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# ON PERIODIC WAVES OF THE NONLINEAR SYSTEMS

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**ABSTRACT.** The paper is concerned with the solvability and approximate solution of the nonlinear partial differential equation describing the periodic wave propagation.

Necessary and sufficient conditions for the existence of the periodic wave solutions are obtained.

An approximate method for solving the equation is presented.

As an illustrative example, the equation of periodic waves of the electric cables is considered.

**1.** Consider the nonlinear wave equation of the form

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + b^2 y + \varepsilon f\Big(y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\Big), \tag{1.1}$$

where $\varepsilon$ is a small parameter, $a$ and $b$ are two positive constants; The function $f\Big(y, \dfrac{\partial y}{\partial t}, \dfrac{\partial y}{\partial x}\Big)$ is assumed to be continuous and Lipschitzian with respect to its all arguments.

When $\varepsilon = 0$ the equation (1.1) can be expressed in the form (degenerate equation)

$$\frac{\partial^2 y_0}{\partial t^2} = a^2 \frac{\partial^2 y_0}{\partial x^2} + b^2 y_0, \tag{1.2}$$

with the initial conditions of the problem given in the type

$$\begin{aligned} y_0(x,0) &= A\cos\omega(x+\varphi),\\ \frac{\partial y_0(x,0)}{\partial t} &= \lambda A \sin\omega(x+\varphi), \end{aligned} \tag{1.3}$$

where $\lambda^2 = a^2\omega^2 - b^2$; $A$, $\varphi$ are arbitrary constants. It is easy to see that the equation (1.2) admits the family of periodic solutions depended upon two constants in the form

$$y_0(x.t) = A\cos\omega(x - ct + \varphi), \tag{1.4}$$

where $c = \frac{\lambda}{\omega}$ is velocity of wave propagation, $\omega$ is the phase constant.

Our investigation is following:

a) To show the necessary and sufficient conditions for the existence of the periodic wave solution of the nonlinear equation (1.1) corresponding to the certain solution of the family (1.4) that is the obtained periodic wave solution of the equation (1.1) reduces to one solution of the family (1.4) when $\varepsilon \to 0$.

b) To give the constructing approximate expression of the periodic wave solution of the nonlinear equation (1.1). Analogous investigation for the periodic oscillatory solutions of the nonlinear equation of the form (1.1) has been dealt with in [1, 2, 4].

**2.** For this aim we consider first the following associative equation:

$$\left[(c+\varepsilon\alpha)^2 - a^2\right]\frac{d^2u}{d\psi^2} - b^2u = \varepsilon F\left[u, -(c+\varepsilon\alpha)\frac{du}{d\psi}, \frac{du}{d\psi}\right] \tag{2.1}$$

where $u = u(\psi, \varepsilon)$; $\psi = x - (c+\varepsilon\alpha)t + \varphi$; $\alpha = \alpha(\varepsilon)$; $F = f$ after the substitution $y$ by $u$; $\frac{\partial y}{\partial t}$ by $-(c+\varepsilon\alpha)\frac{du}{d\psi}$; $\frac{\partial y}{\partial x}$ by $\frac{du}{d\psi}$, $\alpha$ is now an indeterminate constant.

With $\varepsilon = 0$ the equation (2.1) has the form (degenerate equation for (2.1))

$$(c^2 - a^2)\frac{d^2u_0}{d\psi^2} - b^2u_0 = 0$$

or

$$\frac{d^2u_0}{d\psi^2} + \omega^2u_0 = 0. \tag{2.2}$$

Obviously the equation (2.2) admits the periodic solution

$$u_0(\psi) = A\cos\omega\psi; \tag{2.3}$$

where $\psi = x - ct + \varphi$, $A$ is an arbitrary constant. We obtain here the following results:

**Theorem 1.** *If $u = u(\psi, \varepsilon)$ is a periodic solution of the equation* (2.1) *then $y$ defined by the relation $y(x,t,\varepsilon) = u[(x - (c+\varepsilon\alpha)t + \varphi, \varepsilon]$ is a periodic wave solution of the equation* (1.1).

The proof is straightforward.

**Theorem 2.** *Let $u = u(\psi, \varepsilon)$ be a periodic solution with respect to $\psi$ of period $2\pi/\omega$ of the equation* (2.1) *corresponding to a certain solution of the family* (1.4).

*One obtains then the relations*

$$\int\limits_0^{2\pi/\omega} F_0 \sin\omega\psi d\psi = 0,$$
$$\int\limits_0^{2\pi/\omega} F_0 \cos\omega\psi d\psi + 2\pi c\alpha_0\omega^2 A = 0 \tag{2.4}$$

*where*

$$F_0 = F\Big(u_0, -\frac{du_0}{d\psi}\cdot c, \frac{du_0}{d\psi}\Big)\,. \tag{2.5}$$

*Proof.* Suppose that the equation (2.1) has the periodic solution $u = u_*(\psi,\varepsilon)$. It is easy to see that the periodic function $u_*(\psi,\varepsilon)$ is also the periodic solution of the linear nonhomogenuous differential equation of the form

$$\frac{d^2u}{d\psi^2} + \omega^2 u = -\frac{\varepsilon\omega^2}{b^2}\Big[F_* - (2c\alpha + \varepsilon\alpha^2)\frac{d^2u_*}{d\psi^2}\Big], \tag{2.6}$$

where

$$F_* = F\Big[u_*, -(c+\varepsilon\alpha)\frac{du_*}{d\psi}\,, \frac{du_*}{d\psi}\Big]$$

Consequently the following equalities will be held:

$$\int\limits_0^{2\pi/\omega} \Big[F_* - (2c\alpha + \varepsilon\alpha^2)\frac{d^2u_*}{d\psi^2}\Big] \sin\omega\psi d\psi = 0,$$
$$\int\limits_0^{2\pi/\omega} \Big[F_* - (2c\alpha + \varepsilon\alpha^2)\frac{d^2u_*}{d\psi^2}\Big] \cos\omega\psi d\psi = 0, \tag{2.7}$$

Giving $\varepsilon \to 0$, $u_*(\psi,\varepsilon)$ reduces to $u_0(\psi)$, $F_*$ reduces to $\alpha_0$ and the equalities (2.7) will have the form (2.4).

The equalities (2.4) are the system of finite equations determining the constants $A$ and $\alpha_0$, $\varphi$ is still an arbitrary constant. Let $A = A_0$, $\alpha_0$ be the roots of this system. We have here

$$\int\limits_0^{\frac{2\pi}{\omega}} F\big(A_0\cos\omega\psi, c\omega A_0 \sin\omega\psi, -\omega A_0\sin\omega\psi\big)\sin\omega\psi d\psi = 0, \tag{2.8}$$

$$\alpha_0 = -\frac{1}{2\pi c\omega^2 A_0} \int_0^{2\pi/\omega} F\left(A_0 \cos\omega\psi, c\omega A_0 \sin\omega\psi, -\omega A_0 \sin\omega\psi\right) \cos\omega t$$

From the theorems 1 and 2 we can state.

**Theorem 3.** The necessary condition for the existence of the periodic wave solution $y = y(x, t, \varepsilon)$ of the nonlinear equation (1.1) corresponding to the solution $y_0(x, t) = A_0 \cos\omega(x - ct + \varphi)$ of the linear equation (2.2) is that $A_0$ and $\alpha_0$ satisfy the finite equations (2.8).

To the equation (2.1) we associate now an auxiliary system of integro - differential equations of the form

$$\begin{aligned}
\frac{d^2V}{d\psi^2} + \omega^2 V &= -\frac{\varepsilon\omega^2}{b^2}\left[F - (2c\alpha + \varepsilon\alpha^2)\frac{d^2V}{d\psi^2}\right] + P\cos\omega\psi + Q\sin\omega\psi, \\
P &= -\frac{\varepsilon\omega}{\pi} \int_0^{2\pi/\omega} \left[F - (2c\alpha + \varepsilon\alpha^2)\frac{d^2V}{d\psi^2}\right] \cos\omega\psi d\psi, \\
Q &= -\frac{\varepsilon\omega}{\pi} \int_0^{2\pi/\omega} \left[F - (2c\alpha + \varepsilon\alpha^2)\frac{d^2V}{d\psi^2}\right] \sin\omega\psi d\psi.
\end{aligned} \tag{2.9}$$

We have

**Theorem 4.** *The system of equation (2.9) always has the family of $\frac{2\pi}{\omega}$-periodic solutions depended upon two arbitrary constants $A$ and $\alpha$.*

*These solutions reduce to (2.3) when $\varepsilon \to 0$.*

*Proof.* The theorem will be proved by the method of iterations.

For the approximation of 0-th order of the solutions of the system of equations (2.9) we take

$$\begin{aligned}
V_0(\psi) &= A_0 \cos\omega\psi, \\
P_0 &= Q_0 = 0.
\end{aligned} \tag{2.10}$$

For the approximation of $n$-th order of these solutions we take the $\frac{2\pi}{\omega}$ periodic solutions of the following linear system of equations:

$$\frac{d^2V_n}{d\psi^2} + \omega^2 V_n = -\frac{\varepsilon\omega^2}{b^2}\left[F_{n-1} - (2c\alpha + \varepsilon\alpha^2)\frac{d^2V_{n-1}}{d\psi^2}\right] + $$
$$+ P_n \cos\omega\psi + Q_n \sin\omega\psi,$$
$$P_n = -\frac{\varepsilon\omega}{\pi}\int\limits_0^{2\pi/\omega}\left[F_{n-1} - (2c\alpha + \varepsilon\alpha^2)\frac{d^2V_{n-1}}{d\psi^2}\right]\cos\omega\psi d\psi, \tag{2.11}$$
$$Q_n = -\frac{\varepsilon\omega}{\pi}\int\limits_0^{2\pi/\omega}\left[F_{n-1} - (2c\alpha + c\alpha^2)\frac{d^2V_{n-1}}{d\psi^2}\right]\sin\omega\psi d\psi,$$

where

$$F_{n-1} = F\left[V_{n-1}, -(c+\varepsilon\alpha)\frac{dV_{n-1}}{d\psi}, \frac{dV_{n-1}}{d\psi}\right].$$

The system of equation (2.11) obviously has the family of $2\pi/\omega$ periodic solutions depended upon two arbitrary constants $A$ and $\alpha$.

We establish now the sequences of functions $\{V_n(\psi,\varepsilon,A,\alpha)\}$, $\{P_n(\varepsilon,A,\alpha)\}$, $\{Q_n(\varepsilon,A,\alpha)\}$ and prove their uniform convergence if the functions $V_*(\psi,\varepsilon,A,\alpha)$, $P_*(\varepsilon,A,\alpha)$, $Q_*(\varepsilon,A,\alpha)$ are their limits, then it is easy to see that these functions satisfy the system of equations (2.11).

For this purpose we make the transformation

$$V_n - V_{n-1} = X_n\cos\omega\psi + Y_n\sin\omega\psi,$$
$$\frac{dV_1}{d\psi} - \frac{dV_{n-1}}{d\psi} = -X_n\omega\sin\omega\psi + Y_n\omega\cos\omega\psi. \tag{2.12}$$

The system of the equations (2.11) then can be denoted in the form

$$X_n(\psi) = X_n(\psi_0) + \frac{\varepsilon\omega}{b^2}\int\limits_{\psi_0}^{\psi}\Phi_n\sin\omega\psi d\psi,$$
$$Y_n(\psi) = Y_n(\psi_0) - \frac{\varepsilon\omega}{b^2}\int\limits_{\psi_0}^{\psi}\Phi_n\cos\omega\psi d\psi, \tag{2.13}$$

where

$$\Phi_n = F_n - F_{n-1} - (2c\alpha + \varepsilon\alpha^2)\Big(\frac{d^2V_{n-1}}{d\psi^2} - \frac{d^2V_{n-2}}{d\psi^2}\Big) +$$
$$+ (P_n - P_{n-1})\cos\omega\psi + (Q_n - Q_{n-1})\sin\omega\psi,$$
$$P_n - P_{n-1}$$
$$= -\frac{\varepsilon\omega}{\pi}\int_0^{2\pi/\omega}\Big[F_{n-1} - F_{n-2} - (2c\alpha + \varepsilon\alpha^2)\Big(\frac{d^2V_{n-1}}{d\psi^2} - \frac{d^2V_{n-2}}{d\psi^2}\Big)\Big]\cos\omega\psi,$$
$$Q_n - Q_{n-1}$$
$$= -\frac{\varepsilon\omega}{\pi}\int_0^{2\pi/\omega}\Big[F_{n-1} - F_{n-2} - (2c\alpha + \varepsilon\alpha^2)\Big(\frac{d^2V_{n-1}}{d\psi^2} - \frac{d^2V_{n-2}}{d\psi^2}\Big)\Big]\sin\omega\psi d.$$

Setting an evaluation on ther quantities $|X_n(\psi)|+|Y_n(\psi)|$, $|P_n-P_{n-1}|$; $|Q_n-Q_{n-1}|$ we can assert that the sequences $\{V_n(\psi,\varepsilon,A,\alpha)\}$, $\{(P_n(\varepsilon,A,\alpha)\}$, $\{Q_n(\varepsilon,A,\alpha)\}$ are uniformly convergent. Suppose that $P_*(\varepsilon,A,\alpha)$, $Q_*(\varepsilon,A,\alpha)$ are their limits. Obviously they satisfy the system of equations (2.11) and $V_*(\psi,\varepsilon,A,\alpha)$ are periodic in $\psi$ of the period $2\pi/\omega$.

From this result one deduces.

**Theorem 5.** *The necessary and sufficient condition for the existence of the $2\pi/\omega$ - periodic solution in $\psi$ of the equation (2.1) i.e. of the periodic wave solution of the equation (1.1) is that*

$$\begin{aligned} P(\varepsilon,A,\alpha) &= 0,\\ Q(\varepsilon,A,\alpha) &= 0. \end{aligned} \tag{2.14}$$

*Here are the system of finite equations with respect to $A$ and $\alpha$, which satisfy the conditions $A(0) = A_0$, $\alpha(0) = \alpha_0$. If the following inequality is held*

$$\Big|\frac{\partial(P,Q)}{\partial(A,\alpha)}\Big|_{A=A_0,\alpha=\alpha_0} \neq 0, \tag{2.15}$$

*then from one property of the implicit functions it is easy to deduce that the periodic wave solution of the equation (1.1) will be unique.*

Suppose now that the functions $f$ is analytic with respect to all arguments. In this case one can express $V(\psi,\varepsilon,A,\alpha)$. $P(\varepsilon,A,\alpha)$, $Q(\varepsilon,A,\alpha)$ in the power series in $\varepsilon$

$$\begin{aligned} V &= V_0(\psi) + \varepsilon V_1(\psi) + \varepsilon^2 V_2(\psi) + \dots\\ P &= P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots\\ Q &= Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots \end{aligned} \tag{2.16}$$

Substituting (2.16) into system (2.9) we can obtain the systems of equations determining the functions $V_0(\psi), V_1(\psi), P_0, P_1, \ldots, Q_0, Q_1, \ldots$

$$\begin{aligned} V_0(\psi) &= A\cos\omega\psi, \\ P_0 &= Q_0 = 0. \end{aligned} \tag{2.17}$$

$$\begin{aligned} \frac{d^2V_1}{d\psi^2} + \omega^2 V_1 &= -\frac{\omega}{b^2}\left(F_0 + 2c\alpha\omega^2 A\cos\omega\psi + P_1\cos\omega\psi + Q_1\sin\omega\psi\right) \\ P_1 &= -\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}(F_0 + 2c\alpha\omega^2 A\cos\omega\psi)\cos\omega\psi d\psi, \\ Q_1 &= -\frac{\omega}{\pi}\int\limits_0^{2\pi/\omega}(F_0 + 2c\alpha\omega^2 A\cos\omega\psi)\sin\omega\psi d\psi. \end{aligned} \tag{2.18}$$

**3.** As an application of this method we consider the following equation described the nonlinear wave propagation in the electric cables [3]:

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} + b^2 y + \varepsilon\left[\beta y^3 + \gamma(1-\sigma y^2)\frac{\partial y}{\partial t}\right], \tag{3.1}$$

Where $\varepsilon$ is a small parameter, $a$, $b$, $\beta$, $\gamma$, $\sigma$ are the positive constants.

With $\varepsilon = 0$ the equation (3.1) has the form

$$\frac{\partial^2 y_0}{\partial t^2} = a^2\frac{\partial^2 y_0}{\partial x^2} + b^2 y_0, \tag{3.2}$$

which admits the periodic wave solution

$$y_0(x,t) = A\cos\omega(x - ct + \varphi).$$

Using the above method the associative equation for (3.1) has the form

$$\left[(c+\varepsilon\alpha)^2 - a^2\right]\frac{d^2u}{d\psi^2} - b^2u = \varepsilon\left[\beta u^3 - \gamma(1-\sigma u^2)(c+\varepsilon\alpha)\frac{du}{d\psi}\right] \tag{3.3}$$

and the auxiliary system of integro-differential equations corresponding to (3.3) has the form

$$
\begin{gathered}
\left[(c+\varepsilon\alpha)^2 - a^2\right]\frac{d^2u}{d\psi^2} - b^2V = \varepsilon\Big[\beta V^3 - \gamma(1-\sigma V^2)(c+\varepsilon\alpha)\frac{dV}{d\psi}\Big] + \\
P\cos\omega\psi + Q\sin\omega\psi,
\end{gathered} \tag{3.4}
$$

$$
P = -\frac{\varepsilon\omega}{\pi}\int\limits_0^{2\pi/\omega}\Big[\beta V^3 - \gamma(12-\sigma V^2)(c+\varepsilon\alpha)\frac{dV}{d\psi} - (2c\alpha+\varepsilon\alpha^2)\frac{d^2V}{d\psi^2}\Big]\cos\omega\psi d\psi,
$$

$$
Q = -\frac{\varepsilon\omega}{\pi}\int\limits_0^{2\pi/\omega}\Big[\beta V^3 - \gamma(1-\sigma V^2)(c+\varepsilon\alpha)\frac{dV}{d\psi} - (2c\alpha+\varepsilon\alpha^2)\frac{d^2V}{d\psi^2}\Big]\sin\omega\psi d\psi.
$$

For the first approximation of the solution of the system (3.4) we have

$$
\begin{aligned}
V(\psi,\varepsilon) &= A\cos\omega\psi + \frac{\varepsilon A^3}{32b^2}(\beta\cos 3\omega\psi - \gamma\sigma\omega\sin 3\omega\psi) + \dots \\
P &= -A\Big(\frac{3}{4}\beta A^2 + 2c\alpha\omega^2)\varepsilon + \dots \\
Q &= -\omega\gamma A\Big(1 - \frac{\sigma}{4}A^2\Big)\varepsilon + \dots
\end{aligned} \tag{3.5}
$$

Giving $P = 0$, $Q = 0$ one finds

$$
A_0 = \frac{2}{\sqrt{\sigma}}, \quad \alpha_0 = -\frac{3\beta}{2c\sigma\omega^2}.
$$

Substituting these results and $\psi = x - (c+\varepsilon\alpha_0)t + \varphi$ into $V(\psi,\varepsilon)$ one finds the periodic wave solution of the equation (3.1) in the form

$$
\begin{aligned}
y(x,t,\varepsilon) = \frac{2}{\sqrt{\sigma}}\cos\omega\Big[x - \Big(c - \frac{3\beta\varepsilon}{2c\sigma\omega^2}t\Big) + \varphi\Big] + \\
+ \frac{\varepsilon}{4b^2\sigma^{3/2}}\Big\{\beta\cos 3\omega\Big[x - \Big(c - \frac{3\beta\varepsilon}{2c\sigma\omega^2}t\Big) + \varphi\Big] - \\
- \gamma\sigma\omega\sin 3\omega\Big[x - \Big(c - \frac{3\beta\varepsilon}{2c\sigma\omega^2}t\Big) + \varphi\Big]\Big\},
\end{aligned} \tag{3.6}
$$

here $\varphi$ is an arbitrary constant.

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VỀ SÓNG TUẦN HOÀN TRONG CÁC HỆ PHI TUYẾN

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Đã thu được điều kiện cần và đủ về sự tồn tại của nghiệm sóng tuần hoàn. Đã giới thiệu một phương pháp giải gần đúng phương trình trên.

Để minh họa, trong công trình cũng khảo sát phương trình truyền sóng tuần hoàn trong dây cáp điện.