2 - \Delta)a + pa\cos 2\theta + e\cos(\theta - \sigma)\Big\}. \end{aligned} \tag{3.2}$$

Stationary oscillations of constant amplitude and phase will be determined from the equations:

$$\begin{aligned} f &= f_0\cos\theta - g_0\sin\theta = \big[(\gamma a^2 - \Delta) + p\big]a\sin\theta + \alpha\omega(a^2 - 1)a\cos\theta - e\sin\sigma = 0, \\ g &= f_0\sin\theta = \alpha\omega a(a^2 - 1)\sin\theta - \big[(\gamma a^2 - \Delta) - p\big]a\cos\theta + e\cos\sigma = 0. \end{aligned} \tag{3.3}$$

The critical region is characterized by the equality:

$$D_0 = p^2 - (\gamma a^2 - \Delta)^2 - \alpha^2\omega^2(a^2 - 1)^2 = 0. \tag{3.4}$$

The coordinates (Δ_*, a_*^2) of the critical singular point I_* will be determined from:

$$\begin{aligned} \Delta_* &= \gamma a_*^2 + p\cos 2\sigma, \\ \alpha\omega_*(a_*^2 - 1) &= -p\sin 2\sigma. \end{aligned} \tag{3.5}$$

liminating Δ_* leads to the equation:

$$\gamma\alpha^2(a_*^2-1)^3 + \alpha^2(1+\gamma+p\cos 2\sigma)(a_*^2-1)^2 - p^2\sin^2 2\sigma = 0. \tag{3.6}$$

If α, p, γ are relatively small in comparison with unity, we have:

- for $\sigma = 0$, $a_*^2 = 1$, $\Delta_* = \gamma + p$,
- for $\sigma = \pi$, $a_*^2 = 1$, $\Delta_* = \gamma - p$,
- for $\sin 2\sigma > 0$, not more one solution $0 < a_*^2 < 1$,
- for $\sin 2\sigma < 0$, not more one solution $a_*^2 > 1$.

In Fig. 5, for $\sigma = 0$; $\alpha = 0.1$; $p = 0.05$; $\gamma = 0.05$, the curve (0) corresponds to $e = 0$ and the rves (1) (2) correspond to $e = 0.012$; 0.02 respectively. The resonance curves lean to the right. 's the main effect of the cubic non-linearity of hard kind ($\gamma > 0$).

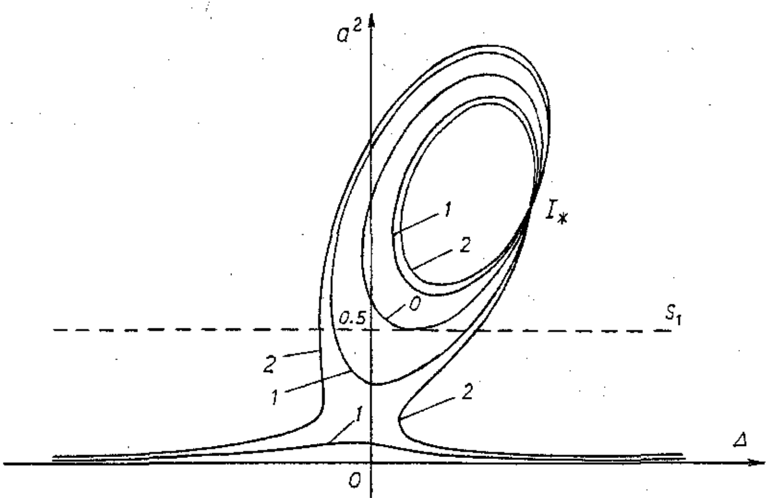

Fig. 5

Stability conditions

To study the stability of the stationary oscillations, we use the variational equations:

$$\delta\dot{a} = \frac{-\varepsilon}{2\omega}\frac{\partial f_0}{\partial a}\delta a - \frac{e}{2\omega}\frac{\partial f_0}{\partial\theta}\delta\theta, \quad a\delta\dot{\theta} = -\frac{\varepsilon}{2\omega}\frac{\partial g_0}{\partial a}\delta a - \frac{\varepsilon}{2\omega}\frac{\partial g_0}{\partial\theta}\delta\theta, \tag{4.1}$$

ıere δa, $\delta\theta$ are small perturbations of a, θ respectively.

The characteristic equation is of the form:

$$a\rho^2 + \frac{\varepsilon}{2\omega}S_1\rho + \frac{\varepsilon^2}{4\omega^2}S_2 = 0, \tag{4.2}$$

d stationary oscillations are asymptotically stable if:

$$S_1 = a\frac{\partial f_0}{\partial a} + \frac{\partial g_0}{\partial\theta} - > 0, \quad S_2 = \frac{\partial f_0}{\partial a}\frac{\partial g_0}{\partial\theta} - \frac{\partial f_0}{\partial\theta}\frac{\partial g_0}{\partial a} > 0. \tag{4.3}$$

The first stability condition is:

$$a^2 > \frac{1}{2}\,. \tag{4.4}$$

In figs 1-5, the dashed line S_1 is of equation $a^2 = \dfrac{1}{2}$ and the region lying above S_1 satisfies (4.4).

The second stability condition can be written as:

$$S_2 = \frac{\partial f}{\partial a}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial a} > 0 \tag{4.5}$$

or in compact form:

$$\frac{1}{D_0} \cdot \frac{\partial W}{\partial a^2} > 0. \tag{4.6}$$

The latter form is valid for ordinary stationary oscillations and determines, in the ordinary part C_1, stable portions of the resonance curve with vertical tangents at the ends. The critical nodal point is of the same stability character as the ordinary portion considered as containing it.

Conclusions

The Vanderpol's system subjected simultaneously to external and parametric excitations in the resonances of orders 1 and 1/2 has been examined. By using critical singular points different forms of the resonance curve can be distinguished.

This publication is completed with financial support from the National Basic Research Programme in Natural Sciences

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Received January 3, 1997

HỆ VANDERPOL DƯỚI KÍCH ĐỘNG CƯỠNG BỨC VÀ THÔNG SỐ Ở CỘNG HƯỞNG BẬC 1 VÀ 1/2

Khảo sát hệ Vanderpol chịu đồng thời các kích động cưỡng bức và thông số tương ứng ở cộng hưởng bậc 1 và 1/2. Sử dụng điểm tới hạn tương ứng dao động dừng tới hạn đã phân loại các dạng đường cộng hưởng của hệ.